Topologies of Stable Strategic Networks with Localized Payoffs

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Abstract— **There are numerous types of networks in the real-world which involve strategic actors: supply chain networks, logistics networks, company networks, and social networks. In this investigation, we explore the topologies of decentralized networks that will be formed by strategic actors who interact with one another. In particular, we analyze a network formation game in a strategic setting where payoffs of individuals depend only on their immediate neighbourhood. These localized payoffs incorporate the social capital emanating from bridging positions that nodes hold in the network. Using this novel and appealing model of network formation, our study explores the structure of networks that form, satisfying pairwise stability or efficiency or both. We derive sufficient conditions for the pairwise stability of several interesting network structures. We characterize topologies of efficient networks by applying classical results from extremal graph theory and discover that the Turán graph (or the complete equi-bipartite network) emerges as the unique efficient network under many configurations of parameters. We examine the tradeoffs between topologies of pairwise stable networks and efficient networks using the notion of price of stability. We identify several parameter configurations where the price of stability is 1 (or at least lower bounded by 0.5) in the proposed model. This leads to another key insight of this paper: under mild conditions, efficient networks will form when strategic individuals choose to add or delete links based on only localized payoffs. We study the dynamics of the proposed model by designing a simple myopic best response updating rule and implementing it on a customized network formation test-bed.**

I. Introduction

In many network settings, the behavior of the system is driven by the actions of a large number of autonomous agents, each motivated by self-interest and optimizing an individual objective function. There are numerous types of networks in the real-world which involve strategic actors: global supply chain networks, logistics networks, company networks, and social networks. A primary reason for such networks to be formed is that every person or node gets certain benefits from the network and these benefits take different forms in different types of networks. However, these benefits do not come for free. Every node in the network has to pay a certain cost for maintaining links with its immediate neighbors or direct friends. This cost takes the form of time, money, or effort depending on the type of network. Owing to the tension between benefits and costs, self-interested or rational nodes think strategically while choosing their immediate neighbors. Most often, local information rather than global information plays the central role in choosing these connections. A stable network that forms out of this process will have a topological structure as dictated by the individual utilities and best response strategies of the nodes.

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The topology of these networks often plays a crucial role in deciding the ease and speed with which certain information driven tasks can be accomplished using these networks. Typical examples of these tasks include enabling optimal communication among nodes for maximum efficiency (knowledge management), extracting certain critical information from the nodes (information retrieval), broadcasting some information to the nodes (information diffusion), collaborating on a large task to accomplish the task efficiently and fast, etc. Thus being able to predict the topology of a network formed by strategic nodes is extremely useful. The global performance of such networks, which are the equilibrium outcomes of decentralized strategic interactions, can be worse than that of a network that is enforced by a central authority. In the literature, networks that are enforced by a central authority are known as efficient networks. Understanding the compatibility between the equilibrium networks and efficient networks is the primary focus of research in network formation ([1], [2], [3]).

Most models of network formation require the agents to know the global structure of the network to compute their respective utilities. This is a demanding requirement as, in several real life decentralized applications, it is unlikely that any individual agent knows the global structure of the network. For example, in friendship networks, an individual more often does not even know who are all the friends of his friends. Thus, it is very important to study the process of network formation where each individual agent knows only its immediate neighborhood.

To the best of our knowledge, our current study is the first one to explore the tradeoff between pairwise stability and efficiency using the notion of price of stability in the context of strategic *localized* network formation, while accounting for several key factors such as link costs, link benefits, and bridging benefits. In the rest of the paper, we refer to this setting as *Network Formation with Localized Payoffs (NFLP)*. For convenience, we use the terms *graph* and *network* interchangeably throughout the paper.

A. Relevant Work

The modeling of strategic network formation in a general setting was first studied in [4]. The authors defined a notion of equilibrium called *pairwise stability* and study the tension between efficiency and pairwise stability by deriving various conditions under which efficiency and pairwise stability are compatible. The authors in [3] (using an agent-based simulation approach) and [5] identified certain pairwise stable structures that are more specific than those anticipated by the analytical findings of [4]. However, the main shortcoming of these works was that each node needed to know global

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topology information to maximize its utility. We note that there is recent research like [6], [8] in the literature that are close to our proposed approach. The model in [6] requires each individual node to know just its immediate neighbors (or 1-hop neighborhood) to optimize its own utility. The authors conducted a systematic analysis of tradeoffs between equilibrium and efficiency. However, the model captures the cost to nodes and 2-hop indirect benefits, but ignores various benefits that nodes can derive from the network such as direct benefits from the neighbors and the bridging benefits. The main focus of the model in [8] was to characterize the structure of stable networks with *Nash equilibrium* as the notion of stability. The authors proposed a polynomial time algorithm for a node to determine its best response in a given graph as nodes can choose to link to any subset of other nodes. They also showed that stable networks have a rich combinatorial structure. However, the model needs each individual node to know its 2-hop neighborhood (the set of all individuals that are reachable within two hops) to optimize its own utility. The model works with Nash equilibrium and our proposed model works with pairwise stability as the notion of equilibrium. Moreover, our model also studies the tradeoffs between the topologies of stable and efficient networks.

B. Our Contributions

The following are our specific contributions.

- We propose a strategic form game to model the process of network formation with localized payoffs and we term the game as *network formation (game) with localized payoffs* (NFLP). The payoff of each player in the proposed game takes into account not only the benefits (δ) that arise from routing information to and from its neighbours but also the cost (c) to maintain a link to each of its neighbours.
- We derive sufficient conditions for pairwise stability of certain standard network topologies using the NFLP model. Some of the networks that we consider for analysis include the cycle, star, complete, and null networks. In addition, we also derive pairwise stability conditions for certain classes of k-partite networks namely bipartite complete networks, complete equi-tri-partite networks and complete equi-k-partite networks. We note that our findings extend the possible topologies for pairwise stable networks compared to that of other models in the literature.
- Next, we analytically characterize topologies of efficient networks by drawing upon classical results from extremal graph theory. Our work leads to sharp deductions about the efficient networks in NFLP. A striking discovery of our study here is that the equi-bi-partite graph (popularly known as the Turán graph) emerges as the unique efficient network under many regions of values of δ and c.
- The quality of optimal (in terms of the sum of payoffs of the individuals in the network) pairwise stable networks is best understood through the notion of price of stability (PoS). PoS allows us to explore the middle ground between centrally enforced solution and completely unregulated anarchy [9]. In most real-world applications, the nodes are not completely unrestricted in their strategic behavior but rather agree upon a prescribed equilibrium solution. In such scenarios, the prescription can be chosen to be the best equilibrium thus making the price of stability an important issue to study. We study several parameter configurations with good PoS values in NFLP. Intriguingly, we find that PoS is 1 for almost all configurations of δ and c. This implies, under mild conditions on δ and c, that the proposed NFLP model produces pairwise stable networks that are efficient.
- Next, we undertake a simulation study to investigate the existence of any non-trivial dynamic process of network formation that yields the theoretically proven pairwise stable and efficient networks in the paper. We propose a simple best response updating rule and simulate strategic dynamics in NFLP to understand how pairwise stable networks evolve over time. Our simulation results support our analytical deductions and also reveal additional interesting insights on the topologies of pairwise stable networks. We observe that there are under suitable configurations, many of the pairwise stable and efficient networks are indeed emergent which highlights the practicality of our theoretical results. In addition, we study the evolution of pairwise stable network and its properties like the clustering co-efficient and convergence time over different configuration parameters.

C. Outline of The Paper

The rest of the paper is organized as follows. In Section II, we propose a strategic form game to model NFLP. Section III analyzes pairwise stability of various network topologies in NFLP. Section IV discusses efficient networks and studies the price of stability in NFLP. Section V examines the dynamic process of network formation in NFLP using a custom-built social network simulator. We finally summarize and discuss possible avenues of future work in Section VI.

II. A Model for Network Formation with Localized Payoffs (NFLP)

We model network formation with only local information using a strategic form game [10]. We consider a network setup with n players denoted by $N = \{1, 2, \ldots, n\}$. $|N|$ denotes the cardinality of set N. A strategy s_i of a player i is any subset of players with which the player would like to establish links. We assume that the formation of a link requires the consent of both the players. However, deletion of a link is unilateral. Assume that S_i is the set of strategies of player i. Let $s = (s_1, s_2, \ldots, s_n)$ be a profile of strategies of the players. Also, let S be the set of all such strategy profiles. Each strategy profile s leads to an undirected graph and we represent it by $G(s)$. If there is no confusion, we just use G. If players x and y form a link (x, y) in G, then we represent the new graph by $G + (x, y)$. If players x and y delete the link (x, y) between them in G, then we represent the new graph by $G - (x, y)$.

We denote d_i to represent the number of neighbors of node $i \in N$ in the given graph G. If nodes i and j are connected by a link, then we assume that the link incurs a cost $c \in (0,1)$ to each node. If nodes i and j are connected by a link, then we assume that node i and node j gain a benefit of $\delta \in (0,1)$ each. Assume that nodes j and k are two neighbours of node i such that j and k are not connected by a direct link. Suppose that nodes j and k communicate using the length 2 path through node i, then (i) we assume that a benefit of δ^2 arises due to this communication, and (ii) we also assume that the benefit δ^2 entirely goes to node i. We refer to δ^2 as the bridging benefit to node i.

The main motivation for such bridging benefits arises in sociological studies suggesting that in practice most of the bridging benefits arise from bridging the communication between pairs of non-neighbor nodes in the network ([11]). In this framework, we define the utility of node i such that it depends on the benefits from immediate neighbors, the costs to maintain links to these immediate neighbors, and the bridging benefits. More formally, for any $i \in N$, the utility u_i of node i in an undirected graph G is defined:

$$u_i(G) = d_i(\delta - c) + d_i\left(1 - \frac{\sigma_i}{\binom{d_i}{2}}\right)\delta^2 \tag{1}$$

where σ_i is the number of links among the neighbors of node i in G. There are two terms in this utility function. The first term specifies the net benefit to node i from its immediate neighbors. The second term specifies the sum of bridging benefits to node i. Here $\left(1 - \frac{\sigma_i}{\binom{d_i}{2}}\right)$ is the fraction of pairs of neighbors of node i that are non-neighbors and d_i normalizes the bridging benefits that node i gains in the network. For example, the fraction of pairs of neighbors of node 1 that are non-neighbors in both $G1$ and $G3$ in Figure 1 is 1.0. However, the degree of node 1 in $g1$ is $d_1 = 5$ and the degree of node 1 in $G3$ is $d_1 = 2$. The normalization term d_i ensures that the bridging benefit for node 1 is higher in $G1$ than in $G3$.

A. The Strategic Form Game

The above framework defines a strategic form game $\Gamma = \Big(N, (S_i)_{i\in N}, (u_i)_{i\in N}\Big)$ that models network formation with local information. As mentioned before, we refer to this as network formation game with local information (NFLP).

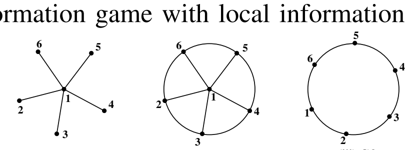

Fig. 1: An illustrative example

Example 1: Assume that $N = \{1,2,3,4,5,6\}$. If $s_1 = \{2,3,4,5,6\}$, $s_2 = \{1\}$, $s_3 = \{1\}$, $s_4 = \{1\}$, $s_5 = \{1\}$, $s_6 = \{1\}$, then the resultant graph $G1$ is the star graph as shown in Figure 1(i). Note that an edge forms with the consent of both the nodes. Following the NFLP model, the payoffs of the players in the star graph are as follows: $u_1(G1) = 5(\delta - c) + 5\delta^2$ and $u_2(G1) = u_3(G1) = u_4(G1) = u_5(G1) = u_6(G1) = (\delta - c)$. If $s_1 = \{2,3,4,5,6\}$, $s_2 = \{1,3,6\}$, $s_3 = \{1,2,4\}$, $s_4 = \{1,3,5\}$, $s_5 = \{1,4,6\}$, $s_6 = \{1,2,5\}$, then the resultant graph $G2$ is the wheel graph as shown in Figure 1(ii). Following the NFLP model, the payoffs of the players in the wheel graph are as follows: $u_1(G2) = 5(\delta - c) + \frac{5\delta^2}{2}$ and $u_2(G2) = u_3(G2) = u_4(G2) = u_5(G2) = u_6(G2) = 3(\delta - c) + \delta^2$. On similar lines, if $s_1 = \{2,6\}$, $s_2 = \{1,3\}$, $s_3 = \{2,4\}$, $s_4 = \{3,5\}$, $s_5 = \{4,6\}$, $s_6 = \{1,5\}$, then the resultant graph $G3$ is the cycle graph as shown in Figure 1(iii). Following the NFLP model, the payoffs of the players in the cycle graph are as follows: $u_1(G3) = u_2(G3) = u_3(G3) = u_4(G3) = u_5(G3) = u_6(G3) = 2(\delta - c) + 2\delta^2$.

After having discussed the network formation model in detail, we now proceed to understand the equilibrium concept of pairwise stability. Specifically, we examine ways to derive sufficient conditions for pairwise stability of certain interesting network structures under the NFLP model.

III. Structure of Pairwise Stable Networks

In this section, we first recall the notion of pairwise stability. Then, we derive sufficient conditions for certain standard networks to be pairwise stable. We first note that the notion of pairwise stability is defined in [4]. Formally, we call an undirected graph $G = (V,E)$ pairwise stable if

(i)$\forall (i,j) \in E, u_i(G) \geq u_i(G - (i,j))$ **and** $u_j(G) \geq u_j(G - (i,j))$
(ii)$\forall (i,j) \notin E,$ **if** $u_i(G) < u_i(G + (i,j))$ **then** $u_j(G) > u_j(G + (i,j))$

Proposition 1: For all $k \geq 3$, the complete k-partite network is pairwise stable if (i) $\delta = c$, and (ii) $a_i = a, \forall i \in \{1,2,...,k\}$ where a_i is the number of nodes in partition i in k-partite network and a is any positive integer.

Proof: We start with a complete k-partite graph, G, satisfying condition (ii) given in the statement of this proposition. Consider a node i in the p^{th} partition of G where $1 \leq p \leq k$. We construct the proof in two steps.

Step 1 (edge addition): We can see that, in G, the only link that can be added from node i is to a node j in the p^{th} partition. Let $\overline{G}$ be the network obtained after a new link (i,j) is added to G. It can be seen that the condition $u_i(\overline{G}) - u_i(G) \leq 0$ is equivalent to pairwise stability condition for edge addition. This implies

$$(\delta - c) + (d_i + 1)\delta^2\left(1 - \frac{\sigma_i^{'}}{\binom{d_i+1}{2}}\right) - d_i\delta^2\left(1 - \frac{\sigma_i}{\binom{d_i}{2}}\right) \leq 0$$

where $\sigma_i^{'}$ is the number of links among the neighbours of node i in $\overline{G}$ and σ_i is the number of links among the neighbours of node i in G. Note that $d_i = d_j$ since nodes i and j belong to the same partition in G. Now we get that $\sigma_i^{'} = \sigma_i + d_j = \sigma_i + d_i$. Simplifying, we get

$$u_i(\overline{G}) - u_i(G) = (\delta - c) - \delta^2 + \delta^2\left(\frac{2\sigma_i}{d_i(d_i - 1)}\right) \tag{2}$$

Since the term $\frac{2\sigma_i}{d_i(d_i-1)}$ lies in the interval $[0,1]$ and the fact that $\delta = c$ (given in the statement of this lemma), we get that expression (2) is non-positive. This implies that no pair of nodes can form a link to improve their respective utilities.

Step 2 (edge deletion): In G, consider that node i in p^{th} partition deletes a link to a node j in the q^{th} partition where

$1 \leq q \leq k$ and $p \neq q$. Let $\overline{G}$ be the network obtained after the link (i, j) has been deleted from G. It can be seen that the condition $u_i(\overline{G}) - u_i(G) \leq 0$ is equivalent to pairwise stability condition for edge deletion. This implies

$$-(\delta - c) + (d_i - 1)\delta^2 \left(1 - \frac{\sigma_i^{'}}{\binom{d_i-1}{2}}\right) - d_i \delta^2 \left(1 - \frac{\sigma_i}{\binom{d_i}{2}}\right) \leq 0$$

where $\sigma_i^{'}$ denotes the number of links among the neighbours of node i in $\overline{G}$. We can see that $\sigma_i^{'} = \sigma_i - d_j + a_i$. Simplifying,

$$-(\delta - c) - \delta^2 + \delta^2 \underbrace{\left(\frac{-2\sigma_i + 2d_j - 2a_i}{d_i - 2} + \frac{2\sigma_i}{d_i - 1}\right)}_{expr_1} \leq 0 \tag{3}$$

We know that $d_i = \sum_{j \neq i} a_j$.

$$\sigma_i = \binom{d_i}{2} - \sum_{j \neq i} \binom{a_j}{2} = \frac{d_i^2 - \sum_{j \neq i} a_j^2}{2} \tag{4}$$

Now, using the above expression for σ_i, we can show that $expr_1 \leq 1$ using proof by contradiction. We are given that $\delta = c$. Thus, from equation (3),

$$-\delta^2 + \delta^2 \underbrace{\left(\frac{-2\sigma_i + 2d_j - 2a_i}{d_i - 2} + \frac{2\sigma_i}{d_i - 1}\right)}_{\leq 1} \leq 0 \Rightarrow u_i(\overline{G}) - u_i(G) \quad \leq 0$$

So, node i does not have any incentive to add an edge to G or delete an edge from G when the conditions given in the statement of the lemma are satisfied. As node i is chosen arbitrarily from G, we have that G is pairwise stable. ■

Applying a similar technique, we can prove the stability results for other standard networks. Due to space constraints, we only summarize these results in Table I.

TABLE I: Pairwise Stability in NFLP

Parameter Region	Additional Conditions	P.S. Networks[1]
(1) $\delta > c$	**(1a)** $(\delta - c) \geq \delta^2$	Complete (Unique)
	(1b) $(\delta - c) < \delta^2$	Complete, C.B.P [4]
	(1c) $(\delta - c) < 1/2\delta^2$	C.E.T.P [6], Complete, C.B.P
(2) $\delta = c$		Complete, C.B.P, Star C.E.K.P [5], Null
(3) $\delta < c$	**(3a)** $(c - \delta) > 2\delta^2$	Null
	(3b) $(c - \delta) \leq \delta^2$	C.B.P, Null
	(3c) $\delta^2 \leq (c - \delta) \leq 2\delta^2$	Cycle, Null
	(3d) $(c - \delta) < 1/2\delta^2$	C.E.T.P, Null, C.B.P

[1]**P.S:** Pairwise Stable [4]**C.B.P:** Complete BiPartite
[5]**C.E.K.P :** Complete Equi k-Partite [6]**C.E.T.P :** Complete Equi Tri-Partite

IV. Structure of Efficient Networks

In this section, we study the structure of efficient networks, i.e., networks that maximize the overall utility, under various conditions of δ and c. First, we begin by introducing some very useful classical results in extremal graph theory which will be used later in our analysis.

A. Triangles in a Graph

If three nodes i, j, and k in $G(V, E)$ are such that i and j, j and k, k and i are connected by edges, then we say that nodes i, j, k form a triangle in G. The number of triangles in a simple graph G plays a crucial role in the computation of utilities to the nodes and we state here some classical results. We know from Turán's theorem [12], that it is possible to have a triangle free graph if $e \leq \left\lfloor \frac{n^2}{4} \right\rfloor$. Here e denotes the number of edges and n the number of vertices of the graph. Moreover, we know that the number of triangles, T, can be lower bounded [13], if the number of edges exceed the above value $\lfloor \frac{n^2}{4} \rfloor$, by $T \geq \frac{n(4e-n^2)}{9}$.

In the rest of this paper, we refer to the graph having maximum number of edges with no triangles as the *Turán Graph* and we represent it by $G_{Turán}$. It can be verified that such a graph is a complete bipartite graph, and the the number of vertices in each partition differs at most by 1.

B. Finding the Efficient Graph

Definition 1 (Efficient Graph): The utility $(u(G))$ of a given network G is defined as the sum of utilities of all the nodes in that network. That is, $u(G) = \sum_{i=1}^{n} u_i(G)$. A graph that maximizes the above expression (i.e. sum of utilities of nodes) is called an efficient graph.
We now present results on the topologies of efficient networks using the proposed framework. Due to space constraints, we do not provide proofs for all the results.

Proposition 2: When $\delta < c$ and $\delta^2 < (c - \delta)$, the null graph is the unique efficient graph.

Proof: For any node i, $d_i > 0$ implies that the utility of that node is negative thus reducing the overall network utility. This follows from $(\delta - c + \delta^2)$ being negative. ■

Proposition 3: When $\delta = c$, the Turán graph is the unique efficient graph.

Proof: We will analyze the efficiency of an arbitrary graph (denoted by G) as follows.

$$\begin{aligned} u(G) = \sum_{i=1}^{n} u_i(G) &= \sum_{i=1}^{n} d_i \delta^2 \left(1 - \frac{\sigma_i}{\binom{d_i}{2}}\right) = \delta^2 \sum_{i=1}^{n} d_i - \delta^2 \sum_{i=1}^{n} \frac{2\sigma_i}{(d_i - 1)} \\ &\leq \delta^2 \sum_{i=1}^{n} d_i - \frac{\delta^2}{(n-2)} \sum_{i=1}^{n} 2\sigma_i = \delta^2 \sum_{i=1}^{n} d_i - \frac{\delta^2}{(n-2)} (6 \times T_3(G)) \end{aligned} \tag{5}$$

where, $T_3(G)$ is the number of triangles in the graph G. The last step of the above simplification is due to the fact that the number of links among the neighbours of a node i is the number of triangles in the graph in which node i is one of the vertices of the triangle. The factor 3 in the last step is due to the fact that every triangle contributes to the σ_i of 3 nodes. We know that, for an efficient graph, equation (5) should be maximized and that happens when the number of triangles in a graph is minimized while simultaneously maximizing the number of edges in the graph.

The Turán graph is a graph with maximum edges that has no triangles. So an efficient graph must have an efficiency greater than or equal to that of a Turán graph. Thus, it is clear that there is no need to consider graphs with edges lesser than that of a Turán graph. Let us consider the case when a graph (denoted by $\overline{G}$) has more edges than the Turán graph. Let $\overline{G}$ have $\lfloor \frac{n^2}{4} \rfloor + x$ edges where $x > 0$. From equation (5),

$$\begin{aligned} u(\overline{G}) = \sum_{i=1}^{n} u_i(G) &= \delta^2 \sum_{i=1}^{n} d_i - \delta^2 \sum_{i=1}^{n} \frac{2\sigma_i}{(d_i - 1)} \\ &\leq \delta^2 \left(2\left(\left\lfloor \frac{n^2}{4} \right\rfloor + x\right)\right) - \frac{\delta^2}{(n-2)} (6T_3(\overline{G})) \end{aligned} \tag{6}$$

where $T_3(\overline{G})$ is the number of triangles in $\overline{G}$. From the Turán theorem, we have

$$u(\overline{G}) \leq \delta^2 \left(2\left(\left\lfloor \frac{n^2}{4} \right\rfloor + x\right)\right) - \frac{\delta^2}{(n-2)} \left(6n\left(\frac{4e - n^2}{9}\right)\right) \tag{7}$$

Since $T_3(G_{Turán}) = 0$, the efficiency of the Turán graph is:

$$u(G_{Turán}) = \sum_i u_i(G_{Turán}) = \delta^2 \left(2 \times \left\lfloor \frac{n^2}{4} \right\rfloor\right) \quad (8)$$

The change in efficiency (Δu) between the two graphs is

$$\Delta u = u(\overline{G}) - u(G_{Turán}) \leq 2\delta^2 \left(x - \frac{n}{(n-2)} \frac{4x}{3}\right) \quad (9)$$

which is clearly negative for any $x > 0$. This implies that the Turán graph is the unique efficient graph. ■

We summarize results on efficiency in Table II.

TABLE II: Efficient Networks in NFLP

Parameter Range	Efficient Topologies
$\delta < c$ and $\delta^2 < (c - \delta)$	Null network
$\delta < c$ and $\delta^2 > (c - \delta)$	Turán network
$\delta = c$	Turán network
$\delta > c$ and $\delta^2 > 3(\delta - c)$	Turán network
$\delta > c$ and $(\delta - c) > 2\delta^2$	Complete network

C. Price of Stability in NFLP

Price of stability(PoS) is the ratio of the sum of payoffs of the players in an optimal (in terms of sum of payoffs of the players) pairwise stable network to that of an efficient network. The following proposition can be obtained using the results summarized in Table I and Table II.

Proposition 4: PoS is 1 in each of the following scenarios:

(i) $\delta > c, (\delta - c) > 2\delta^2$ **(ii)** $\delta > c, \delta^2 > (\delta - c), \delta^2 \geq 3(\delta - c)$

(iii) $\delta = c$ **(iv)** $\delta < c, \delta^2 > (c - \delta)$

In addition to this, we can determine other parameter ranges where the PoS is lower bounded by 0.5. We present one such result below. Due to space constraints, we do not provide the details of the proof.

Proposition 5: When $\delta > c, (\delta - c) \leq \delta^2 < 3(\delta - c)$, PoS is lower bounded by 0.5.

Thus, we can observe that, under mild conditions, the proposed NFLP produces pairwise stable networks that are efficient (or close to efficient) which is very desirable from a system designer's perspective.

V. Simulations

So far, we have examined various networks which satisfy properties of pairwise stability and/or efficiency. Our results show that, under many configurations, the set of pairwise stable networks need not be unique, so even converging to a particular pairwise stable network is in itself non-trivial task. Further, this hints at the difficulty of designing dynamics that select a "good" equilibrium. Further, as shown in the studies on PoS, there is no reason to expect equilibria to be efficient in all parameter regions. Instead of focusing on one of the standard static equilibrium concept of pairwise stability discussed in earlier sections, we investigate, through simulations, whether there is some nontrivial network formation process that yields the pairwise stable and efficient networks discussed so far in the paper.

1) Simulation Setup: We start with a random initial network consisting of n nodes. The number of edges between these nodes is determined by the parameter $density(\gamma)$. For example, if $\gamma = 0$, we start with an empty network; if $\gamma = 0.35$, we start with a network that contains 35% of the possible $\binom{n}{2}$ edges. These edges are chosen uniformly at random. Each node is given an opportunity to act, based on a random schedule. We have run simulations for networks with 5, 10 and 20 nodes. However, due to space constraints, we only discuss the results for 20-node networks. Each node, when scheduled, considers three actions - namely, add an edge to a node that it is not directly connected to, delete an existing edge to a node, or do nothing. Each node chooses the action that maximizes his individual payoff, breaking ties randomly. Note that node i, when adding an edge to node j, is allowed to do so only if it is beneficial to both or if node j is not worse off. However, node i, when deleting an existing edge to node j, is allowed to do so unilaterally. We define one *iteration* as a scenario in which each node has been given exactly one opportunity to act.

At some stage, the network could evolve into a stable state where no node has any incentive to modify the network. This is the case of normal termination of a simulation run. However, there may be cases where the network does not emerge into a stable state and cycles through previously visited states even after many iterations (the case of *dynamic-equilibrium* as noted in Hummon [3]). The parameter *Max-Iterations* (= 1000 in our simulations) indicates the number of iterations before we forcibly terminate the simulation run. The parameter *Num-Repetitions* (=100 in our simulations) indicates the number of times the experiment was repeated. The simulations were averaged out over different initial conditions and random schedules.

2) Simulation Results: We examine various snapshots (Figure 2(a) to Figure 2(g)) during the network formation process of a single simulation run which is repeated for a fixed parameter of δ and c. We fix δ=c=0.5. In this configuration, we can observe from our proposed payoff model (Equation 1) that the net benefits from direct links is 0 and so, nodes try to maximize the benefits due to bridging behavior. The nodes form/delete links such that they emerge as a bridge in connecting their unconnected neighbours. Hence, we would expect the final pairwise stable network to be consisting of nodes who are filling the positions of structural holes in the network and hence, the emergent pairwise stable graph should be *triangle-free* as nodes form links with nodes who are themselves not connected with each other. We can observe that initially the nodes are forming links in such a way that triangles are not present but eventually triangles do form due to the cumulative action of other nodes in the network. When triangles emerge in the neighbourhood of a node, it leads to deletion of links from that node (as the node will benefit strictly from deletion) and the final emergent network is the triangle-free Turán network (shown in Figure 2(f)). An isomorphic representation of this network is also given in Figure 2(g) for better visualization.

We also study how the clustering coefficient (measure of number of triangles in the graph) changes as the network evolves through the different phases shown in Figure 2(a) to Figure 2(f). We plot this result in Figure 2(h). We see that clustering coefficient is 0 upto time epoch 50. Later, there is an increase in the value which is followed by the reduction in the clustering coefficient back to 0 (at time epoch 150) when the pairwise stable network emerges.

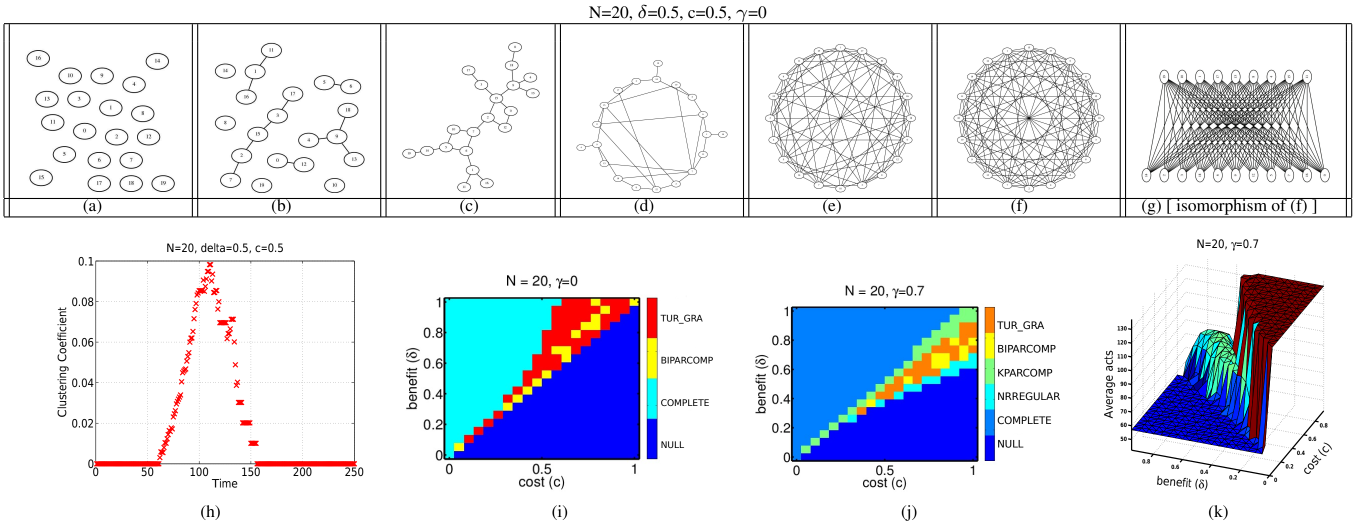

Fig. 2: Simulation Results

As explained before, this is indeed the expected behaviour during the network formation process for the parameters $\delta = c = 0.5$. Figure 2(i) and Figure 2(j) shows the simulation results for 20-node networks. The initial network densities ($\gamma = 0$ and $\gamma = 0.7$) are marked in the figures. The vertical axis of each plot in these two figures is the benefit value (δ), discretized as $\{0, 0.05, 0.1, 0.15, ..., 1\}$, and the horizontal axis represents the cost parameter (c), also discretized as $\{0, 0.05, 0.1, 0.15, ..., 1\}$. We repeat the simulation for *Num-Repetitions* for every (c, δ) pair. Each repetition for the simulation results in a network that can be classified as one of the structures mentioned in the theoretical analysis. We plot the most frequent (*modal*) network structure as determined by the frequency with which each of the network structures resulted in *Num-Repetitions* simulation runs. Some abbreviations used are:

TUR_GRA	Turán Graph	BIPARCOMP	BiPartite Complete
NRREGULAR	Near-Regular	KPARCOMP	KPartite Complete

To classify a network as Near-Regular, we compute the sorted degree vector and calculate the total mean squared deviation from an appropriate regular network (as in [3]).

We will now examine the effect the initial network density has on the effort needed by the nodes to achieve convergence to a pairwise stable network. A single addition of an edge or a single deletion of an edge by a node is considered to be a single 'act' by that player. We now study the mean number of acts performed by the players to converge to a pairwise stable network starting from initial random network with $\gamma = 0.7$. When $\delta < c$, in most configurations, the stable network is a sparse network. Thus, we can see from Figure 2(k) that the number of changes to the network is more in the $\delta < c$ region and this is because the initial network is a dense network ($\gamma = 0.7$) and the players need to perform a lot more deletions to the network before reaching the final stable network. When $\delta > c$, in most cases, the stable network is the dense network and hence, in most parameter configurations under $\delta > c$, players need to make less changes and only few acts are needed to reach the stable network.

VI. Discussion

In this paper, we proposed a network formation game with local information (NFLP) and studied topologies of pairwise stable and efficient networks. Based on this analysis, we studied the tradeoffs between pairwise stability and efficiency. In particular, we computed the PoS of the proposed NFLP. We observed that NFLP produced pairwise stable networks that are also efficient (or close to efficient) under mild conditions. This framework can be extended to the case of directed graphs and weighed graphs. This involves certain challenges such as defining utility model appropriately. Second, the setting in this paper can extended by varying the notions of stability and efficiency. We note that there are several possible notions of stability and efficiency that exist in the literature. The choice of an appropriate notion of stability as well as efficiency is a topic of debate.

References

[1] M. O. Jackson, *Social and Economic Networks*. PUP, Aug. 2008.
[2] S. Goyal, *Connections: an introduction to the economics of networks*. PUP, 2007.
[3] N. P. Hummon, "Utility and dynamic social networks," *Social Networks*, vol. 22, no. 3, pp. 221 – 249, 2000.
[4] M. Jackson and A. Wolinsky, "A Strategic Model of Social and Economic Networks," *JET*, vol. 71, no. 1, pp. 44–74, Oct. 1996.
[5] P. Doreian, "Actor network utilities and network evolution," *Social Networks*, vol. 28, no. 2, pp. 137–164, May 2006.
[6] V. Buskens and A. Van De Rijt, "Dynamics of networks if everyone strives for structural holes," *AJS*, vol. 114, no. 2, pp. 371–407, 2008.
[7] E. Arcaute, R. Johari, and S. Mannor, "Local two-stage myopic dynamics for network formation games," in *WINE 2008*, pp. 263–277.
[8] J. Kleinberg, S. Suri, E. Tardos, and T. Wexler, "Strategic network formation with structural holes," in *EC 2008*, ACM, pp. 284–293.
[9] E. Anshelevich, A. Dasgupta, J. Kleinberg, E. Tardos, T. Wexler, and T. Roughgarden, "The price of stability for network design with fair cost allocation," *SIAM Journal of Computing*, vol. 38, pp. 1602–1623, November 2008.
[10] R. B. Myerson, *Game Theory: Analysis of Conflict*. HUP, 1991.
[11] R. S. Burt, "Secondhand brokerage : Evidence on the importance of local structure for managers , bankers , and analysts," *Academy of Management Journal*, vol. 50, no. 1, pp. 119–148, 2007.
[12] P. Turán., "On an extremal problem in graph theory," *Matematikai es Fizikai Lapok*, 1941.
[13] E. Nordhaus and B. Stewart, "Triangles in ordinary graph," *Canadian Journal of Mathematics*, vol. 15, no. 1, pp. 33–41, 1963.