Article

Strategic Network Formation with Localized Pay-offs

Studies in Microeconomics
2(1) 63–119

SAGE Publications
Los Angeles, London,
New Delhi, Singapore,
Washington DC
DOI: 10.1177/2321022214522732
http://mic.sagepub.com

Rohith D. Vallam
C.A. Subramanian
Ramasuri Narayanam
Y. Narahari
N. Srinath

Abstract

In this investigation, we analyze a network formation game in a strategic setting where pay-offs of individuals depend only on their immediate neighbourhood. These localized pay-offs incorporate the social capital emanating from bridging positions that nodes hold in the network. Using this simple and novel model of network formation, our study explores the structure of networks that form, satisfying pairwise stability or efficiency or both. We derive sufficient conditions for the pairwise stability of several interesting network structures. We characterize topologies of efficient networks by drawing upon classical results from extremal graph theory and discover that the Turan graph (or the complete equi-bipartite network) emerges as the unique efficient network under many configurations of parameters. We examine the trade-offs between topologies of pairwise stable networks and efficient networks using the notion of price of stability. Interestingly, we find that price of stability is equal to 1 for almost all configurations of parameters in the proposed model; and for the rest of the configurations, we obtain a lower bound of 0.5. This leads to another key insight of this article: under mild conditions, efficient networks will form when strategic individuals choose to add or delete links based on only localized pay-offs. We study the dynamics of the

Rohith D. Vallam (corresponding author), Department of Computer Science and Automation, Indian Institute of Science, Bengaluru, India. Email: rohithdv@gmail.com
C.A. Subramanian, Orca Radio Systems, Bengaluru, India. Email: subbu.ca@gmail.com
Ramasuri Narayanam, IBM India Research Labs, Bengaluru, India. Email: ramasurn@in.ibm.com
Y. Narahari, Department of Computer Science and Automation, Indian Institute of Science, Bengaluru, India. Email: hari@csa.iisc.ernet.in
N. Srinath, Department of Electronic Systems Engineering, Indian Institute of Science, Bengaluru, India. Email: srinathnarasimha@gmail.com

proposed model by designing a simple myopic best response updating rule and implementing it on a customized network formation testbed.

Keywords

Network formation, localized pay-offs, social networks, pairwise stability, efficiency (social-welfare maximization), price of stability, myopic best response dynamics

Introduction

Several real-world networks such as the Internet, social networks, organizational networks, biological networks, food webs, co-authorship networks, citation networks and many more exhibit complex network structures. Complex networks, generally modelled as graphs in most of the mathematical literature, have been extensively studied in recent years and they are pervasive in today's science and technology (Barrat et al., 2008; Newman, 2003; Newman et al., 2006; Strogatz, 2001). Studying the properties of the complex network structures helps to understand the underlying phenomena and developing new insights into the system such as small-world phenomena, scale-free topology and structural holes (Burt, 1992, 2004; Newman, 2003; Reka and Barabási, 2002; Song et al., 2005; Watts and Strogatz, 1998).

Complex networks have also been studied extensively in the social sciences (Brandes and Erlebach, 2005; Easley and Kleinberg, 2010; Newman, 2003; Wasserman and Faust, 1994) (and the references therein). These studies reveal that complex social networks play an important role in spreading information (Boorman, 1975; Cooper, 1982; Rogers, 2003; Schelling, 1978; Strang and Soule, 1998; Valente, 1995). Individuals that participate in the process of information dissemination in such networks receive various kinds of social and economic incentives and at the same time they also incur costs in forming and maintaining the contacts (i.e., links) with other individuals in terms of time, money and effort. For this reason, individuals do act strategically while selecting their neighbours. Thus, in several contexts, the behaviour of the system is driven by the strategic actions of a large number of individuals, each motivated by self-interest and optimizing an individual objective function. Thus, it is important to study the dynamics of strategic interaction among the individuals in complex social networks in order to understand how such networks form and this is one of the primary motivations for this article.

Many recent studies on network formation have used game theoretic approaches (Borgs et al., 2011; Brautbar and Kearns, 2011; Demange and Wooders, 2005; Dutta et al., 1998; Goyal, 2007; Jackson, 2008; Jackson and Dutta, 2000; Myerson, 1991; Slikker and Nouweland, 2001) based on the observation that individuals are strategic and are interested in maximizing their pay-offs from the social interactions. These models capture the strategic interactions among individuals and the analysis of these models satisfactorily deduces the topologies of equilibrium networks. In this domain, networks that are enforced by a central authority are known as efficient networks. Understanding the compatibility between equilibrium networks and efficient networks has been the primary focus of research in network formation (Corbo and Parkes, 2005; Doreian, 2006; Elias et al., 2011; Galeotti et al., 2006; Goyal, 2007; Hummon, 2000; Jackson, 2008; Jackson and Watts, 2002).

The crux of most of the models for network formation in the literature (Anshelevich et al., 2003; Anshelevich et al., 2008; Corbo and Parkes, 2005; Elias et al., 2011; Fabrikant et al., 2003; Galeotti et al., 2006; Jackson and Wolinsky, 1996) is that the underlying strategic form game where the players, strategies and pay-offs are defined as follows: (i) the individual agents in the complex network are the players, (ii) the strategy of each agent is a subset of other agents with which it wishes to form links and (iii) the pay-off of each agent depends on the structure of the network. Another key aspect of most of the existing work in the literature is that the process of network formation is modelled in a decentralized fashion where the individuals in the network take autonomous decisions regarding whether to form or delete links with other agents.

Importance of Local Benefits

However, most of these models require the agents to know the complete global structure (that is, information about all nodes as well as all the links between the nodes) of the network to compute their respective pay-offs. This, in effect, tries to capture the philosophy that an individual is benefited not only by his immediate friends/contacts but also by his/her friends's friends, friends' friends' friends, etc. Naturally, this approach leads to complicated pay-off calculations for the nodes which can potentially be very tedious and intractable task. In fact, we can observe such computational constraints in several real-world examples like distributed sensor networks and real-world social networks. In distributed sensor networks, coalitions of sensors can work together to track targets of interest and each sensor is constrained on its energy and will typically be aware of only its immediate neighbourhood. Real-world online social networks like Facebook, Twitter, etc., are typically large in size and it may not be practical for an individual to know the entire topology to compute his/her pay-off. Further, most of the benefits an individual experiences will be primarily dominated by his/her immediate contacts. This observation has also justified by empirical evidence such as (Burt, 1992, 2004, 2007) which has clearly shown that a significant fraction of the perceived social and economic benefits for the individuals is derived from their 1-hop neighbourhood. This occurs because the benefits between nodes which are more than one hop away are in general too 'long-range' to confer significant benefits.

Importance of Structural Holes

Further, in many scenarios, it may be beneficial for individuals to form links with others who are not connected among themselves. This is counter-intuitive in scenarios like friendship networks, etc., which exhibit a large number of triads. In such friendship networks, it is very often that an individual's friend's friend is also a direct friend of the individual. However, in many other networks, this may not necessarily represent the true observations. Let us examine the following observation from (Burt, 2004):

> Opinion and behavior are more homogeneous within than between groups, so people connected across groups are more familiar with alternative ways of thinking and behaving. Brokerage across the structural holes between groups provides a vision of options otherwise unseen, which is the mechanism by which brokerage becomes social capital.

Basically, (Burt, 2004) came to the conclusion that in organizational networks, people who bridge multiple unconnected nodes are in a more advantageous position than others who are closely knit within a group. Such positions are termed 'structural holes' as they bridge the gap between multiple, diverse groups. Thus, people whose networks span structural holes have early access to diverse, often contradictory, information and interpretations, which gives them a competitive advantage in seeing good ideas.

Mehra et al. (2001) studied the influence of structural positions on promotions and performance evaluation in organizations. They argue that the differences in structural location of individuals, in particular, whether they bridge structural holes in the social network, explains a significant part of the variation in promotion timing of otherwise similar people. Similar studies on the importance of structurally advantageous network positions in the context of intra-organizational mobility have been mentioned in Podolny and Baron (1997) while Ahuja (2000) considered a detailed study on the influence of a firm's position in inter-organizational networks on its innovativeness and overall performance.

Thus, it may be worthwhile to develop a 'localized' model of network formation wherein there are benefits and costs for each link formed along with a need for individuals to be part of structural holes in the network. To the best of our knowledge, very few works have tried to capture these aspects in modelling strategic network formation. We briefly review these works in the literature review section-and put forth the key differences between these models and our proposed model.

Overview

In our work, we model network formation wherein a rational individual's benefits are determined only by its immediate neighbourhood or 1-hop neighbourhood information. Beyond the benefits of having a direct links to other individuals, there is also a form of synergy. This synergy occurs due to the bridging benefits that the individual experiences due to his/her sparse neighbourhood. A sparse neighbourhood leads to higher synergistic benefits due to structural holes. We refer to this setting as 'Network Formation with Localized Pay-offs (NFLP)'. We will go through the model in detail in Section 3.

An example scenario where this model can be applicable is in the context of interaction networks. These interaction networks can be among researchers in an university, interactions among employees in an organization, etc., wherein an undirected connection exists between two individuals if they interact for a period of time on some subject of mutual interest. Here, there is a benefit that is obtained from the interaction when individuals interact with another individual. At the same time, there is a cost accrued to both the individuals in the form of time or effort spent on the interaction. An individual can also synergistically benefit from having interaction with persons from diverse areas which potentially may lead to 'good', unbiased, creative ideas being developed by the individual which falls back to Burt's hypothesis about the importance of structural holes in the network. These ideas will be more 'biased' if there is more connectivity among the contacts of an individual. Also, in these scenarios, the pay-off to an individual will be typically due to the immediate contacts of the individual with whom he/she has an interaction even though the individual may be 'aware' of the existence of other individuals outside his 1-hop neighbourhood. Our model, thus, assumes that the individuals are aware of the presence of other individuals in the network even though they might not be directly connected with them. However, our model does not assume that individuals know the link structure of the entire network.

We also note that our model assumes that a link forms with the consent of both the individuals (refer to Section 3), as social contacts usually emerge in this manner. This assumption is widely considered in several models of network formation in the literature (Doreian, 2006; Hummon, 2000; Jackson, 2003; Jackson and Wolinsky, 1996; Xie and Cui, 2008a, 2008b). In such situations, an appropriate choice for the notion of equilibrium is 'pairwise stability' (Jackson and Wolinsky, 1996). Informally, we call a network pairwise stable if no agent can improve its pay-off by deleting any link and no two unconnected individuals can form a link to improve their respective pay-offs. We call a network 'efficient' if the sum of pay-offs of the individuals is maximal. In this framework, our objective is to investigate the trade-off between topologies of pairwise stable and efficient networks.

The primary contribution of our work is to come up with a game theoretic model in the above setting and study the topologies of the equilibrium networks and efficient networks that emerge in such a network formation process. We next examine the trade-offs between topologies of equilibrium networks and efficient networks using the notion of price of stability (Anshelevich et al., 2008). Informally, price of stability is the ratio of the sum of pay-offs of the players in an optimal (in terms of sum of pay-offs of the players) pairwise stable network to that of an efficient network. Interestingly, we find that price of stability is 1 for almost all configurations of the parameters in the proposed model; and for the rest of the configurations of the parameters in the proposed model, we obtain a lower bound of 0.5 on price of stability. This indicates that, when some mild conditions are satisfied, efficient networks will form when strategic individuals choose to add or delete links based on localized pay-offs. Further, we propose a simple best response updating rule in order to understand the dynamics of the network formation process under the proposed model. Through extensive simulations, we investigate various effects on the network formation based on varying initial conditions, parameter configurations, etc. One of the primary observations that we make is that many of the theoretically proven efficient and pairwise stable networks do, in fact, emerge as a result of this dynamic process which indicates the practicality of the results obtained in the article.

In the rest of the article, we use the terms 'graph' and 'network' interchangeably. We thus use the terms 'nodes' and 'individuals' interchangeably throughout the article. As a game-theoretic approach is used, we sometimes use the terms 'players' and 'individuals' interchangeably throughout the article.

Literature Review

The field of network formation has been extensively studied in diverse fields such as sociology, physics, computer science, economics, mathematics and biology (Bloch and Jackson, 2007; Borgs et al., 2011; Brautbar and Kearns, 2011; Buskens and Van De Rijt, 2008; Calvo-Armengol, 2004; Demange and Wooders, 2005; Doreian, 2006; Doreian, 2008a, 2008b; Dutta et al., 1998; Galeotti et al., 2006; Gilles and Johnson, 2000; Goyal, 2007; Goyal and Vega-Redondo, 2007; Hummon, 2000; Jackson, 2003, 2005, 2008; Jackson and Dutta, 2000; Jackson and van den Nouweland, 2005; Jackson and Watts, 2002; Jackson and Wolinsky, 1996; Kleinberg et al., 2008; Slikker and Nouweland, 2001). In this section, we have included a discussion of the models that are most relevant to our work.

The modelling of strategic formation in a general network setting was first studied in the seminal work of (Jackson and Wolinsky, 1996). They basically consider a value function and an allocation rule model where the value function defines a value to each network and the allocation rule distributes this value to the nodes in the network. They investigate whether efficient networks will form when self-interested individuals can choose to form links and/or break links. The authors define two stylized models. For these models, the authors observe that for high and low costs the efficient networks are pairwise stable, but not always for medium level costs. They also examine the tension between efficiency and stability and derive various conditions and allocation rules for which efficiency and pairwise stability are compatible. An important feature their model does not capture is that of the intermediary benefits that nodes gain by being intermediaries lying on the paths between non-neighbour nodes. In particular, they do not capture the benefits due to structural holes.

Hummon (2000) carries out several interesting investigations to unravel more specific topologies using a particular model proposed by Jackson and Wolinsky (1996). Two different agent-based simulation approaches, the multi-thread model and the discrete event simulation model, are used in the analysis done by Hummon (2000) to explore the dynamics of network evolution based on a model proposed in Jackson and Wolinsky (1996). Hummon identifies certain pairwise stable structures that are more specific than those anticipated by the formal analysis of Jackson and Wolinsky (1996). Doreian (2006) explores the same issue in a systematic manner and establishes the conditions under which different pairwise structures are generated. Some gaps in the analysis of Doreian (2006) are addressed by Xie and Cui (2008a, 2008b).

Jackson (2003) reviews several models of network formation in the literature with an emphasis on the trade-offs between efficiency with stability. This work also studies the relationship between pairwise stable and efficient networks in a variety of contexts and under three different definitions of efficiency. A later paper by Jackson (2005) presents a family of allocation rules (for example, networkolus) that incorporate information about alternative network structures when allocating the network value to the individual nodes. The author provides a general method of defining allocation rules in network formation games.

Goyal and Vega-Redondo (2007) propose a non-cooperative game model in which a node i can benefit from serving as an intermediary between a pair of nodes x and y. In their model, a node i could lie on an arbitrarily long path between x and y. The authors assume, however, that the benefits from farther nodes are not subject to decay. They also assume that the benefit of communication between any pair of nodes is always one unit. This unit is distributed to the two communicating nodes and only to certain so called essential nodes (Goyal and Vega-Redondo, 2007) on the paths between the two communicating nodes. In this setting, the authors show that a star graph is the only non-empty robust equilibrium graph. The authors also study the implications of capacity constraints in the ability of individual nodes to form links to other nodes and show that a cycle network emerges.

Narayanam and Narahari (2011) propose a generic model of network formation that essentially builds on the model of Jackson and Wolinsky (1996). This model simultaneously captures four key determinants of network formation: (i) benefits from immediate neighbours through links, (ii) costs of maintaining the links, (iii) benefits from non-neighbouring nodes and decay of these benefits with distance and (iv) intermediary benefits that arise from multi-step paths. Narayanam and Narahari (2011) analyze the proposed model to determine the topologies of stable and efficient networks.

The aforementioned models of network formation have the limitation that each individual (or node) needs to know global information about the structure of the network in order to compute its pay-off. A few recent models (Arcaute et al., 2008;

Buskens and Van De Rijt, 2008; Kleinberg et al., 2008) in the literature make an attempt to overcome the above limitation.

1. Buskens and Van De Rijt (2008) propose a model that requires each individual agent to know just its immediate neighbours (or 1-hop neighbourhood) to optimize its own pay-off. However, the model captures only the cost to nodes and ignores various benefits that nodes can derive from the network such as direct benefits from the neighbours and the bridging benefits.
2. Arcaute et al. (2008) study the myopic dynamics in network formation games. A key aspect of the dynamics studied in this model is the local information and the authors show that these dynamics converge to efficient or near efficient outcomes. However, the model does not identify the topologies of equilibrium and efficient networks. Moreover, the model works with Pareto efficiency whereas we work with a more natural notion of efficiency, namely social-welfare maximization.
3. Kleinberg et al. (2008) propose a game-theoretic network formation model where the pay-off of each node is based on local neighbourhood information. This model captures direct link benefits along with intermediary benefits that nodes accrue by being on the 2-hop path between two unconnected nodes (say v and w). This model assumes that the intermediary benefits decreases in the number of other length-2 paths between v and w. Also, they assume that the benefits a node accrues due to bridging activity is additive in the number of pairs it bridges. Also, they consider a model wherein the link cost is borne only by one of the endpoints of the link. Kleinberg et al. (2008) also characterize the structure of stable networks with 'Nash equilibrium' as the notion of stability. The authors propose a polynomial time algorithm for a node to determine its best response in a given graph as nodes can choose to link to any subset of other nodes. They also show that stable networks have a rich combinatorial structure. Also, the model in Kleinberg et al. (2008) works with Nash equilibrium, while our proposed model works with the more natural notion of pairwise stability as the notion of equilibrium. Pairwise stability incorporates the effects of both unilateral and bilateral deviations unlike the notion of Nash stability which considers robustness due to only unilateral deviations. Our model additionally investigates the trade-off between the topologies of stable networks and the topologies of efficient networks through the natural notion of price of stability as well as through the result of myopic best response updating rules.

Our Contributions

To the best of our knowledge, our current study is the first one to comprehensively explore the trade-off between pairwise stability and efficiency using the notion of price of stability in the context of strategic *localized* network formation, while accounting for several key factors such as link costs, link benefits and bridging benefits. The following are the specific contributions of our article:

1. *Section 3: An Elegant Model for Network Formation with Localized Pay-offs:* We propose a strategic form game to model the process of network formation with localized pay-offs and we term the game as *network formation (game) with localized pay-offs* (NFLP). The pay-off of each player in the proposed game takes into account not only the benefits (δ) that arise from routing information to and from its neighbours but also the cost (c) to maintain a link to each of its neighbours.
2. *Section 4: Sufficient Conditions for Pairwise Stability of Network Topologies:* We derive sufficient conditions for pairwise stability of certain standard network topologies using the NFLP model. Some of the networks that we consider for analysis include the cycle, star, complete and null networks. In addition, we also derive pairwise stability conditions for certain classes of k-partite networks namely bipartite complete networks, complete equi-tripartite networks and complete equi-k-partite networks. We note that our findings extend the possible topologies for pairwise stable networks compared to that of other models in the literature.
3. *Section 5: Characterization of Topologies of Efficient Networks:* Next, we analytically characterize topologies of efficient networks by drawing upon classical results from extremal graph theory. Our work leads to sharp deductions about the efficient networks in NFLP. A striking discovery of our study here is that the equi-bipartite graph (popularly known as the Turan graph) emerges as the unique efficient network under many regions of values of δ and c.
4. *Section 6: Price of Stability Investigations:* The quality of optimal (in terms of the sum of pay-offs of the individuals in the network) pairwise stable networks is best understood through the notion of price of stability (PoS). PoS allows us to explore the middle ground between centrally enforced solution and completely unregulated anarchy (Anshelevich et al., 2008). In most real-world applications, the nodes are not completely unrestricted in their strategic behaviour but rather agree upon a prescribed equilibrium solution. In such scenarios, the prescription can be chosen to be the best equilibrium thus making the price of stability an important issue to study. We study the PoS in NFLP to reveal trade-offs between pairwise stable networks and efficient networks. Intriguingly, we find that PoS is 1 for almost all configurations of δ and c. For the remaining configurations of δ and c, we obtain a lower bound of $\frac{1}{2}$ on PoS. This implies, under mild conditions on δ and c, that the proposed NFLP model produces pairwise stable networks that are efficient.
5. *Section 7: Convergence of Pairwise Stable/Efficient Networks Through Best Response Dynamics:* Next, we investigate, through simulations, about

the existence of any non-trivial dynamic process of network formation that yields the theoretically proven pairwise stable and efficient networks in the article. We propose a simple best response updating rule and simulate strategic dynamics in NFLP to understand how pairwise stable networks evolve over time. Our simulation results support our analytical deductions and also reveal additional interesting insights on the topologies of pairwise stable networks. We observe that there are under suitable configurations, many of the pairwise stable and efficient networks are indeed emergent which highlights the practicality of our theoretical results. In addition, we study the evolution of pairwise stable network and its properties like the clustering co-efficient, convergence time, etc., over different configuration parameters.

We now begin our investigation by proposing the details of NFLP model in the next section. In order to enhance readability, we delegate most of the proofs of the results in the different sections to the appendices.

A Model for Network Formation with Localized Pay-offs (NFLP)

We model network formation using a strategic form game as proposed by Myerson (1991). We consider a network setup with n players denoted by $N = \{1, 2,..., n\}$. A strategy $s_i \subseteq N \setminus \{i\}$ of a player i is any subset of players with which the player would like to establish links. Assume that S_i is the set of all possible strategies of player i. Let $s = (s_1, s_2,..., s_n)$ be a profile of strategies of the players. Also let $S = \times_{i=1}^{n} S_i$ be the set of all such strategy profiles. For notational convenience, we represent $s \in S$ as $s = (s_i, s_{-i})$ given a player $i \in N$. We also represent $s \in S$ as $s = (s_i, s_j, s_{-ij})$ given a player $i, j \in N$. Here s_{-i} is the strategy profile of all players except player i. Similarly, s_{-ij} is the strategy profile of all players except player i and player j. Note that each strategy profile $s \in S$ induces a corresponding underlying network which we denote by $G(s)$. If there is no confusion about the underlying strategy profile s, we just use G to represent $G(s)$. We also say that the profile s supports G. Note that G represents the outcome of a strategic game where nodes play the strategy profile s. We denote the vertex and edge sets of the graph G by V and E respectively.

Let (i, j) represent an ordered pair of vertices i and j. We assume that the formation of a link requires the consent of both the players i.e., given a strategy profile $s = (s_1, s_2,... s_n)$, the outcome of the game is the undirected network G defined by $(i, j) \in E$ if and only if $i \in s_j$ and $j \in s_i$. Also, an edge between i and j in G is either represented by (i, j) or (j, i) as all edges in G are undirected. Every node in the game derives a pay-off from the outcome of the game. We represent the pay-off of a node i by the real-valued function u_i: G → R where G is the set of all possible undirected graphs with n nodes and R is the set of real numbers. Note that if players x and y form a link (x, y) in a graph G, then we represent the new graph by $G + (x, y)$. If players x and y delete an existing link (x, y) in a graph G, then we represent the new graph by $G - (x, y)$. Also we use the terms graph or network interchangeably throughout the article. We also use the terms players and nodes interchangeably throughout the article.

Definition 1 (wolinsky: 96): *We call an undirected graph $G = (V, E)$ pairwise stable if*

$$\text{(i) } \forall(i, j) \in E,\ u_i(G) \geq u_i(G - (i, j)) \text{ and } u_j(G) \geq u_j(G - (i,j)) \tag{1}$$

$$\text{(ii) } \forall(i, j) \notin E \text{ if } u_i(G) < u_i(G + (i, j)) \text{ then } u_j(G) > u_j(G + (i, j)) \tag{2}$$

Note that Equation (1) represents the condition for stability under deletion of an existing link and Equation (2) represents the condition for stability under addition of a new link. Thus, Equation (1) represents the scenario where a node is not better off by unilaterally deleting one of its existing links. The condition specified by Equation (2) represents a scenario where no new link can be formed in a pairwise stable network. This condition uses the fact that link addition requires mutual consent and in a pairwise stable network, even if one of the nodes has consented to establish a link (as forming the link will strictly benefit the node), the link will not be formed as the other node will be strictly worse-off forming the link.[1]

In network formation scenarios, link formation is inherently bilateral: the consent of two nodes is required to form a single link. This is the reason we use pairwise-stability as the equilibrium concept in this work as it is robust to both unilateral and bilateral deviations. We now describe some of the basic parameters of the network formation game.

Degree of Node: The degree $d_i(G)$ of node i in the graph G represents the number of neighbours of node i in G. Node j is a neighbour of node i in G if there exists an undirected link (i, j) in G. For simplicity of notation, we denote the degree of node i by d_i whenever the underlying graph is implied by the context.

Costs: If nodes i and j are connected by a link in G, then we assume that each node incurs a cost $c \in (0, 1)$ for maintaining that link i.e., if the degree of node i is d_i, then node i incurs a cost of cd_i.

Benefits from Immediate Neighbours: Assume that $\delta \in (0, 1)$. If node i is connected to a node j by a direct link, then we assume that both node i and node j accrue a benefit of δ from this link i.e., if the degree of node i is d_i, then node i gains a benefit of δd_i from its immediate neighbours.

Bridging Benefits: Consider a node i. Assume that nodes j and k are two neighbours of node i such that j and k are not connected by a direct link. Suppose that nodes j and k communicate using the length 2 path through node i, then (i) we assume that a benefit of δ^2 arises due to this communication, and (ii) we also assume that the benefit δ^2 entirely goes to node i. We refer to δ^2 as the bridging benefit[2] to node i. The main motivation for this kind of bridging benefits is by sociological studies suggesting that in practice most of the benefits arise from bridging the communication between pairs of non-neighbour nodes in the network (Burt, 2007). We assume that players in the network communicate using shortest paths—this is a standard assumption used in the literature for ease of modelling.

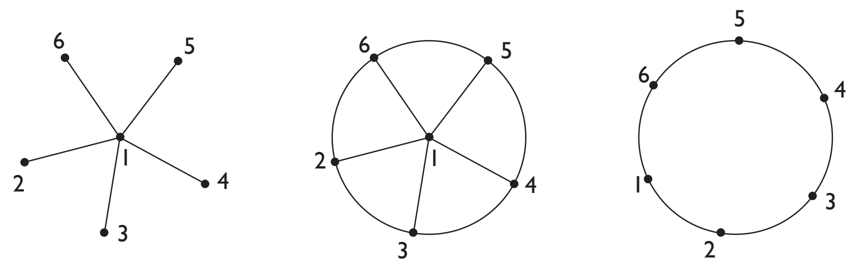
Pay-off Model

In this strategic form game described above, we define the pay-off of node *i* such that it depends on the benefits from immediate neighbours, the costs to maintain links to these immediate neighbours and the bridging benefits. Let σ_i be the number of links among the neighbours of node *i* in *G*. In the graph $G, \frac{\sigma_i}{\binom{d_i}{2}}$ represents the clustering co-efficient of the node *i* in *G*. Basically, the clustering co-efficient of node *i* represents the fraction of pairs of neighbours of node *i* that are neighbours and thus it is a measure of the density of the neighbourhood of node *i*. Formally, for any $i \in N$, the pay-off u_i of node *i* in an undirected graph *G* is defined as follows:

$$u_i(G) = \underbrace{d_i(\delta - c)}_{(a)} + \underbrace{\underbrace{d_i(1 - \frac{\sigma_i}{\binom{d_i}{2}})}_{(c)}\ \delta^2}_{(b)} \tag{3}$$

There are two terms in this pay-off function. The first term (represented by (*a*) in Equation (3) specifies the net benefit to node *i* from its immediate neighbours. The second term (represented by (*b*) in Equation (3) specifies the bridging benefits to node *i*. We elaborate this below.

As described in the introduction, players obtain lower pay-off for bridging nodes in a densely connected neighbourhood than in a sparsely connected neighbourhood. We use degree-weighted inverse clustering coefficient (given by (*c*) in Equation (3) for this purpose. Basically, the intuition behind this expression is the following. If a node has contacts who are themselves unconnected, then the node accrues a bridging benefit of $d_i\delta^2$. On the other extreme, if all the neighbours of node *i* are completely connected among themselves, then the node accrues a bridging benefit of 0 as the clustering coefficient will be 1 in this scenario. For example, in the context of interaction networks, a node with a densely connected neighbourhood will suffer a reduction in bridging benefits as a result of lower quality of ideas generated. The low quality of ideas can be attributed to the presence of biased and redundant opinions in the neighbourhood of node *i*. Also, note that d_i weight in (*c*) of Equation (3) normalizes the level of bridging benefits that node *i* gains in the network. It enables more bridging benefits to nodes who have more contacts and vice versa.



(i) G1 (ii) G2 (iii) G3

Figure 1. An Illustrative Example

Source: Developed by the Authors.

For example, consider Figure 1. The fraction of pairs of neighbours of node 1 that are non-neighbours in both $G1$ and $G3$ in Figure 1is 1.0. However the degree of node 1 in $G1$ is $d_i = 5$ and the degree of node 1 in $G3$ is $d_1 = 2$. The normalization term d_i ensures that the bridging benefit for node 1 is higher in $G1$ than in $G3$.

NFLP – An Example

The above framework defines a strategic form game $\Gamma = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ that models network formation with localized pay-offs. We refer to this as network formation game with localized pay-offs (NFLP). The following example illustrates NFLP.

Example 1: *Assume that* $N = \{1, 2, 3, 4, 5, 6\}$. *If* $s_1 = \{2, 3, 4, 5, 6\}$, $s_2 = \{1\}$, $s_3 = \{1\}$, $s_4 = \{1\}$, $s_5 = \{1\}$, $s_6 = \{1\}$, *then the resultant graph* $G1$ *is the star graph as shown in Figure 1(i). Note that an edge forms with the consent of both the nodes.*

Following the NFLP model, the pay-offs of the players in the star graph are as follows: $u_1(G1) = 5(\delta - c) + 5\delta^2$ and $u_2(G1) = u_3(G1) = u_4(G1) = u_5(G1) = u_6(G1) = (\delta - c)$.

If $s_1 = \{2, 3, 4, 5, 6\}$, $s_2 = \{1, 3, 6\}$, $s_3 = \{1, 2, 4\}$, $s_4 = \{1, 3, 5\}$, $s_5 = \{1, 4, 6\}$, $s_6 = \{1, 2, 5\}$, then the resultant graph $G2$ is the wheel graph as shown in Figure 1(ii). Following the NFLP model, the pay-offs of the players in the wheel graph are as follows: $u_1(G2) = 5(\delta - c) + \frac{5\delta^2}{2}$ and $u_2(G2) = u_3(G2) = u_4(G2) = u_5(G2) = u_6(G2) = 3(\delta - c) + \delta^2$.

On similar lines, if $s_1 = \{2, 6\}$, $s_2 = \{1, 3\}$, $s_3 = \{2, 4\}$, $s_4 = \{3, 5\}$, $s_5 = \{4, 6\}$, $s_6 = \{1, 5\}$, then the resultant graph $G3$ is the cycle graph as shown in Figure 1(iii). Following the NFLP model, the pay-offs of the players in the cycle graph are as follows: $u_1(G3) = u_2(G3) = u_3(G3) = u_4(G3) = u_5(G3) = u_6(G3) = 2(\delta - \text{c}) + 2\delta^2$.

After having discussed the network formation model in detail, we now proceed to understand the equilibrium concept of pairwise stability. Specifically, we examine ways to derive sufficient conditions for pairwise stability of certain interesting network structures under the NFLP model.

Sufficient Conditions for Pairwise Stability of Network Topologies

In this section, we first recall the notion of pairwise stability. We derive sufficient conditions for pairwise stability for certain standard network topologies considered in the literature. Note that we set forth to derive only sufficient conditions for certain network topologies to be pairwise stable. This is due to the fact that under certain parameter configurations, there may be multiple network structures which may be simulataneouly stable and hence, due to combinatorial size of possible network structures, it is tractable to only derive sufficient conditions for certain standard and interesting network structures to be pairwise stable. Further, in our article, we discover new pairwise stable structures through extensive simulations. These results will be presented in Section 7. However, one of the interesting observations that emerges out of the analysis in this section is that the complete network emerges as the unique pairwise stable network under certain parameter configuration i.e., this configuration serves as both necessary and sufficient condition for the complete network to be pairwise stable.

Examining Pairwise Stability of Network Topologies

We now focus on examining the pairwise stability of the certain standard topologies in the framework in NFLP. Pairwise stability under various network formation models has been addressed in the literature Jackson (2008), Goyal (2007), Galeotti et al. (2006) Buskens and Van De Rijt (2008), Goyal and Vega-Redondo (2007), Kleinberg et al. (2008), Fabrikant et al. (2003), Corbo and Parkes (2005), Jackson and Wolinsky (1996), Doreian (2006), Doreian (2008a, 2008b). In our approach, we consider the topologies of certain standard networks (such as complete network, cycle network, star network, multi-partite networks) and then study whether such topologies are pairwise stable following the framework of NFLP. We now present few results to establish certain standard networks are pairwise stable in the framework of NFLP.

Proposition 1: *The cycle network is pairwise stable under the following conditions:*

1. For cycle of length 3, $(\delta - c) \geq 0$
2. For cycle of length 4, $-2\delta^2 \leq (\delta - c) \leq \delta^2$

3. For cycle of length 5, $-2\delta^2 \leq (\delta - c) \leq \delta^2$
4. For cycle of length 6 or greater, $-2\delta^2 \leq (\delta - c) \leq -\delta^2$

Proposition 2: *If* $-\delta^2 \leq (\delta - c) \leq \delta^2$*, then the complete bipartite network is pairwise stable.*

Proposition 3: *The null (empty) network is pairwise stable if* $(\delta - c) \leq 0$.

Proposition 4: *The complete network is pairwise stable if* $(c - \delta) \leq 0$.

Proposition 5: *When* $(\delta - c) > \delta^2$*, the complete network is the unique pairwise stable network.*

Proposition 6: *The star network is pairwise stable only when* $\delta = c$.

Proposition 7: *Consider a tripartite complete graph denoted by G. Let* a_i, $\forall i \in \{1, 2, 3\}$ *denote the sizes of the three partitions of G. The condition for G to be pairwise stable is given below.*

$$\frac{-2(a_j-1)}{(a_j+a_k-2)}+\frac{(a_j(a_j-1)+a_k(a_k-1))}{(a_j+a_k-1)(a_j+a_k-2)} \leq \frac{(\delta-c)}{\delta^2} \leq \frac{(a_j(a_j-1)+a_k(a_k-1))}{(a_j+a_k-1)(a_j+a_k)}$$

Further, asymptotically as the number of nodes increases, the pairwise stability condition reduces to

$$-\frac{1}{2} \leq \frac{\delta-c}{\delta^2} \leq \frac{1}{2} \tag{4}$$

when all the partitions in G are of equal size.

Proposition 8: *For* $k \geq 3$*, the complete k-partite network is pairwise stable if (i)* $\delta = c$*, and (ii)* $a_i = a, \forall i \in \{1, 2, ..., k\}$ *where* a_i *is the number of nodes in partition i in k-partite network and a is any positive integer.*

We list the pairwise stability results under the NFLP model in Table 1 and graphically illustrate the parameter regions in Figure 2.

Characterization of Topologies of Efficient Networks

In this section, we study the structure of efficient networks, i.e., networks that maximize the overall pay-off, under various conditions of δ and c. First, we begin by introducing a few useful classical results in extremal graph theory and we use these results later in our analysis.

Triangles in a Graph

If three nodes i, j and k in $G(V, E)$ are such that i and j, j and k, k and i are connected by edges, then we say that nodes i, j, k form a triangle in G. The number of triangles in a simple graph G plays a crucial role in the computation of pay-offs to the nodes

Table 1. Sufficient Conditions for Pairwise Stable Network Topologies in the Proposed Pay-off Model

Parameter Region	Additional Conditions	Pairwise Stable Topologies	Implied by
(1) $\delta > c$	**(1a)** $(\delta - c) \geq \delta^2$	Complete (Unique)	Proposition 4, Proposition 5
	(1b) $(\delta - c) < \delta^2$	Complete C.B.P[1]	Proposition 4 Proposition 2
	(1c) $(\delta - c) < (1/2)\delta^2$	C.E.T.P[3] Complete C.B.P	Proposition 7 Proposition 4 Proposition 2
(2) $\delta = c$		Complete, Null, C.B.P, Star, C.E.K.P[2]	Proposition 4, Proposition 3 Proposition 2, Proposition 6 Proposition 8
(3) $\delta < c$	**(3a)** $(c - \delta) > 2\delta^2$	Null	Proposition 3
	(3b) $(c - \delta) \leq \delta^2$	C.B.P Null	Proposition 2 Proposition 3
	(3c) $\delta^2 \leq (c - \delta) \leq 2\delta^2$	Cycle Null	Proposition 1 Proposition 3
	(3d) $(c - \delta) < (1/2)\delta^2$	C.E.T.P Null C.B.P	Proposition 7 Proposition 3 Proposition 2

Source: Developed by the Authors.
Notes: [1]C.B.P: Complete BiPartitie [2]C.E.K.P: Complete Equi K-Partite 3C.E.T.P: Complete Equi Tri-Partite

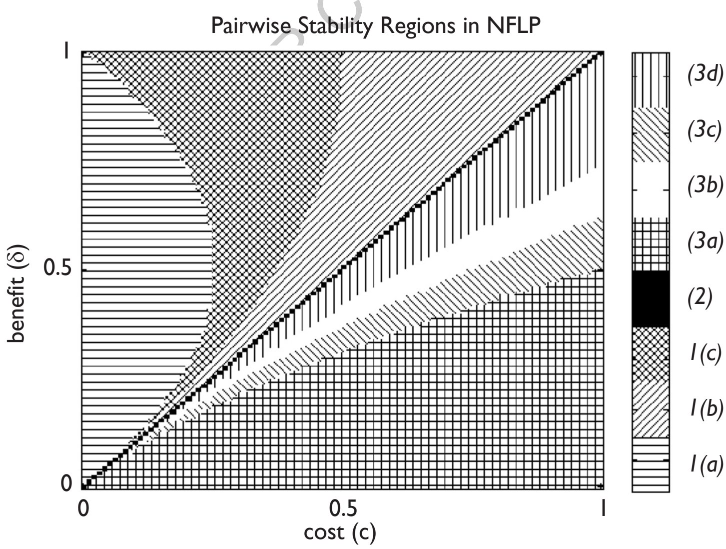


Figure 2. Graphical Illustration of Pairwise Stability Results for the NFLP Model

Source: Developed by the Authors.
Note: The legends in the figure correspond to the numbering specified in Table 1.

and we state here some classical results. We know from Turan's theorem (Turan, 1941), that it is possible to have a triangle free graph if the following holds:

$$e \leq \left\lfloor \frac{n^2}{4} \right\rfloor \tag{5}$$

Here e denotes the number of edges and n the number of vertices of the graph. Moreover, from Nordhaus and Stewart (1963), we know that the number of triangles, T, can be lower bounded, if the number of edges exceed the above value $\left\lfloor \frac{n^2}{4} \right\rfloor$, by

$$T \geq \frac{n(4e - n^2)}{9} \tag{6}$$

In what follows, we refer to the graph having maximum number of edges with no triangles as the *Turan Graph* and we represent it by G_{Turan}. It is easy to verify that such a graph is a complete bipartite graph, and the the number of vertices in each partition differs at most by 1.

Definitions

Let G^N be the set of all undirected graphs that are possible with $N = \{1, ..., n\}$ players. Note that $|G^N| = 2^{n(n-1)/2}$. We now proceed to define two notions of efficiency of networks commonly used in literature namely social-welfare maximization and Pareto-optimality.

Definition 2 (Social-Welfare Maximizing Network): *A network G^* is said to be social-welfare maximizing if it maximizes the sum of pay-offs of all players when compared to any other network G i.e.,*

$$G^* = \underset{G \in G^N}{\arg\max}\ u(G) \quad \text{Where } u(G) = \sum_{i=1}^{n} u_i(G)$$

Definition 3 (Pareto-Optimal Network): *A network G^* is said to be Pareto-optimal if there does not exist any $G' \in G^N$ such that*

$$u_i(G^*) \leq u_i(G'),\ \forall\ i \in N \text{ and } \exists j \in N,\ u_j(G^*) < u_j(G')$$

Lemma 1: *Let G^* be a social-welfare maximizing network. Then G^* is Pareto-optimal.*

Proof: Suppose G^* is not Pareto-optimal. Then, we can find a G' and $j \in N$ such that $u_i(G^*) \leq u_i(G')$, $\forall\ i \in N$ and $u_j(G^*) < u_j(G')$. Thus,

$$u(G') = \sum_{i=1}^{n} u_i(G') > \sum_{i=1}^{n} u_i(G^*) = u(G^*)$$

which contradicts the social-welfare maximizing property of G^*. Hence, G^* is a Pareto-optimal network.

By Lemma 1, we can observe that social-welfare maximizing property is a stronger condition than Pareto-optimality. We will henceforth use the notion of social-welfare maximization to characterize efficient networks in NFLP under different configurations of values of δ and c.

Finding the Efficient Graph

Proposition 9: *When $\delta < c$ and $\delta^2 < (c - \delta)$, the null graph is the unique efficient graph.*

Proof: For any node i, $d_i > 0$ implies that the pay-off of that node is negative thus reducing the overall network pay-off. This follows from $(\delta - c + \delta^2)$ being negative.

Proposition 10: *When $\delta = c$, the Turan graph is the unique efficient graph.*

Proposition 11: *When $\delta < c$ and $\delta^2 > (c - \delta)$, the Turan graph is the unique efficient graph.*

Proposition 12: *When $\delta > c$ and $\delta^2 \geq 3(\delta - c)$, the Turan graph is the unique efficient graph.*

Proposition 13: *When $\delta > c$ and $(\delta - c) > 2\delta^2$, the complete graph is the efficient graph.*

Conjecture 1: *When $\delta > c$ and $(\delta - c) \leq \delta^2 < 3(\delta - c)$, the Turan graph is the efficient graph.*

Conjecture 2: *When $\delta > c$ and $(\delta - c) \leq 2\delta^2$:*

(i) if $(\delta - c) > \frac{n}{n-2}\delta^2$, then the complete graph is the efficient graph.

(ii) if $(\delta - c) < \frac{n}{n-2}\delta^2$, then the Turan graph is the efficient graph.

We summarize the above results on efficiency in Table 2.

Table 2. Characterization of Topologies of Efficient Networks in NFLP

Parameter Range	Efficient Topologies	Implied By
$\delta < c$ and $\delta^2 < (c - \delta)$	Null network	Proposition 9
$\delta < c$ and $\delta^2 > (c - \delta)$	Turan network	Proposition 11
$\delta = c$	Turan network	Proposition 10
$\delta > c$ and $\delta^2 > 3(\delta - c)$	Turan network	Proposition 12
$\delta > c$ and $(\delta - c) > 2\,\delta^2$	Complete network	Proposition 13

Source: Developed by the Authors.

Price of Stability (PoS) of the NFLP Game

Recall that PoS (Anshelevich et al., 2008) is the ratio of the sum of pay-offs of the players in a best pairwise stable network to that of an efficient network. In NFLP, a best pairwise stable network means a pairwise stable network with a maximum value of the sum of pay-offs of the players. By invoking the results derived in the previous sections, we now present our results on PoS for the proposed model.

Theorem 1: *The price of stability (PoS) is* 1 *in each of the following scenarios:*

(i) $\delta > c$ and $(\delta - c) > 2\delta^2$,
(ii) $\delta > c$, $\delta^2 > (\delta - c)$ and $\delta^2 \geq 3(\delta - c)$,
(iii) $\delta = c$,
(iv) $\delta < c$ and $\delta^2 > (c - \delta)$.

This theorem can be proved easily using the results summarized in Table 1 and Table 2.

Note: Since the null network is the only efficient network when $\delta < c$ and $\delta^2 < (c - \delta)$, PoS is not defined in this region.

In view of Conjecture 1, the following result presents bounds on PoS.

Proposition 14: *When* $\delta > c$ *and* $(\delta - c) \leq \delta^2 < 3(\delta - c)$, *PoS* $> \frac{1}{2}$.

Proof: We know that, under the conditions $\delta > c$ and $(\delta - c) < \delta^2 < 3(\delta - c)$, the pairwise stable graph with the highest pay-off is the Turan graph (as seen from Table 1). Let Conjecture 1 be false. In this scenario, let us denote the efficient graph by $\overline{G}$. We will now evaluate an upper bound on the maximum efficiency of $\overline{G}$. $\overline{G}$ has to have more direct links than the Turan graph (as $\delta > c$) to be a candidate for efficient graph. Let $\overline{G}$ have $\left(\left\lfloor \frac{n^2}{4} \right\rfloor + x\right)$ edges where $x > 0$.

$$u(\overline{G}) = \sum_{i=1}^{n} u_i(\overline{G}) = (\delta - c)\sum_{i=1}^{n} d_i + \sum_{i=1}^{n} d_i \delta^2 \left(1 - \frac{\sigma_i}{\binom{d_i}{2}}\right)$$

$$= (\delta - c + \delta^2)\sum_{i=1}^{n} d_i - \delta^2 \left(\frac{2\sigma_i}{d_i - 1}\right)$$

Since d_i can be at most $(n - 1)$,

$$u(\overline{G}) \leq (\delta - c + \delta^2)n(n-1) - \left(\frac{2\delta^2}{n-2}\right)\sum_{i=1}^{n} \sigma_i$$

$$u(\overline{G}) \leq (\delta - c + \delta^2)n(n-1) - \left(\frac{2\delta^2}{n-2}\right)T_3(\overline{G})$$

By Equation (6), we have,

$$u(\overline{G}) \le (\delta - c + \delta^2)n(n-1) - \left(\frac{2\delta^2}{n-2}\right)\left(\frac{n(4e-n^2)}{9}\right)$$
$$= (\delta - c + \delta^2)n(n-1) - \left(\frac{\delta^2 n}{n-2}\right)\left(\frac{8x}{9}\right)$$

Since $\left(\frac{\delta^2 n}{n-2}\right)\left(\frac{8x}{9}\right) > 0$, we have,

$$u(\overline{G}) \le (\delta - c + \delta^2)n(n-1)$$

As mentioned before, the Turan graph is pairwise stable under these conditions (refer Table 1). Hence we get the following:

$$u(G_{Turan}) = (\delta - c + \delta^2)\left(2\left\lfloor \frac{n^2}{4} \right\rfloor\right)$$

$$PoS \ge \frac{u(G_{Turan})}{u(\overline{G})} \ge \frac{(\delta - c + \delta^2)\left(\frac{n^2-1}{2}\right)}{(\delta - c + \delta^2)n(n-1)} = \frac{1}{2} + \frac{1}{2n}$$

This implies that $PoS > \frac{1}{2}$.

Remark: In view of Conjecture 2, it can be noted that a similar bound can be obtained in the region $\delta > c$ and $(\delta - c) \le 2\delta^2$.

From Theorem 1 and Proposition 14 along with the simulation results, we conclude that, under mild conditions, the proposed NFLP produces efficient networks that are pairwise stable. This is desirable from the view of system design.

Convergence to Pairwise Stable/Efficient Networks through Myopic Best Response Dynamics

So far, we have examined various networks which satisfy properties of pairwise stability and/or efficiency. Our results show that, under many configurations, the set of pairwise stable networks need not be unique, so even converging to a particular pairwise stable equilibrium network is in itself non-trivial task. Further, this hints at the difficulty of designing dynamics that select a 'good' equilibrium. Further, as shown in the studies on Price of Stability in Section 6, there is no reason to expect equilibria to be also efficient.

Instead of focussing on one of the standard static equilibrium concept of pairwise stability discussed in earlier sections, we investigate, through simulations, whether there is some non-trivial network formation process that yields the pairwise stable and efficient networks discussed so far in the article. We propose a natural dynamic process of network formation which applies the proposed NFLP model. These dynamics describe the myopic behaviour of a strategic agent, should it be allowed to deviate within a particular set of allowable deviations.

We design a simple, localized best response updating rule for network formation and understand the convergence of such a dynamic updating rule through extensive simulations. The main purpose of this exercise is to get a better understanding of the actual network formation process as theoretical analysis has limited scope in enabling the understanding of the cumulative effects of many of the parameters occurring in real-world scenarios like the initial network, updating order of the nodes, etc., that influence the network formation process.

Starting from some initial configuration of a network, we examine whether it is possible to converge to any of the pairwise stable networks (discussed in Section 4) if self-optimizing nodes of the network follow some simple myopic updating rule. However, since we provided only sufficient conditions for certain network topologies to be pairwise stable (in Section 4), we can deduce that the convergent topology of the proposed myopic best response dynamic process may not be any of the standard networks considered in Section 4. Studies on the convergence of this dynamic process have the potential to reveal certain other topologies that satisfy pairwise stability apart from these standard networks.

Outline of the Section

The outline of the section is as follows:

1. We explain the simulation setup and describe the myopic best response dynamics used in the simulation.
2. We put forth some important metrics which are recorded during the simulations and describe classification criteria of pairwise stable networks.
3. We investigate whether the network topologies considered so far in Section 4 and Section 5 do emerge in our simulations as convergent networks.
4. We investigate whether our simulation process discovers new pairwise stable topologies not considered so far in the article.
5. We investigate a single simulation run in closer detail to gather additional insights about the network formation process.
6. We study the behaviour of convergence time under various parameter configurations of the simulations.

Simulation Setup

We built a custom network formation simulator using the C++ programming language in order to model the network formation process under our proposed network model. To implement the standard graph routines, we used the BOOST C++

libraries (Boost-C++-Libraries, 2012) which has efficient implementations of fundamental graph data structures and routines. We start with a random initial network consisting of n nodes. The number of edges between these nodes is determined by the parameter *density*(γ). For example, if $\gamma = 0$, we start with an empty network; if $\gamma = 0.35$, we start with a network that contains .35% of the possible $\binom{n}{2}$ edges. These edges are chosen uniformly at random. As noted in Section 3, a node obtains a benefit of δ $(0 \leq \delta \leq 1)$ and incurs a cost (c $(0 \leq c \leq 1)$) for maintaining a direct relationship (represented by an edge) with another node. In addition, each node reaps additional indirect benefit because of its potential to bridge its unconnected neighbours (determined by sparsity of relationships among his neighbours).

The Simulation Process: Myopic Best Response Dynamics

A single simulation run starts off with an existing initial network (maybe a null network) among the nodes in the network in the custom network formation simulator. We fix a particular value of δ and c for a single simulation run. We now describe the details of a single simulation run below.

In a particular simulation run, each node is given an opportunity to act, based on a random schedule. Each node, when scheduled, considers three actions—namely, add an edge to a node that it is not directly connected to, delete an existing edge to a node or do nothing. Each node chooses the action that maximizes its individual pay-off (which is based on the parameters δ and c), breaking ties randomly. Node i, when adding an edge to node j, may be allowed to do so only if it is beneficial to both or if node j is at least not worse off (mutual add (MA)). Similarly, node i, when deleting an existing edge to node j, may be allowed to do so unilaterally (unilateral delete). Note that these design of this dynamics follow very naturally from the definition of pairwise stability.

Table 3 lists the various simulation parameters. At some stage in the simulation, the network could evolve into a stable state where no node has any incentive to modify the network. One iteration in which no node modifies the network is an 'idle iteration', and the parameter 'Num-Idle-Terminate' indicates the number of idle iterations before we conclude that the network has reached a stable state. This is the case of normal termination of a simulation run. However, there may be cases where the network does not emerge into a stable state and cycles through previously visited states even after many iterations (the case of 'dynamic-equilibrium' as noted in Hummon (2000). The parameter 'Max-Iterations' indicates the number of iterations before we forcibly terminate the simulation run. However, we have observed that all the simulation runs achieved convergence much before the maximum iterations allowed indicating that the formation of dynamic equilibrium is not possible in our pay-off model. However, we leave the formal proof of this observation as a future work.

Table 3. Simulation Parameters

Parameters	Values
N	3, 4, 5, 10, 20
Cost (c)	0.05 to 1, in steps of 0.05
Benefit (δ)	0.05 to 1, in steps of 0.05
Density (γ)	0, 0.35, 0.7
Experiment	Mutual-Add, Unilateral-Delete
Num-Iterations	1000
Num-Repetitions	100
Num-Idle-Terminate	30

Source: Developed by the Authors.

We run the simulations for each combination of possible values of δ and c as shown in Table 3. Further, each simulation run is repeated multiple times as per the 'Num-Repetitions' parameter. The simulations were averaged out over different initial networks and random schedules.

Dynamic Process Evolution: Emergence of Known Pairwise Stable Networks

We now proceed to understand some of the results of our simulations. First, in this section, we focus only on the networks considered in Section 4 and Section 5. Specifically, we are interested in knowing the following aspects in the simulations:

1. Do the pairwise stable networks identified in Table 1 actually emerge in the simulation process which uses the myopic best response updating rule?
2. If so, under what values of δ and c do they emerge?
3. Do these conditions match with the theoretical results proved in Table 1?

We present the results in Figure 3 and Figure 4.

Figure 3(a)–Figure 3(i) and Figure 4(a)–Figure 4(i) presents the relevant results. The vertical axis of each plot in Figure 3 and Figure 4 is the benefit value (δ), ranging from 0 to 1 (discretized as $\{0, 0.05, 0.1, 0.15, ..., 1\}$), and the horizontal axis represents the cost parameter (c), ranging from 0 to 1, which is also discretized as $\{0, 0.05, 0.1, 0.15, ..., 1\}$).

In general, given a particular value of δ and c, the simulation process may converge to different pairwise stable network structures. The type of network structure emerging in the network formation process depends on a number of factors like the initial network, the scheduling order of the nodes along with the

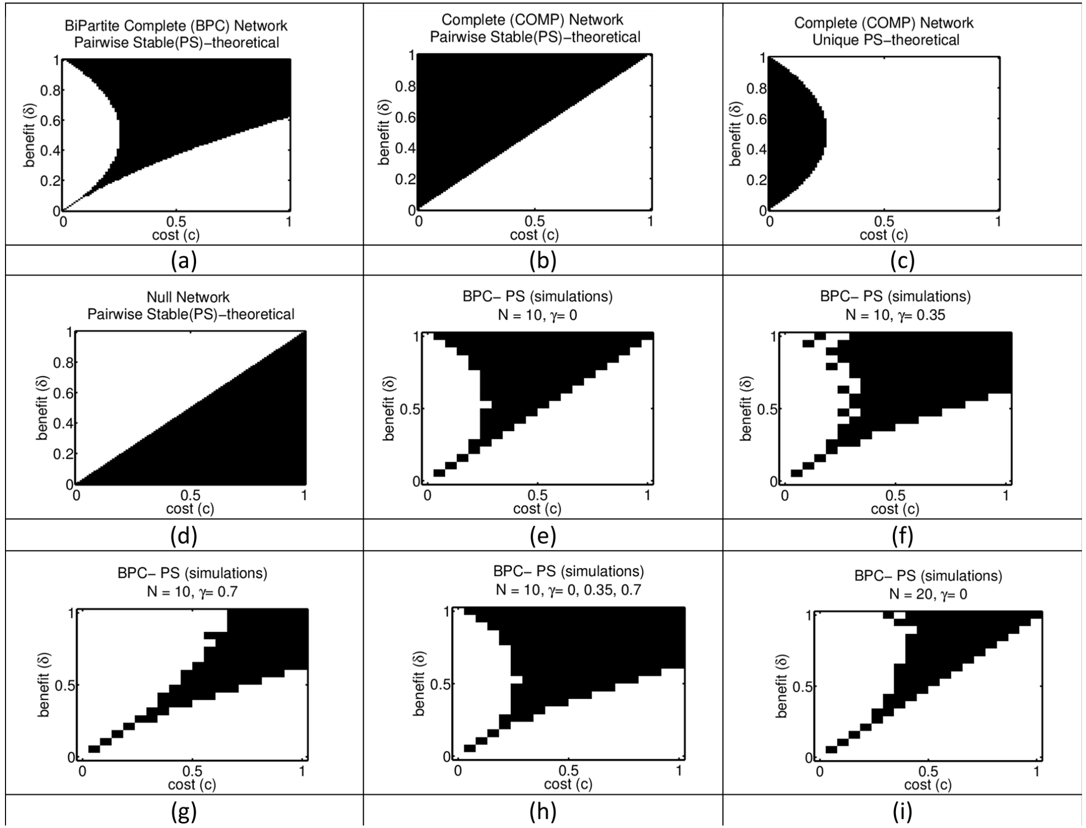
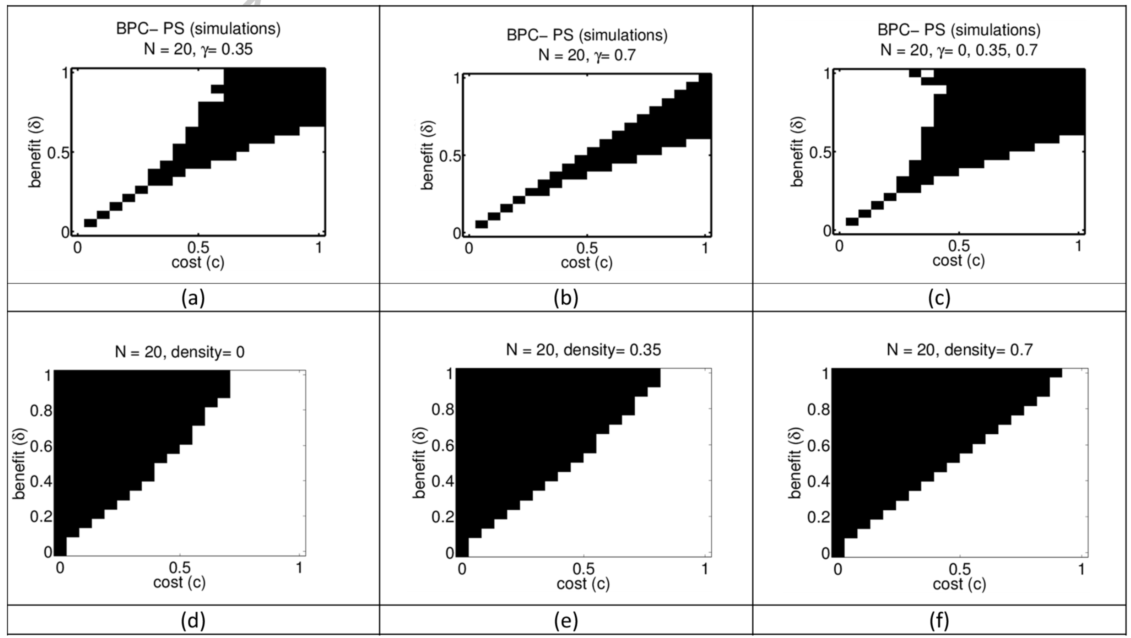


Figure 3. Detailed Results through Simulations (Repetitions = 0: for each (δ, c) pair)

Source: Developed by the Authors.



(Figure 4 Continued)

(Figure 4 Continued)

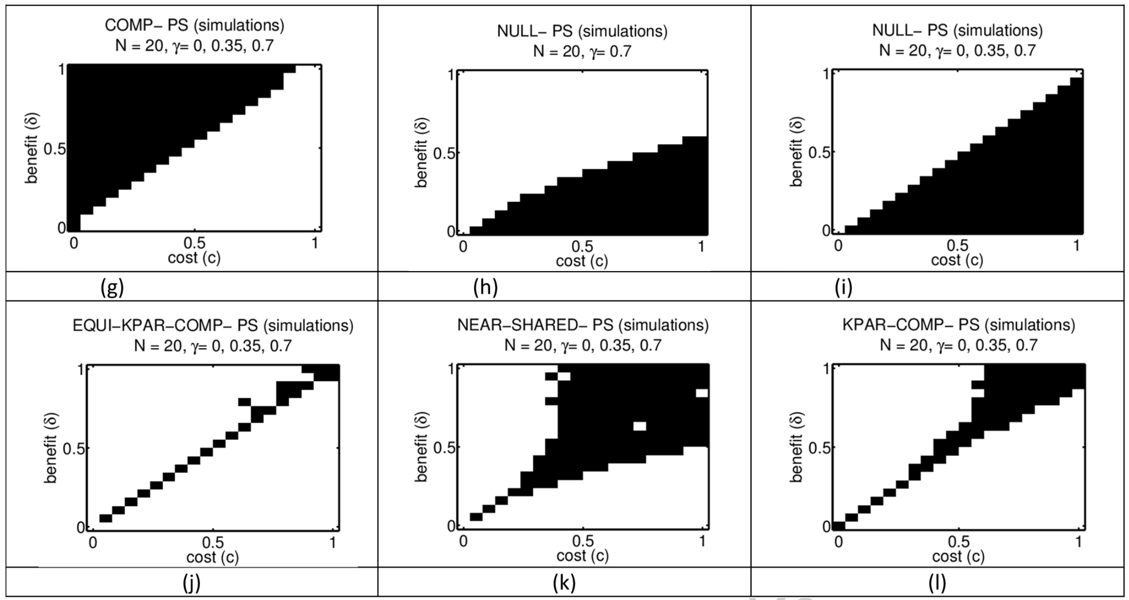


Figure 4. Detailed Results through Simulations (Repetitions = 100: for each (δ, c) pair)

Source: Developed by the Authors.

parameters of δ and c. In particular, we start with three different initial networks with densities (0, 0.35, 0.7) respectively.

Figure 3(a)–(d) plot the pairwise stable regions as proved theoretically (given in Table 1) for three networks namely bipartite complete network, null network and complete network. Figure 3(e)–Figure 3(i) and Figure 4(a)–Figure 4(l) show the simulation results. Now, we will describe these plots in more detail. Note in Figure 3(a)–(d), the region shaded in black corresponds to the values of δ and c for which the corresponding network has been proved to be pairwise-stable theoretically (as given in Table 1). Similarly, the region shaded in black in the plots given in Figure 3(e)–Figure 3(i) and Figure 4(a)–Figure 4(l) (which represent the results of the simulation process) corresponds to the values of δ and c for which the simulation process converged to the corresponding pairwise stable network.

Figure 3(e) shows the regions where the Bipartite Complete (BPC) network emerged as one of the pairwise stable network when the simulation run was started with number of nodes (N = 10) and initial network with density(γ = 0). Clearly, we can see that BPC does not emerge as pairwise stable in the regions where $\delta < c$ as the null network (which coincides with the initial network) is also pairwise stable and the nodes prefer not to add any links to the initial network. However, Figure 3(f) and Figure 3(g) show that if the starting network is already having some existing links then nodes try to form BPC network even in the regions where $\delta < c$. This shows the importance of the initial network in the network formation process.

Figure 3(h) is obtained by merging all the regions of Figure 3(e)–(g) and this closely corresponds to the theoretical regions of BPC stability shown in Figure 3(a). Figure 3(i) and Figure 4(a)–(c) similarly show results for N =20. In this case, however, we observe that Figure 4(c) is not as close to Figure 3(a) which is due to the fact that there may be many more pairwise stable topologies that may emerge as the number of nodes increase which illustrates a fundamental difficulty in identifying *all* pairwise stable networks for *every* possible value of number of nodes (N).

Another observation is that the complete network is theoretically proven to be the unique pairwise stable network in the region shown in Figure 3(c). We can clearly see the simulation results in Figure 3(h) and Figure 4(c) that this region is clearly excluded from the BPC stable region as starting with any initial network, only the complete graph emerges as unique the pairwise stable network in the region specified by Figure 3(c).

We similarly show the stability regions for complete and null networks in Figure 4(d) and Figure 4(f) respectively which corresponds to the theoretical results of Figure 3(b) and Figure 3(d) respectively. As explained earlier, Figure 4(e) again illustrates the importance of initial network in making the null network as the pairwise stable network.

As shown in Proposition 8, the equi-k-partite network is stable when $\delta = c$ and Figure 4(j) shows that indeed in this region, the equi-k-partite network does emerge as the pairwise stable network when $N = 20$. Proposition 8 was only a sufficient condition, we observe from the figure that there are other regions of δ and c (which we have not analytically studied) at which equi-k-partite network emerges as the pairwise stable network.

As explained earlier, the pairwise stable network structures as shown in Table 1 is not exhaustive and hence, we used simulations to depict the region of stability for important types of network structures namely the near-shared network and k-partite complete network. We show the results in Figure 4(k) and Figure 4(l).

Dynamic Process Evolution: Discovery of New Pairwise Stable Networks

Figure 5 shows the simulation results for 10-node and 20-node networks. The exact parameter configurations and the initial network densities are marked within the individual plots in the figure. The vertical axis of each plot in Figure 5 is the benefit value (δ), ranging from 0 to 1, and the horizontal axis represents the cost parameter (c), ranging from 0 to 1. As noted earlier, for a $<c, \delta>$ pair, we repeat the simulation for *Num-Repetitions*. Each repetition for the simulation results in a network that can be classified as one of the structures mentioned in the theoretical analysis. We plot the most frequent (*modal*) network structure as determined by the frequency with which each of the network structures resulted in *Num-Repetitions* simulation runs. The experiment was repeated starting with different network densities, $\gamma = 0{,}0.3$ and 0.7. We list some of the abbreviations used in the legends of the plots in Table 4.

In each of the plots in Figure 5, we observe that the complete graph is the resultant pairwise stable network (when $\delta > c$, $(\delta - c) \geq \delta^2$) which concurs with the theoretical deductions that the complete graph is the unique pairwise stable network in this region (Table 1 and Figure 3(c)).

We can also infer there is an overlap in the stability regions among complete and complete bipartite (see Figure 3(a) and Figure 3(b)) and also between null and complete bipartite networks (see Figure 3(a) and Figure 3(d)). However, as observed through simulations (Figure 5), we see that the complete bipartite network emerges as the *modal* pairwise stable network in its regions of overlap with the aforementioned networks. This can be attributed to the fact there are a large number of possible bipartite graphs whereas there is only one null network and one complete network. Hence, the likelihood of the null and complete emerging in a region where the bipartite network is also pairwise stable, is small.

We also observe from some of the plots in Figure 5 that near-shared and k-partite complete networks emerge as pairwise stable networks under some regions of the parameters. As explained in earlier sections, this can be attributed to the fact that our analytical results (as shown in Table 1) is not exhaustive and there exist some new topologies (which we identify as near-shared or k-partite complete networks) which are also pairwise stable.

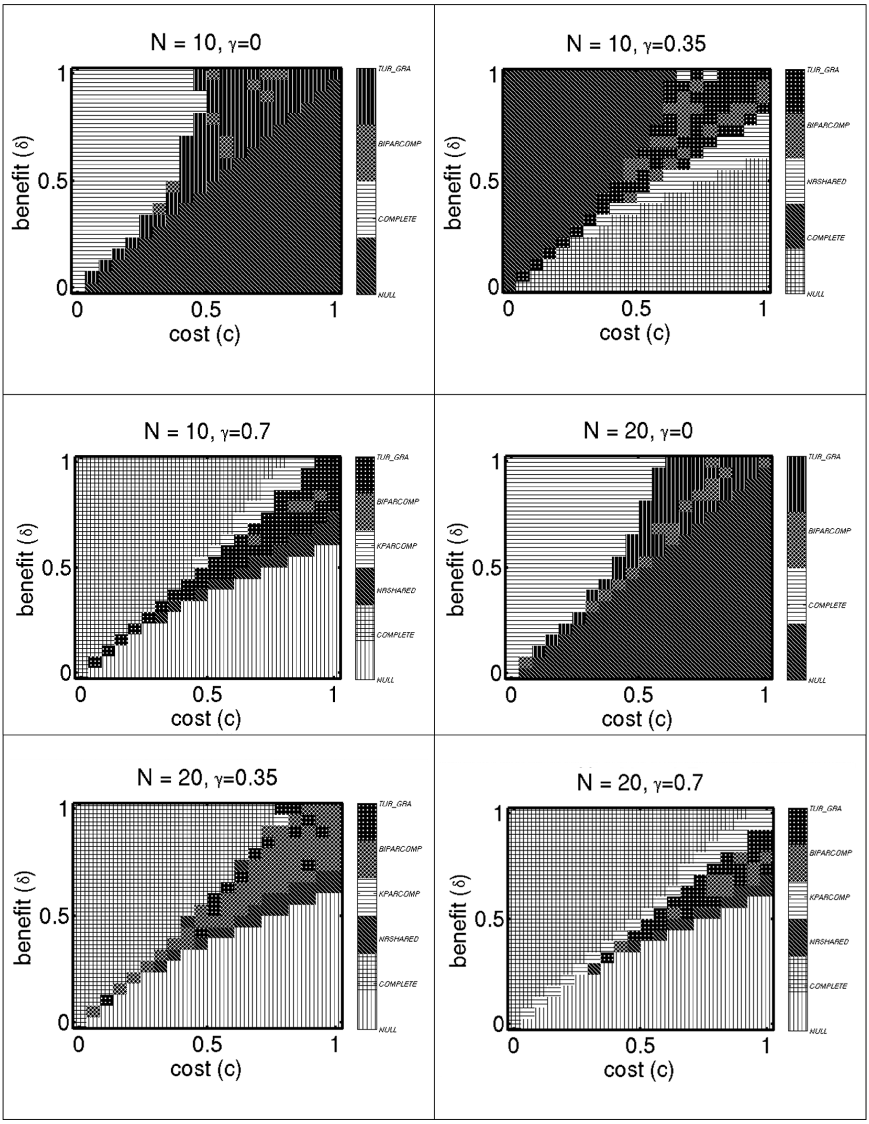


Figure 5. Network Topologies Obtained during Simulations

Source: Developed by the Authors.

Table 4. Some Abbreviations used in Figure 5

TUR_GRA	Turan Graph	BIPARCOMP	BiPartite Complete
NRSHARED	Near-Shared	KPARCOMP	KPartite Complete

Source: Developed by the Authors.

Dynamic Process Evolution: Examining a Single Simulation Run

Having studied the macroscopic behaviour of our simulations, we investigate the network formation process from a microscopic viewpoint. We examine various snapshots during the network formation process of a single simulation run which is repeated just once for a fixed parameter of δ and c. We consider $\delta = c = 0.5$ as our parameter configuration. We can observe from the our proposed pay-off model (Equation (3)) that for this configuration the benefits from direct links is 0 and so, nodes try to maximize the benefits due to bridging behaviour. The nodes form/delete links such that they emerge as a bridge in connecting their unconnected neighbours. Hence, we would expect the final pairwise stable network to be consisting of nodes who are filling the positions of structural holes in the network and hence, the emergent pairwise stable graph should be a 'triangle-free' as nodes form links with nodes who are themselves are not connected with each other.

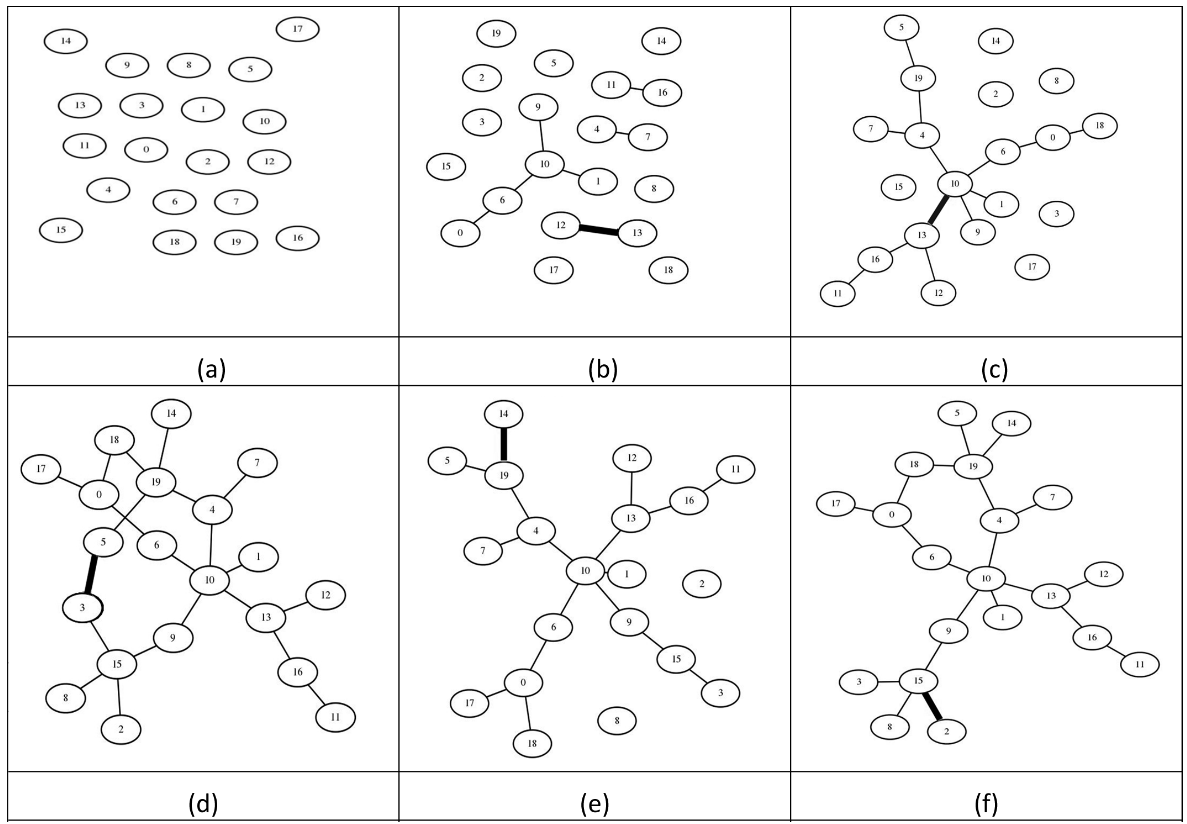
We depict the snapshots of network formation process at different instances of time in Figure 6. We can see that initially the nodes are forming links in such a way that triangles are not present but eventually triangles eventually do form due to the cumulative action of other nodes in the network. When triangles emerge in the neighbourhood of a node, it leads to deletion of links from that node (as the node will benefit strictly from deletion) and the final emergent network (Figure 6(l)) is a equi-bipartite complete network (which is triangle-free), also known as the Turan network). A better visualization of the bipartite structure of the emergent Turan network (given in Figure 6(l)) is provided through Figure 7(b).

In complex network literature, the number of triangles in the network is a important parameter which was first studied by (Watts and Strogatz, 1998) by definition the notion of 'clustering', sometimes also known as network transitivity. Clustering refers to the increased propensity of pairs of people to be acquainted with one another if they have another acquaintance in common. Watts and Strogatz (1998) define a 'clustering coefficient' (denoted by C) that measures the degree of clustering in a undirected unweighted graph.

$$C = \frac{3 \times \text{Number of triangles on the graph}}{\text{Number of connected triples of vertices}}$$

The factor three accounts for the fact that each triangle can be seen as consisting of three different connected triples, one with each of the vertices as central vertex, and assures that $0 \leq C \leq 1$. A triangle is a set of three vertices with edges between each pair of vertices; a connected triple is a set of three vertices where each vertex can be reached from each other (directly or indirectly), i.e., two vertices must be adjacent to another vertex (the central vertex).

It can be observed from the pay-off model proposed in equation (3) in Section 3 that $\frac{\sigma_i}{\binom{d_i}{2}}$ component in the pay-off model corresponds to the clustering coefficient of node i. Thus, in our pay-off model, nodes benefit from having lesser clustering



(Figure 6 Continued)

(Figure 6 Continued)

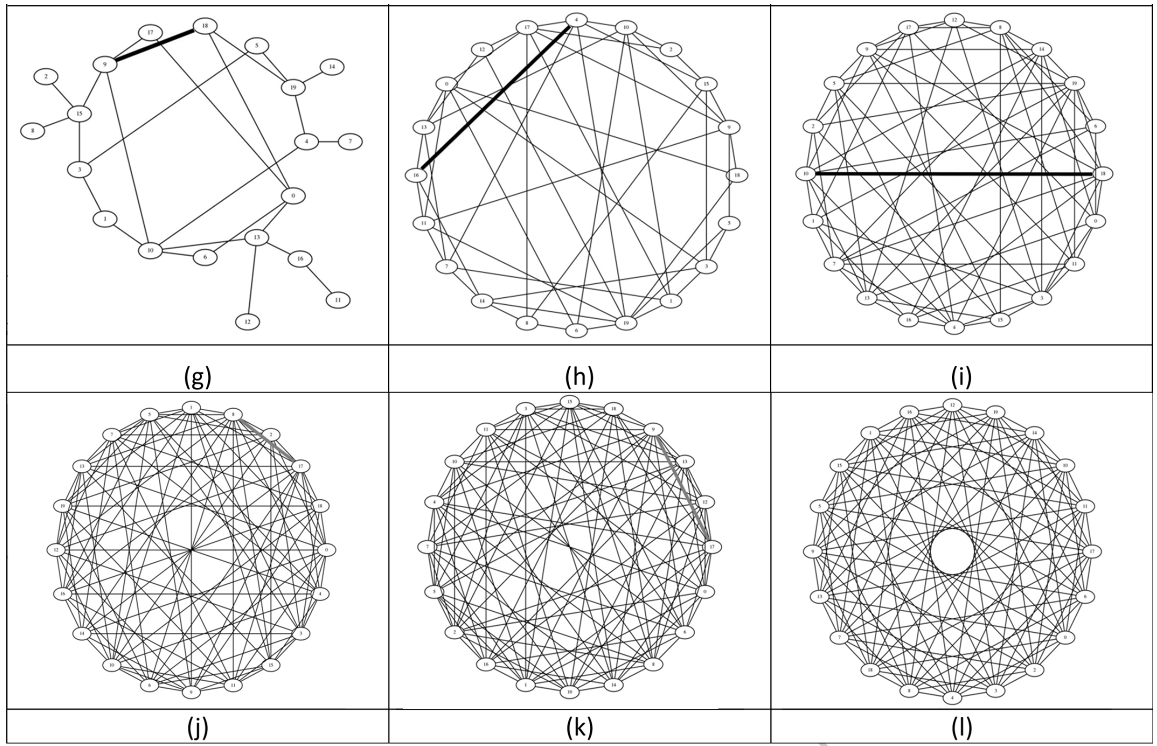


Figure 6. Evolution of the Network Formation Process (N = 20, δ = 0.5, c = 0.5, *seed* = 24336).

Source: Developed by the Authors.

Notes: Thick black edge denotes link addition by the scheduled node in the current round. Similarly, thick grey edge indicates deletion of the link by the scheduled node in the current round. Thin Black edges denote links that have been formed from previous rounds.

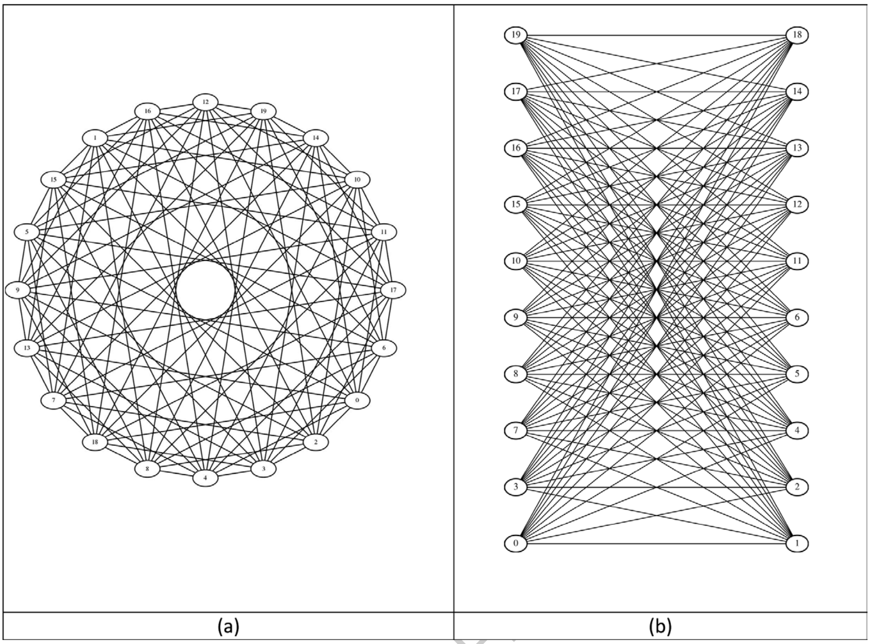


Figure 7. Isomorphic Representation of the Unique Efficient and Pairwise Stable Network known as the Turan Network (i.e., Complete Equi-Bipartite Network) when $N = 20$, $\delta = 0.5$, $c = 0.5$.

Source: Developed by the Authors.

Note: Figure 7(a) is the same network as the evolved network from the single simulation run given in Figure 6(l) and Figure 7(b) is its isomorphic representation.

coefficient as this will lead to the formation of structural holes, which in turn leads to increase in the pay-off for the node.

We now study how the clustering coefficient changes as the network evolves through the different phases shown in Figure 6. We plot this result in Figure 8(a). We see that upto time epoch 50 clustering coefficient is 0. Later there is a increase in the value which is followed by the reduction in the clustering coefficient back to 0 (at time epoch 150) when the pairwise stable network emerges. As explained before, this is indeed the expected behaviour during the network formation process for the parameters $\delta = c = 0.5$.

We also study the average clustering co-efficient in all the pairwise stable networks that emerge for different values of δ and c. We take the average over running 'Num-repetitions' number of times. The result is shown in the 3d plot in Figure 8(b). We can see that the clustering coefficient assumes value of 1 in the regions where the complete network is stable and 0 when the null network is stable. In other regions, the clustering coefficient value is between 0 and 1 which indicates a trade-off between the benefits from direct links and the benefits from bridging benefits to the nodes in the network.

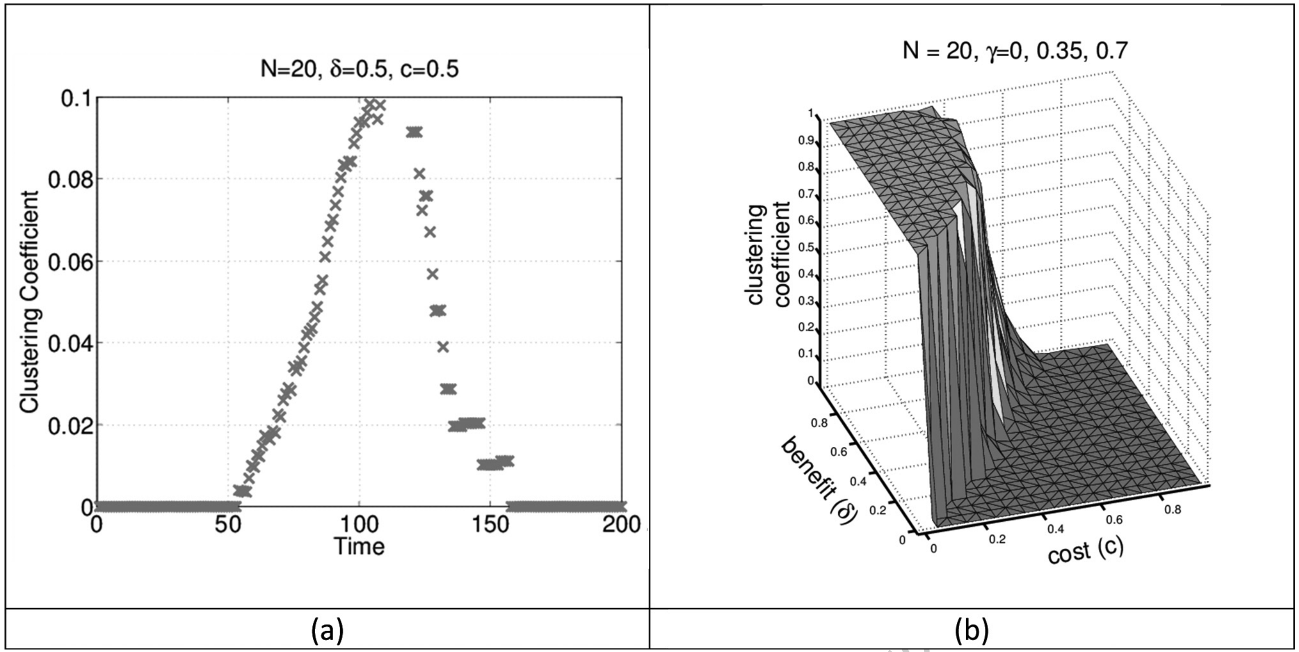


Figure 8. Study of Clustering Coefficient (N = 20)

Source: Developed by the Authors.

Dynamic Process Evolution: Studying Convergence Time

In this section, we will study the effect the initial network density has on the effort needed by the nodes to achieve convergence to a pairwise stable network. A single addition of an edge or a single deletion of an edge by a node is considered to be a single 'act' by that player. We now study the mean number of acts performed by the players to converge to a pairwise stable network starting from various initial random networks. We can see from Figure 9(a) that the number of changes to the network is more when the $\delta > c$ region and this is because the initial network is a null network and the players need to perform a lot more additions/deletions to the network before reaching the final stable network which is the complete network. When $\delta < c$, the players need not perform any change to the network as the initial null network is already pairwise stable. In fact, we can observe from the Figure 9 that the number of acts needed to reach the complete network is maximum (about 180) when starting with null network than when compared to other scenarios of $\gamma = 0.35$ and $\gamma = 0.7$ (mean acts is about 130).

We observe a reversal of the work needed to reach null network in Figure 9(c) where more number of changes is needed to reach null network than reaching the complete network. This can be attributed to the fact that the initial network is already a dense network to start with and it takes relatively less effort to reach the complete network than the null network under appropriate configurations of δ and c.

Initial network density of 0.35 corresponds to a medium-dense network (Figure 9(b)) and hence there is a non-zero effort to reach any of the pairwise stable network under any parameter configuration. However, as in Figure 9(a), it takes more effort for players to reach the complete network than the null network.

Summary

Recent studies have indicated that social structure has an important role in impacting economic outcomes. In our investigation, we have pursued this observation in more detail in the context of network formation problem. Typically, nodes in a social structure tend to undertake decisions based on the local information available to them. We investigate the effect of local information that strategic agents possess on the eventual equilibrium network formed among the nodes.

We proposed a model for network formation game in which strategic agents use their local neighbourhood information to links they want to form or delete. Following the justification of Burt, (1992, 2004, 2007) regarding the hypothesis that people/nodes who stand near the holes of a social structure have a higher chance of having good ideas which result in higher social capital, we incorporate brokerage into the pay-off model of strategic agents. Nodes, thus, obtain a higher pay-off if they are able to bridge disconnnected pairs of nodes. Having formulated the model, we examine equilibrium properties in the model. It has been observed in recent literature that pairwise stability is a natural way to think about link formation in a

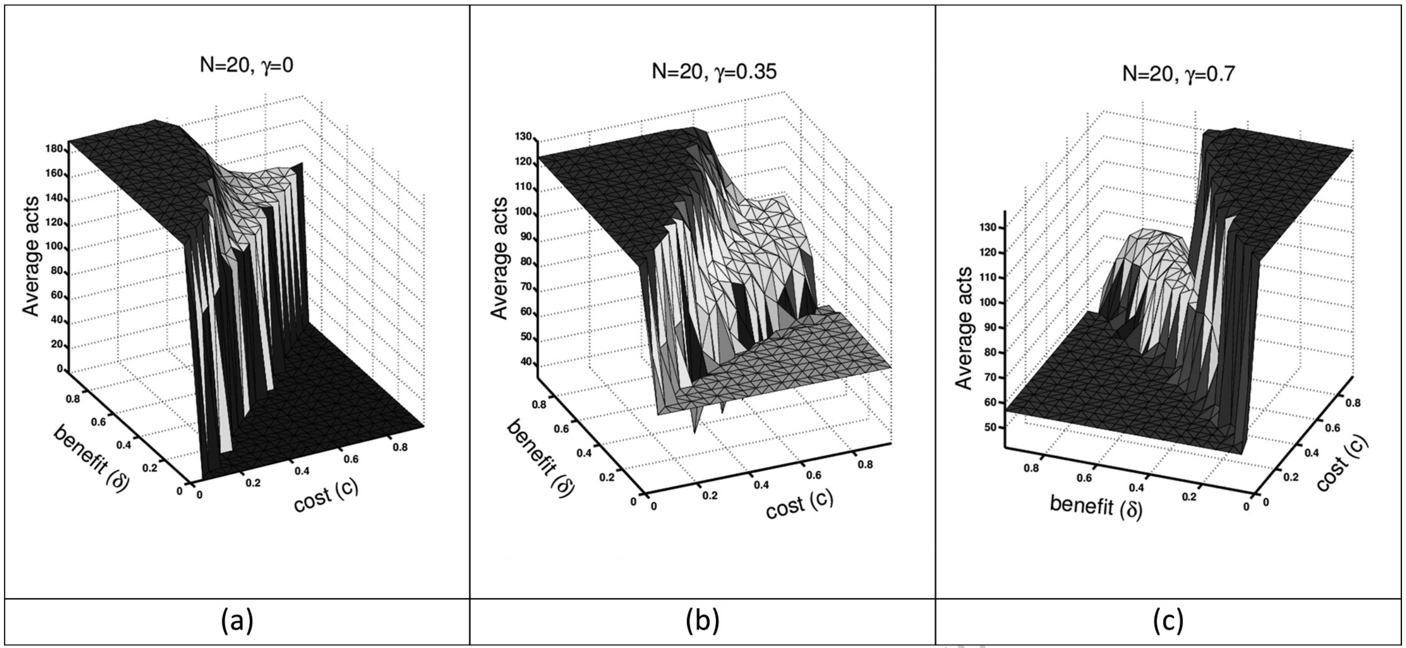


Figure 9. Study of Number of Acts before Converging to Pairwise Stable Network (N = 20)

Source: Developed by the Authors.

number of social and economic contexts such as the formation of friendship ties, co-authorships (Jackson and Wolinsky, 1996), collaboration between firms, trading links between buyers and sellers and free trade agreements between nations (Goyal and Joshi, 2003). Hence, in our work, we used the notion of pairwise stability as the relevant equilibrium concept as it incorporates the effects of both unilateral and bilateral deviations unlike the notion of Nash stability which considers robustness due to only unilateral deviations.

We derived sufficient conditions for pairwise stability for various network topologies exploiting the symmetry of the corresponding network topologies. Some of the networks considered for pairwise stability included the following networks: complete, complete bipartite, complete tripartite, complete equi-k-partite, null, star and cycle.

Next, using the notion of social-welfare maximization for identifying efficient networks, we characterized efficient networks in the NFLP model using techniques from the area of extremal graph theory. We also studied the trade-off between pairwise stability and efficiency using the notion of Price of Stability (PoS). In particular, we computed the PoS of the proposed NFLP model. Except for a few configurations of δ and c, we have shown that PoS is 1. This means that, under mild conditions, that NFLP produces efficient networks that are pairwise stable.

We then investigated whether the pairwise stable and efficient networks could actually emerge through a non-trivial network formation process which starts from a random initial network. We studied the outcome of myopic best response dynamics by developing a custom network formation simulator to capture the network formation process. We varied the simulation process under different values of parameters of the simulation namely initial network density, scheduling order among the nodes of the network and various values of δ and c. We observed that the simulation process converged to the theoretically proven pairwise stable networks under many parameter settings. Further, it also led to discovery of new pairwise stable networks which were not considered under the theoretical investigations. This discovery reinforced the importance of the two approaches followed in our investigation namely theoretical analysis and simulation studies. We also gained additional insights regarding convergence time and network evolution pattern in the network formation process.

Summarizing, our work investigated the strategic network formation problem in detail and highlighted the attracting and drawing forces that exists between equilibrium networks and efficient networks where the network formation model is based on realistic assumptions of local information and brokerage benefits.

Discussion

In the pay-off function we defined in Section 3; the pay-off of any node had two components, benefit from direct links and benefit from bridging. The pairwise stable network topologies of our model, Section 4, shows that there are no bridges in the equilibrium networks. Bridges can also be considered as bottlenecks of information flow. Since every node is striving to obtain a bridging position there are no bridges in the equilibrium networks, this suggests that the proposed pay-off model avoids bottlenecks in decentralized network formation. Here are a few pointers for future work. First, the framework in this article can be extended to the case of directed graphs and weighed graphs. This involves certain challenges such as defining the pay-off model appropriately. Second, the setting in this article can be extended by varying the notions of stability and efficiency. We note that there are several possible notions of stability and efficiency that exist in the literature. The choice of an appropriate notion of stability as well as efficiency is a topic of debate.

Further, our model gives us some valuable hints at the networks formed in real-world as well. Some noted work in complex network literature has observed the emergence of bipartite graphs in real-world scenarios (Reka and Barabási, 2002; Newman et al., 2001). An important example has been the class of collaboration networks. It has been observed that the network of actors basically is a uni-mode bipartite graph (Newman et al., 2001). Other important examples of real-world bipartite networks include boards of directors of companies, co-ownership networks of companies and collaboration networks of scientists and movie actors. In the analysis of our proposed model in this article, we have seen the emergence of important graph structures like the Turan graph and in general, bipartite graphs and k-partite graphs during the network formation process under many configurations. Though our model does not precisely solve the difficult problem of identification of *all* parameters affecting network formation, it nevertheless offers valuable hints about some of the important parameters affecting real-world network formation. The studies on our pay-off model of network formation also offers strong evidence that incorporation of important game theoretic concepts like pairwise stability is vital to the understanding of complex network formation behaviour.

Acknowledgement

A preliminary version of this research work appears in the proceedings of the ninth annual IEEE International Conference on Automation Science and Engineering (IEEE CASE 2013), Madison, WI, USA.

Notes

1. Note that the notion of pairwise stability can also be equivalently defined in terms of the strategy vectors of the underlying network formation game. Since we are primarily interested in the pairwise stability of different network topologies in this article, we will focus on the pairwise stability definition with respect to the induced undirected network.
2. Assume that node i bridges the communication between j and k; and a benefit of δ^2 is generated. In the literature, there are three well known ways of distributing the benefit δ^2 to nodes i, j and k: (i) only node i gets entire δ^2, (ii) node i gets 0 and (iii) nodes i, j and k get equal share of δ^2. In this article, we work with scenario (i). A similar approach is utilized in Kleinberg et al. (2008) as well. We note that the analysis that we perform using scenario (i) can also be extended in a similar way to other two scenarios.

Appendix

Proofs For Pairwise Stability

Proposition 1: The cycle network is pairwise stable under the following conditions:

1. For cycle of length 3, $(\delta - c) \geq 0$
2. For cycle of length 4, $-2\delta^2 \leq (\delta - c) \leq \delta^2$
3. For cycle of length 5, $-2\delta^2 \leq (\delta - c) \leq \delta^2$
4. For cycle of length 6 or greater, $-2\delta^2 \leq (\delta - c) \leq -\delta^2$

Proof:

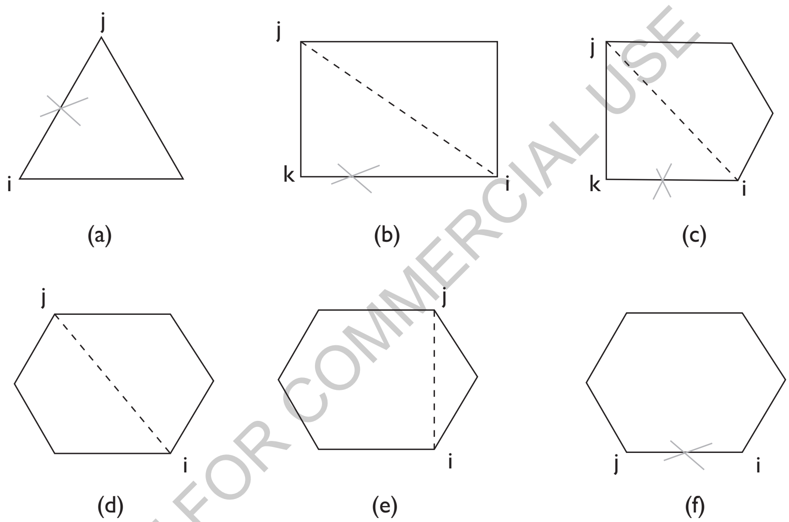


Figure 10. Cycle graph Pairwise Stability - Various Cases

Source: Developed by the Authors.

We will use a common notation throughout this proof. Let g be the graph under consideration. Let g' be the graph obtained from g after either an addition of a link or deletion of an existing link as the case may be. Let $u_i(g)$ and $(u_i(g')$ be the corresponding pay-offs of i is graphs g and g'.

Case 1: Consider a cycle of length =3 as shown in Figure 10(a). We will analyze pairwise stability under addition and deletion of links.

- Addition of new link: This is meaningless as all possible links are already present.
- Deletion of link: Consider a node i which wants to delete its link to node j. We compute the pay-offs of the node i before and after deletion of the link (i, j).

$$u_i(g) = 2(\delta - c) + 2 \times \delta^2 \times 0 = 2(\delta - c)$$

$$u_i(g') = (\delta - c) + 2 \times \delta^2 \times 0 = (\delta - c)$$

For pairwise stability, we need,

$$u_i(g) \geq u_i(g') \Rightarrow (\delta - c) \geq 0 \tag{7}$$

Case 2: Consider a cycle of length = 4 as shown in Figure 10(b). We will analyze pairwise stability under addition and deletion of links.

- Addition of new link: Consider a node i which wants to add a link to node j as shown in Figure 10(b). We compute the pay-offs of the node i before and after addition of the link (i, j).

$$u_i(g) = 2(\delta - c) + 2 \times \delta^2 \times 1 = 2(\delta - c) + 2 \times \delta^2$$

$$u_i(g') = 3 \times (\delta - c) + 3 \times \delta^2 \times \left(1 - \frac{2}{\binom{3}{2}}\right) = 3 \times (\delta - c) + \delta^2$$

For pairwise stability, we need,

$$u_i(g) \geq u_i(g')$$
$$\Rightarrow 2(\delta - c) + 2 \times \delta^2 \geq 2(\delta - c) + 2 \times \delta^2$$
$$\Rightarrow (\delta - c) \leq \delta^2 \tag{8}$$

- Deletion of link: Consider a node i which wants to delete its link to node k as shown in Figure 10(b). We compute the pay-offs of the node i before and after deletion of the link (i, k).

$$u_i(g) = 2(\delta - c) + 2 \times \delta^2 = 2(\delta - c) + 2\delta^2$$
$$u_i(g') = (\delta - c) + 2 \times \delta^2 \times 0 = (\delta - c)$$

For pairwise stability, we need,

$$u_i(g) \geq u_i(g') \Rightarrow (\delta - c) \geq -2\delta^2 \tag{9}$$

Combining Equation (8) and Equation (9), we get the following pairwise stability conditions for cycle of length 4.

$$-2\delta^2 \leq (\delta - c) \leq \delta^2 \tag{10}$$

Case 3: Consider a cycle of length = 5 as shown in Figure 10(b). We will analyze pairwise stability under addition and deletion of links.

- Addition of new link: Consider a node i which wants to add a link to node j as shown in Figure 10(c). We compute the pay-offs of the node i before and after addition of the link (i, j).

$$u_i(g) = 2(\delta - c) + 2 \times \delta^2$$
$$u_i(g) = 3 \times (\delta - c) + 3 \times \delta^2 \times \left(1 - \frac{2}{3}\right) = 3 \times (\delta - c) + \delta^2$$

For pairwise stability, we need,

$$u_i(g) \geq u_i(g') \Rightarrow (\delta - c) \leq \delta^2 \tag{11}$$

- Deletion of link: Consider a node i which wants to delete its link to node k as shown in Figure 10(c). We compute the pay-offs of the node i before and after deletion of the link (i, k).

$$u_i(g) = 2(\delta - c) + 2\delta^2 u_i(g') = (\delta - c)$$

For pairwise stability, we thus need,

$$(\delta - c) + 2\delta^2 \geq 0 \tag{12}$$

Thus, the pairwise stability conditions for a cycle of length 5 is the intersection of the conditions specified in Equation (11) and Equation (12) which is given below

$$-2\delta^2 \leq (\delta - c) \leq \delta^2 \tag{13}$$

Case 4: Consider a cycle of length ≥ 6 as shown in Figure 10(d)–Figure 10(f). We will analyze pairwise stability under addition and deletion of links.

- Addition of new link: Unlike the above cases, we can study stability under addition of links for a cycle of length ≥ 6 by considering two types of addition operations.
- Addition of link to a non-two-hop neighbour: Consider a node i which wants to add a link to node j as shown in Figure 10(d). We compute the pay-offs of the node i before and after addition of the link (i, j).

$$u_i(g) = 2(\delta - c) + 2\delta^2 u_i(g') = 3(\delta - c) + 3\delta^2$$

For pairwise stability, we thus need,

$$u_i(g) \geq u_i(g') \Rightarrow (\delta - c + \delta^2) \leq 0 \tag{14}$$

- Addition of link to a two hop neighbour: Consider a node i which wants to add a link to node j as shown in Figure 10(e). We compute the pay-offs of the node i before and after addition of the link (i, j).

$$u_i(g) = 2(\delta - c) + 2\delta^2 u_i(g') = 3(\delta - c) + 2\delta^2$$

For pairwise stability, we thus need,

$$u_i(g) \geq u_i(g') \Rightarrow (\delta - c) \leq 0 \tag{15}$$

Combining Equation (14) and Equations (15), we get the following condition for pairwise stability under addition of links for a cycle of length ≥ 6.

$$(\delta - c + \delta^2) \leq 0 \tag{16}$$

- Deletion of link: Consider a node i which wants to delete its link to node j as shown in Figure 10(f). We compute the pay-offs of the node i before and after deletion of the link (i, j).

$$u_i(g) = 2(\delta - c) + 2\delta^2 u_i(g') = (\delta - c)$$

For pairwise stability under deletion, we thus need

$$\delta - c + 2\delta^2 \geq 0 \tag{17}$$

Thus, the pairwise stability conditions for a cycle of length ≥ 6 is the intersection of the conditions specified in Equation (16) and Equation (17) which is given below

$$-2\delta^2 \leq (\delta - c) \leq -\delta^2 \tag{18}$$

Proposition 2: If $-\delta^2 \leq (\delta - c) \leq \delta^2$, then the complete bipartite network is pairwise stable.

Proof: As usual, we will use a common notation throughout this proof. Let g be a complete bipartite network, with a_1 and a_2 nodes respectively in the two partitions. Let g' be the graph obtained from g after either an addition of a link or deletion of an existing link as the case may be. Let $u_i(g)$ and $u_i(g')$ be the corresponding pay-offs of i is graphs g and g'.

- Addition of new link: Consider a node i which wants to add a link to node j. We compute the pay-offs of the node i before and after addition of the link (i, j).

$$u_i(g) = a_2(\delta - c) + a_2\delta^2$$
$$u_i(g') = (d_i + 1)(\delta - c) + (d_i + 1)\delta^2 \left(1 - \frac{a_2}{\binom{(d_i+1)}{2}}\right)$$
$$= (d_i + 1)(\delta - c) + (d_i + 1)\frac{(a_2 - 1)}{(a_2 + 1)}\delta^2$$
$$= (a_2 + 1)(\delta - c) + (a_2 - 1)\delta^2$$

For pairwise stability under addition, we have,

$$u_i(g) \geq u_i(g')$$
$$a_2(\delta - c) + a_2\delta^2 \geq (a_2 + 1)(\delta - c) + (a_2 - 1)\delta^2$$
$$\delta^2 \geq (\delta - c) \tag{19}$$

- Deletion of link: Consider a node i which wants to delete its link to node j as shown. We compute the pay-offs of the node i before and after deletion of the link (i, j).

$$u_i(g) = a_2(\delta - c) + a_2\delta^2$$
$$u_i(g') = (\delta_i - 1)(\delta - c) + (d_i - 1)\delta^2$$
$$= (a_2 - 1)(\delta - c) + (a_2 - 1)\delta^2$$

For pairwise stability under deletion, we have,

$$u_i(g) \geq u_i(g')$$
$$a_2(\delta - c) + a_2\delta^2 \geq (a_2 - 1)(\delta - c) + (a_2 - 1)\delta^2$$
$$\delta^2 \geq (c - \delta) \tag{20}$$

Combining Equation (19) and Equation (20), we get the pairwise stability conditions for bipartite complete network and is given below.

$$-\delta^2 \leq (\delta - c) \leq \delta^2 \tag{21}$$

Proposition 3: The null (empty) network is pairwise stable if $(\delta - c) \leq 0$.

Proof: As usual, We will use a common notation throughout this proof. Let g be the graph under consideration. Let g' be the graph obtained from g after either an addition of a link or deletion of an existing link as the case may be. Let $u_i(g)$ and $u_i(g')$ be the corresponding pay-offs of i is graphs g and g'.

- Addition of new link: Consider a node i which wants to add a link to node j. We compute the pay-offs of the node i before and after addition of the link (i, j).

$$u_i(g) = 0$$

$$u_i(g') = (\delta - c)$$

For pairwise stability under addition, we have,

$$0 \geq (\delta - c) \tag{22}$$

- Deletion of link: This does not make sense as the graph is already a null network.

Thus, by Equation (22), we have the following condition for pairwise stability for null network.

$$0 \geq (\delta - c)$$

Proposition 4: The complete network is pairwise stable if $(c - \delta) \leq 0$

Proof: As usual, We will use a common notation throughout this proof. Let g be the graph under consideration. Let g' be the graph obtained from g after either an addition of a link or deletion of an existing link as the case may be. Let $u_i(g)$ and $u_i(g')$ be the corresponding pay-offs of i is graphs g and g'.

- Addition of new link: This does not make sense as the graph is already complete.
- Deletion of link: Consider a node i which wants to delete its link to node j. We compute the pay-offs of the node i before and after deletion of the link (i, j).

$$u_i(g) = d_i(\delta - c)$$

$$u_i\,(g') = (d_i = 1)(\delta - c)$$

For pairwise stability, we need,

$$u_i(g) \geq u_i(g') \Rightarrow (c - \delta) \leq 0 \tag{23}$$

Proposition 5: When $(\delta - c) > \delta^2$, the complete network is the unique pairwise stable network.

Proof: Consider any graph G which is not a complete network. We will show that it is beneficial for an arbitrary node i (who has less than $(n - 1)$ links) to add a link under the above conditions. Once this is proved, it naturally implies that the complete graph is the unique pairwise stable network under the conditions given above. Consider any node i with $d_i < (n - 1)$. Let σ_i be the number of links among its neighbours. Let σ_i' be the number of links among its neighbours after it has added a new link to node j. Similarly, let u_i be the pay-off of node i before addition of a new link and let u_i' be the pay-off of node i after addition of a new link. We know that

$$u_i = d_i \times (\delta - c) + d_i \times \delta^2 \times s_i$$

where $s_i = (1 - \frac{\sigma_i}{\binom{d_i}{2}})$ is the sparsity in the neighbourhood of node i. Note that, when node i adds a link to node j, the number of links among its neighbours can, at most, increase by d_i i.e., $\sigma_i' = \sigma_i + d_i$. Thus, the new pay-off of node i, u_i', is, at least, given by,

$$u_i' = (d_i + 1) \times (\delta - c) + (d_i + 1) \times \delta^2 \times (1 - \frac{(\sigma_i + d_i)}{\binom{(d_i+1)}{2}})$$

We now examine under what conditions will $u_i < u_i'$ and this will suffice for proving the complete graph is unique pairwise stable graph under these conditions.

$$u_i < u_i'$$

$$\Rightarrow d_i \times (\delta - c) + d_i \times \delta^2 \times (1 - \frac{2\sigma_i}{d_i \times (d_i - 1)}) < (d_i + 1) \times (\delta - c) + (d_i + 1) \times \delta^2 \times (1 - \frac{2(\sigma_i + d_i)}{(d_i + 1) \times d_i})$$

Simplifying, we get,

$$\delta^2 \times \underbrace{(1 - \frac{2\sigma_i}{d_i(d_i - 1)})}_{s_i} < (\delta - c)$$

$$\Rightarrow \delta^2 \times s_i < (\delta - c) \tag{24}$$

We know that $s_i \in [0, 1]$ and hence, s_i is upper bounded by 1. Thus, if we have the following stronger condition, then Equation (24) will automatically be satisfied.

$$\delta^2 < (\delta - c)$$

Thus, when $\delta^2 < (\delta - c)$ holds, node i has an incentive to add a new link irrespective of its neighbourhood and connections among its neighbours. Applying this argument repeatedly, we can conclude that any network except the complete network is not pairwise stable.

Proposition 6: The star network is pairwise stable only when $\delta = c$.

Proof:

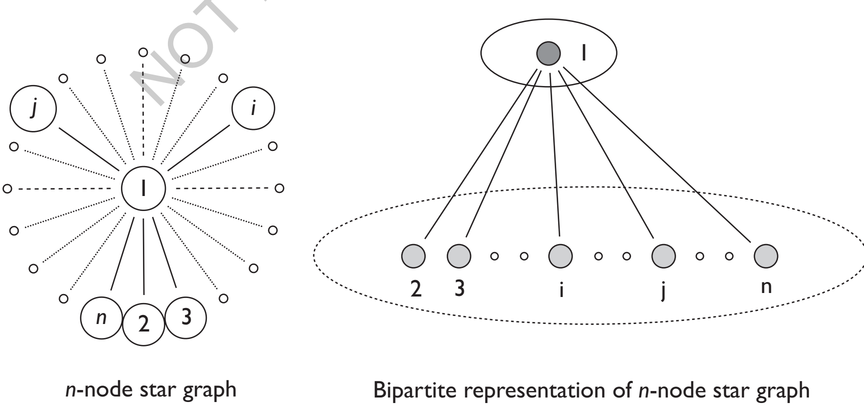


Figure 11. Examining Pairwise Stability of the Star Graph

Source: Developed by the Authors.

Let us consider the n node star graph as given in Figure 11. We can see from the figure that the star graph is isomorphic to the complete bipartite graph where there the central node is in one partition and all the peripheral nodes is in the other partition. Let us consider the central node to be Node 1. We will examine the conditions for pairwise stability of the star graph. We will consider three cases independently:

1. *Case 1:* Node 1 deletes any one of its link to the peripheral nodes, say Node 1.
2. *Case 2:* Peripheral Node i adds a new link to another peripheral node j.
3. *Case 3:* Peripheral Node i deletes the link to the central Node 1.

In Case 1, let u_1 be the pay-off of Node 1 before it deletes a link to a peripheral node. Let u_1' be the pay-off of Node 1 after deleting the link to Node i (shown in Figure 11).

$$u_1 = (n-1) \times (\delta - c) + (n-1)\delta^2$$

$$u_1' = (n-2) \times (\delta - c) + (n-2)\delta^2$$

$$u_i = (\delta - c)$$

$$u_i' = 0$$

By pairwise stability conditions, we require that $u_i \geq u_i'$ and $u_1 \geq u_1'$

$$u_1 \geq u_1' \Rightarrow (\delta - c) + \delta^2 \geq 0 \tag{25}$$

$$u_i \geq u_i' \Rightarrow (\delta - c) \geq 0 \tag{26}$$

In Case 2, let the peripheral Node i adds a new link to another peripheral Node j. Let u_i and u_j be the pay-offs of Node i and Node j before adding the link. Let u_i' and u_j' be the pay-offs of Node i and Node j after adding the link.

$$u_i = u_j = (\delta - c)$$

$$u_i' = u_j' = (2 \times (\delta - c))$$

By pairwise stability conditions, we require that $u_i \geq u_i'$. Hence, we get,

$$\delta - c \leq 0 \tag{27}$$

The scenario in Case 3 is symmetric to Case 2.

We take the intersection of conditions given in Equation (25), Equation (26) and Equation (27) and thus, we get,

$$(\delta - c) \geq 0 \text{ AND } (\delta - c + \delta^2 \geq 0) \text{ AND } (\delta - c) \leq 0$$

$$\Rightarrow \delta = c \tag{28}$$

Thus, the star network is pairwise stable *only* under the condition $\delta = c$.

Proposition 7: Consider a tripartite complete graph denoted by G. Let a_i, $\forall i \in \{1, 2, 3\}$ denote the sizes of the three partitions of G. The condition for G to be pairwise stable is given below.

$$\frac{-2(a_j - 1)}{(a_j + a_k - 2)} + \frac{(a_j(a_j - 1) + a_k(a_k - 1))}{(a_j + a_k - 1)(a_j + a_k - 2)} \leq \frac{(\delta - c)}{\delta^2} \leq \frac{(a_j(a_j - 1) + a_k(a_k - 1))}{(a_j + a_k - 1)(a_j + a_k)}$$

Further, asymptotically as the number of nodes increases, the pairwise stability condition reduces to

$$-\frac{1}{2} \le \frac{\delta - c}{\delta^2} \le \frac{1}{2} \tag{29}$$

when all the partitions in G are of equal size.

Proof: We consider a completely connected tripartite network with each partition having a_1, a_2, a_3 nodes respectively. We derive conditions for pairwise stability of such a network. Consider a node t in partition i where i can be 1, 2, or 3. Its pay-off is dependent on the number of nodes in the other partitions j and k.

$$u_t = (\delta - c) \times (a_j + a_k) + (a_j + a_k) \times \delta^2 \times (1 - \frac{\sigma_t}{\binom{(a_j + a_k)}{2}})$$

$$\text{where } \sigma_t = a_j + \binom{a_k}{2} - \binom{a_j}{2} - \binom{a_k}{2}$$

Substituting and simplifying, we get,

$$u_t = (\delta - c)(a_j + a_k) + \delta^2 \frac{(a_j(a_j - 1) + a_k(a_k - 1))}{(a_j + a_k - 1)} \tag{30}$$

The network can grow from this state if a node in partition i forms a link with a node in the same partition and, if this happens, the number of links among the neighbours of node t, denoted by σ_t', will be given by $\sigma_t' = (\sigma_t + d_t)$ where d_t is the degree of node t in the modified network given by $d_t = (a_j + a_k + 1)$. Substituting and simplifying, the new pay-off of the node t (denoted by u_t') is given by:

$$u_t' = (\delta - c)(a_j + a_k + 1) + \delta^2 \frac{(a_j(a_j - 1) + a_k(a_k - 1))}{(a_j + a_k)} \tag{31}$$

The network is definitely Pairwise stable under link addition if $u_t \ge u_t'$ for all t in the network. Thus, the tripartite network is pairwise stable under addition if: $u_t - u_t' \ge 0$, which implies,

$$-(\delta - c) + \delta^2 \left((a_j(a_j - 1) + a_k(a_k - 1)) \frac{1}{(a_j + a_k)(a_k + a_j - 1)} \right) \ge 0$$

which implies,

$$\frac{(\delta - c)}{\delta^2} \le \frac{(a_j(a_j - 1) + a_k(a_k - 1))}{(a_j + a_k - 1)(a_j + a_k)} \tag{32}$$

One can verify that the RHS is upper bounded by 1 and also lower bounded by $\frac{1}{3}$ when nodes in any partition exceed 1. This has to be true for all nodes t in all partitions 1, 2, 3.

Pairwise Stability under deletion of a link: The network can shrink from this state if a node t in partition i deletes a link with a node in either partition j, or k. Lets assume without loss of generality that it deletes a node in partition j. When this happens the pay-off of the node t would be u_t' given by:

$$u_t' = (\delta - c)(a_j + a_k - 1) + \delta^2 \frac{((a_j - 1)(a_j - 2) + a_k(a_k - 1))}{(a_j + a_k - 2)} \tag{33}$$

The network is definitely Pairwise stable under link deletion if $u_t \geq u_t'$ for all t in the network. Thus, the complete tripartite network is pairwise stable under link deletion if: $u_t - u_t' \geq 0$, which implies,

$$(\delta - c) + \delta^2 \left((a_j(a_j - 1) + a_k(a_k - 1)) \frac{-1}{(a_j + a_k - 1)(a_k + a_j - 2)} + \frac{2(a_j - 1)}{a_j + a_k - 2} \right) \geq 0$$

which implies,

$$\frac{(\delta - c)}{\delta^2} \geq \frac{-2(a_j - 1)}{(a_j + a_k - 2)} + \frac{(a_j(a_j - 1) + a_k(a_k - 1))}{(a_j + a_k - 1)(a_j + a_k - 2)} \tag{34}$$

Again the RHS can be lower bounded by –1 and upper bounded by $-\frac{1}{3}$ when all partitions have more than a single node.

Special Case: All partitions are of equal size

By setting $a_1 = a_2 = a_3 = a$ the condition for the network to be pairwise stable,

$$-\frac{(a-1)}{(2a-1)} \leq \frac{\delta - c}{\delta^2} \leq \frac{(a-1)}{(2a-1)}$$

Asymptotically when the number of nodes increase w.l.g., we can state the condition for pairwise stability as:

$$-\frac{1}{2} \leq \frac{\delta - c}{\delta^2} \leq \frac{1}{2} \tag{35}$$

Proposition 8: For $k \geq 3$, the complete k-partite network is pairwise stable if (i) $\delta = c$, and (ii) $a_i = a, \forall i \in \{1, 2, ..., k\}$ where a_i is the number of nodes in partition i in k-partite network and a is any positive integer.

Proof: We start with a k-partite graph, G, satisfying condition (ii) given in the statement of this proposition. Consider a node i in the p^{th} partition of G where $1 \leq p \leq k$. We construct the proof in two steps.

Step 1 (edge addition): We can see that, in G, the only link that can be added from node i is to a node j in the p^{th} partition. Let $\bar{G}$ be the network obtained after a new link (i, j) is added to G. For pairwise stability, we need $u_i(\bar{G}) - u_i(G) \leq 0$. This implies,

$$(\delta - c) + (d_i + 1)\delta^2 (1 - \frac{\sigma_i'}{\binom{d_i + 1}{2}}) - d_i \delta^2 (1 - \frac{\sigma_i}{\binom{d_i}{2}}) \leq 0$$

where σ_i' is the number of links among the neighbours of node i in $\bar{G}$ and σ_i is the number of links among the neighbours of node i in G. Note that $d_i = d_j$ since nodes i

and j belong to the same partition in G. Now we get that $\sigma_i' = \sigma_i + d_j = \sigma_i + d_i$. Simplifying, we get,

$$u_i(\overline{G}) - u_i(G) = (\delta - c) - \delta^2 + \delta^2\left(\frac{2\sigma_i}{d_i(d_i-1)}\right) \tag{36}$$

Since the term $\frac{2\sigma_i}{d_i(d_i-1)}$ lies in the interval [0, 1] and the fact that $\delta = c$ (given in the statement of this proposition), we get that expression (36) is non-positive. This implies that no pair of nodes can form a link to improve their respective pay-offs.

Step 2 (edge deletion): In G, consider that node i deletes a link to a node j in the q^{th} partition where $1 \le q \le k$ and $p \ne q$. Let $\overline{G}$ be the network obtained after the link (i, j) has been deleted from G. For pairwise stability, we need $u_i(\overline{G}) - u_i(G) \le 0$. This implies,

$$-(\delta - c) + (d_i - 1)\delta^2\left(1 - \frac{\sigma_i'}{\binom{d_i-1}{2}}\right) - d_i\delta^2\left(1 - \frac{\sigma_i}{\binom{d_i}{2}}\right) \le 0$$

where σ_i' denotes the number of links among the neighbours of node i in $\overline{G}$. We can see that $\sigma_i' = \sigma_i - d_j + a_i$. Simplifying,

$$-(\delta - c) - \delta^2 + \delta^2 \underbrace{\left(\frac{-2\sigma_i + 2d_j - 2a_i}{d_i - 2} + \frac{2\sigma_i}{d_i - 1}\right)}_{expr_1} \le 0 \tag{37}$$

Claim: $expr_1 \le 1$.

Proof of the Claim: We know that $d_i = \sum_{j \ne i} a_j$. Now, we derive an expression for σ_i.

$$\sigma_i = \binom{d_i}{2} - \sum_{j \ne i}\binom{a_j}{2} = \frac{d_i(d_i-1)}{2} - \frac{1}{2}\left(\sum_{j \ne i} a_j^2 - \sum_{j \ne i} a_j\right) = \frac{d_i^2 - \sum_{j \ne i} a_j^2}{2} \tag{38}$$

Now, we show that $expr_1 \le 1$. The proof is by contradiction. Suppose $expr_1 > 1$.

$$\begin{aligned}
&\left(\frac{-2\sigma_i + 2d_j - 2a_i}{d_i - 2} + \frac{2\sigma_i}{d_i - 1}\right) > 1 \\
&2(d_j - \sigma_i - a_i)(d_i - 1) + (2\sigma_i)(d_i - 2) > (d_i - 2)(d_i - 1) \\
&(2d_j d_i - 2\sigma_i - 2a_i d_i - 2d_j + 2a_i) > (d_i^2 - 3d_i + 2)
\end{aligned} \tag{39}$$

From condition *(2)* in Proposition 8, we have $a_i = 1, \forall i$ and $d_i = d_j = (k - 1)a$. Also, using Equation (38) in Equation (39) and simplifying, we have,

$$\begin{aligned}
&(k+1)a - (k-1)a^2 > 2 \\
&\Rightarrow (k+1)a > 2 + (k-1)a^2 > (k-1)a^2 \\
&\Rightarrow a < \left(\frac{k+1}{k-1}\right)
\end{aligned} \tag{40}$$

Let $y(k) = \left(\frac{k+1}{k-1}\right)$. As we know that the function $y(k)$ is a decreasing function of k (as derivative of $y(k)$ with respect to k is < 0), we can write,

$$a < y(2) \Rightarrow a < 3$$

So, clearly we can conclude that $expr_1 > 1$ for $0 < a < 3$ (i.e., $a = 2$ and $a = 1$) and $expr_1 \leq 1$ for $a \geq 3$.

Now we will examine what happens when $a = 1$ and $a = 2$. Substituting $a = 1$ in Equation (40) and simplifying, we get $2 > 2$ which is absurd. Substituting $a = 2$ in Equation (40) and simplifying, we get $k < 2$ which violates the hypothesis that $k \geq 3$. Hence, by the above arguments, $expr_1 \leq 1$, $\forall a \in \{1, 2, ...\}$, $\forall k \geq 3$. This completes the proof of the claim.

Note that we are given that $\delta = c$. Thus, from Equation (37),

$$-\delta^2 + \delta^2 \underbrace{\left(\frac{-2\sigma_i + 2d_j - 2a_i}{d_i - 2} + \frac{2\sigma_i}{d_i - 1}\right)}_{\leq 1} \leq 0 \Rightarrow u_i(\overline{G}) - u_i(G) \leq 0$$

Thus, node i does not have any incentive to add an edge to G or delete an edge from G when the conditions given in the statement of the proposition are satisfied. As node i is chosen arbitrarily from G, we have that G is pairwise stable.

Proofs on Efficient Networks

Proposition 10: When $\delta = c$, the Turan graph is the unique efficient graph.

Proof: We will analyze the efficiency of an arbitrary graph (denoted by G) as follows:

$$\begin{aligned} u(G) &= \sum_{i=1}^{n} u_i(G) = \sum_{i=1}^{n} d_i \delta^2 \left(1 - \frac{\sigma_i}{\binom{d_i}{2}}\right) \\ &= \delta^2 \sum_{i=1}^{n} d_i - \delta^2 \sum_{i=1}^{n} \frac{2\sigma_i}{(d_i - 1)} \\ &\leq \delta^2 \sum_{i=1}^{n} d_i - \frac{\delta^2}{(n-2)} \sum_{i=1}^{n} 2\sigma_i \\ &= \delta^2 \sum_{i=1}^{n} d_i - \frac{\delta^2}{(n-2)} (2 \times 3 \times T_3(G)) \end{aligned} \tag{41}$$

where, $T_3(G)$ is the number of triangles in the graph G. The last step of the above simplification is due to the fact that the number of links among the neighbours of a node i is the number of triangles in the graph in which node i is one of the vertices of the triangle. The factor 3 in the last step is due to the fact that every triangle contributes to the σ_i of 3 nodes. We know that, for an efficient graph, Equation (41) should be maximized and that happens when the number of triangles in a graph is minimized while simultaneously maximizing the number of edges in the graph.

The Turan graph (refer Equation (5)) is a graph with maximum edges that has no triangles. So an efficient graph must have an efficiency greater than or equal to that of a Turan graph. Thus, it is clear that there is no need to consider graphs with edges lesser than that of a Turan graph. Let us consider the case when a graph (denoted by $\bar{G}$) has more edges than the Turan graph. Let $\bar{G}$ have $\left\lfloor \frac{n^2}{4} \right\rfloor + x$ edges where $x > 0$. From Equation (41), we know that,

$$u(\bar{G}) = \sum_{i=1}^{n} u_i(G) = \delta^2 \sum_{i=1}^{n} d_i - \delta^2 \sum_{i=1}^{n} \frac{2\sigma_i}{(d_i - 1)}$$
$$\leq \delta^2 \left(2\left(\left\lfloor \frac{n^2}{4} \right\rfloor + x \right) \right) - \frac{\delta^2}{(n-2)}(6T_3(\bar{G})) \tag{42}$$

where $T_3(\bar{G})$ is the number of triangles in $\bar{G}$. From Equation (6), we have,

$$u(\bar{G}) \leq \delta^2 \left(2\left(\left\lfloor \frac{n^2}{4} \right\rfloor + x \right) \right) - \frac{\delta^2}{(n-2)} \left(6n\left(\frac{4e - n^2}{9} \right) \right) \tag{43}$$

Since $T_3(G_{Turan}) = 0$, the efficiency of the Turan graph is:

$$u(G_{Turan}) = \sum_i u_i(G_{Turan}) = \delta^2 \left(2 \times \left\lfloor \frac{n^2}{4} \right\rfloor \right) \tag{44}$$

The change in efficiency (Δu) between the two graphs is,

$$\Delta u = u(\bar{G}) - u(G_{Turan}) \leq 2\delta^2 \left(x - \frac{n}{(n-2)} \frac{4x}{3} \right) \tag{45}$$

which is clearly negative for any $x > 0$. This implies that the Turan graph is the unique efficient graph.

Proposition 11: When $\delta < c$ and $\delta^2 > (c - \delta)$, the Turan graph is the unique efficient graph.

Proof: We prove this by contradiction. Assume that $\bar{G}$ is any graph other than the Turan graph and $\bar{G}$ is efficient. We show below that $\bar{G}$ cannot have lesser number of edges than G_{turan},

$$u(\bar{G}) = \sum_{i=1}^{n} u_i(\bar{G}) = (\delta - c)\sum_{i=1}^{n} d_i + \sum_{i=1}^{n} d_i \delta^2 \left(1 - \frac{\sigma_i}{\binom{d_i}{2}} \right)$$

$$\leq (\delta - c + \delta^2) \sum_{i=1}^{n} d_i$$

$$< u(G_{turan}) \textit{ whenever}, \sum_{i=1}^{n} d_i < 2\left\lfloor \frac{n^2}{4} \right\rfloor$$

And observe, if $\bar{G}$ has same number of edges as G_{turan} and is different from it, it can contain triangles and will have an pay-off less than that of G_{turan}, as the benefit from bridging would go down and the benefit from direct links would remain unchanged.

Thus $\bar{G}$ contains more edges than G_{turan}. Observe, that the benefit from direct links is negative $(\delta - c)\sum_{i=0}^{n} d_i < 0$, and $\bar{G}$ has an higher pay-off compared to that of G_{turan}. It has to be that the bridging benefits in $\bar{G}$ has to be greater than that of the Turan graph, as the pay-off due to direct links term has become more negative compared to its value in G_{turan}

$$u(\bar{G}) = \sum_{i=1}^{n} u_i(\bar{G}) = \underbrace{(\delta - c)\sum_{i=1}^{n} d_i}_{\text{negative}} + \underbrace{\sum_{i=1}^{n} d_i \delta^2 \left(1 - \frac{\sigma_i}{\binom{d_i}{2}}\right)}_{\text{payoff more than } G_{Turan}}$$

This implies that this graph would give a higher pay-off for the $\delta = c$ case, as the first term is 0 there. This contradicts Theorem 10 and so our assumption must be wrong. Hence the Turan graph is efficient.

Proposition 12: When $\delta > c$ and $\delta^2 \geq 3(\delta - c)$, the Turan graph is the unique efficient graph.

Proof: Let $\bar{G}$ be the efficient graph. Using a similar analysis that lead to Equation (43), we can see that,

$$u(\bar{G}) \leq (\delta + c + \delta^2)\left(2\left(\left\lfloor \frac{n^2}{4} \right\rfloor + x\right)\right) - \frac{\delta^2}{(n-2)}\left(6n\left(\frac{4e - n^2}{9}\right)\right)$$

$$= (\delta + c + \delta^2)\left(2\left(\left\lfloor \frac{n^2}{4} \right\rfloor + x\right)\right) - \frac{\delta^2 n}{(n-2)}\left(\frac{8x}{3}\right) \tag{46}$$

For the Turan graph, it can also be seen by simple analysis that

$$u(G_{Turan}) = 2\left\lfloor \frac{n^2}{4} \right\rfloor (\delta - c + \delta^2)$$

$$\Rightarrow u(\bar{G}) - u(G_{Turan}) \leq 2x\left((\delta - c + \delta^2) - \frac{4n\delta^2}{3(n-2)}\right)$$

$$< 2x\left((\delta - c + \delta^2) - \frac{4\delta^2}{3}\right) \tag{47}$$

Thus, when $\delta^2 \geq 3(\delta - c)$, the Turan graph is the unique efficient graph.

Proposition 13: When $\delta > c$ and $(\delta - c) > 2\delta^2$, the complete graph is the efficient graph.

Proof: It can be shown that starting with an arbitrary graph $\bar{G}$ (which is not a complete graph), adding an edge between two nodes i and j (with smallest degree) increases the cumulative pay-off of these two nodes by at least $2\delta^2$. At the same time, there is a decrease in pay-off of a *common* neighbour of nodes i and j, say node k, as there is a decrease in the bridging benefits of node k. It can be shown that the cumulative decrease in pay-off of all such common neighbours formed is $\frac{2\delta^2}{d_k - 1}\min(d_i, d_j)$ which is less than equal to $2\delta^2$. Repeating the above process, we obtain the complete network.

Notes on Convergence using Myopic Best Response Dynamics

Metrics Recorded

At the end of *Num-Repetitions* number of repetitions, a number of metrics were recorded.

1. The network structure (shape) for each repetition
2. The frequency with which each of the network structures in Section 12.2 resulted (across all repetitions)
3. The mean pay-off of the final network (across all repetitions)
4. The mean time to reach the final network (across all repetitions)
5. The mean number of acts to reach the final network (across all repetitions)

Before we present the results, we briefly describe the classification criteria used to identify pairwise stable networks.

Classification of Pairwise Stable Network Structures

Once the network reaches a stable state, we classify the network structure as one of the network structures shown in Table 4. As in (Hummon, 2000), we use the sorted (descending order) degree vector to identify the structure of the stable network. For example, the Null network has a sorted degree vector of (0, 0, ..., 0), the Star network (n–1, 1, 1, ..., 1) and the Complete network (n–1, n–1, ..., n–1). We refer to a network structure a shared network if it is a regular network (i.e., all nodes have same degree) of some uniform degree. For example, a cycle is a 2-regular graph and hence is a shared network.

Also as in Hummon (2000), we use total mean squared deviation (MSD) to classify the resultant stable network as near-'standard network' (for example, near-complete network). Further, if the mean squared deviation is above a certain threshold (τ) then we know its not

Table 5. Possible Network Structures Considered in the Simulations

NULL	STAR	SHARED	COMPLETE
NEAR-NULL	NEAR-STAR	NEAR-SHARED	NEAR-COMPLETE
BI-PARTITE-COMPLETE	TURAN	EQUI-K-PARTITE-COMPLETE	EQUI-K-PARTITE
K-PARTITE-COMPLETE	K-PARTITE		

Source: Developed by the Authors.

close to any of the above topologies, we then colour the graph using a greedy colouring algorithm (Boost-C++-Libraries, 2012) and then classify it either as a general k-partite graph (where k equals the number of colours required to colour the graph) or any of the other network structures shown in Table 12. In our simulations, we use the maximum deviation ($(n-1)^2$) for calculating the τ, i.e., $\tau = 0.1 \times (n-1)^2$.

Note that whenever we classify a network as any type of k-partite network, we implicitly mean that $K \geq 3$. The case of $K = 2$ is the same as bipartite network and is handled as a separately as shown in Table 12. Turan network refers to a complete bipartite network with the sizes of the two partitions to be as equal as possible. If N is even, then the Turan network has equal sized partitions whereas if N is odd, the size of one partition is one less than the other partition.

For classification of a sorted degree network as a near-shared network, we first need to calculate the order of the regular network with which this degree vector needs to be compared. As in Hummon (2000), to compute the total mean squared deviation for the shared structure, the ideal order is defined by average number of ties in the in-out degree vector, rounded to the nearest whole tie. In this example, if the degree vector is (3, 2, 1, 1, 1), the average is 1.6, and the ideal type shared structure is (2, 2, 2, 2, 2). However, note that a cycle network is necessarily a shared network but a shared network need not always be a cycle network.

The following example clarifies this procedure: Consider the 5-node network as shown in Figure 12. Suppose that we would like to classify this network as one of the following standard networks: null, star, shared, complete, near-null, near-star, near-shared or near-complete. This is done as follows. Note that the given network does not classify as any of the first four networks in the list given above. Hence, we try to classify the given network as one of the remaining four networks (i.e., the 'near' type networks).

We know that the sorted degree vector is (4, 3, 3, 2, 2) for the given network. The ideal order for the shared network comparison is calculated by taking the average degree (which is 2.8) and rounding to the nearest integer (which gives 3). This means we have to compare the network to a 3 regular network. The total MSD from the shared network is thus $((4-3)^2 + (3-3)^2 + (3-3)^2 + (2-3)^2 + (2-3)^2))/5 = 0.6$. The total MSD of this network from star network is $((4-4)^2 + (3-1)^2 + (3-1)^2 + (2-1)^2 + (2-1)^2))/5 = 2$. Similarly, the total MSD

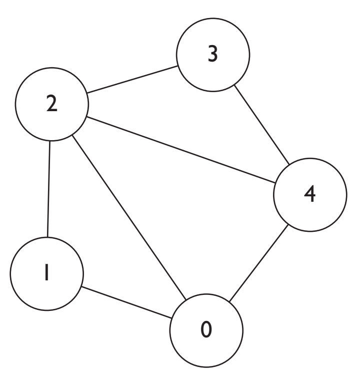


Figure 12. A Stylized 5-node Network

Source: Developed by the Authors.

from null network is 8.4, and the total MSD from the complete network is 2. The value 0.6 being the least among these and less than 10% of maximum deviation 16, we classify the above network structure as near-shared.

Multiple Classification of Pairwise Stable Structures

We note that the classification of pairwise stable network structures according to Table 4 is not mutually exclusive. There can exist networks which can be classified as more than one of the types described in Figure 13. We illustrate a couple of interesting network structures that we encountered during our simulations here. Figure 13(a) refers to a pairwise stable network that emerged when we ran the simulation with random_seed = 6875, $\delta = 0.7$, $c = 0.55$. We observed that this network is both a near-shared network as well as a tripartite complete network whose partitions are (0, 6, 7, 8), (1, 2, 5), (3, 4, 9). In such cases, we classify the network structure as a k-partite complete network.

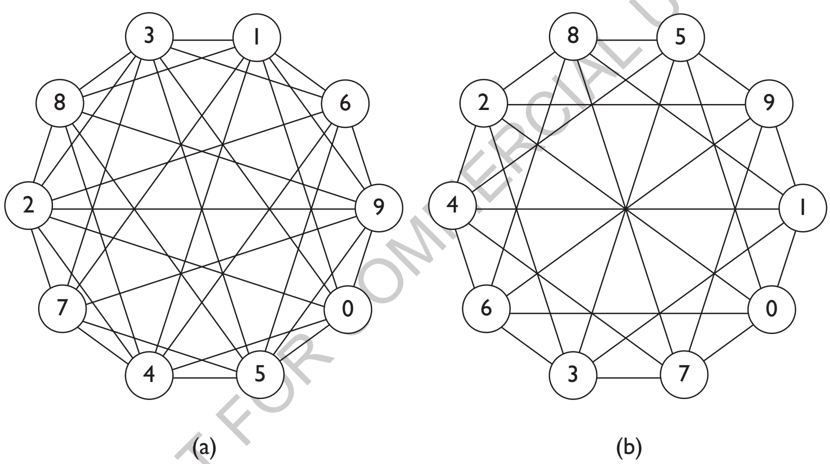


Figure 13. Possibility of Multiple Classifications for a given Network Structure

Source: Developed by the Authors.

Another example is shown in Figure 13(b) which is obtained when running simulations with 'random_seed' = 15256, $\delta = 0.5$, $c = 0.5$. We observe that this graph can be classified as a regular (or Shared) network with degree = 5. However, it turns out that this graph is also an equi-partitioned bipartite network with partitions (0, 3, 4, 8, 9), (1, 2, 5, 6, 7). In such cases, we classify the graph as equi-bipartite network (or the Turan network).

Interpretation of Pairwise Stability

In a pairwise stable network, if a node adds a link to another node and gains strictly from it, the other node should lose strictly. Hence, the addition of the link becomes infeasible in this case. However, nodes in a pairwise stable network can still add links if adding these links does not change the pay-offs of either of the nodes. In this case, the nodes are indifferent about adding the link. In the case of deletion, a node will delete a link from the current network unilaterally if it strictly benefits from doing so. We use this interpretation of pairwise stability during the course of our simulations.

References

Ahuja, G. (2000). Collaboration networks, structural holes, and innovation: A longitudinal study. *Administrative Science Quarterly*, *45*(3), 425–455.

Anshelevich, E., Dasgupta, A., Kleinberg, J., Tardos, E., Wexler, T., & Roughgarden, T. (2008). The price of stability for network design with fair cost allocation. *SIAM Journal of Computing*, *38*(4), 1602–1623.

Anshelevich, E., Dasgupta, A., Tardos, E., & Wexler, T. (2003). Near-optimal network design with selfish agents. *Proceedings of the 35th Annual ACM Symposium on Theory of Computing (STOC)* (pp. 511–520). New York, USA.

Arcaute, E., Johari, R., & Mannor, S. (2008). Local two-stage myopic dynamics for network formation games. *Proceedings of the 4th International Workshop on Internet and Network Economics*, WINE '08 (pp. 263–277). Berlin, Heidelberg: Springer-Verlag.

Barrat, A., Barthlemy, M., & Vespignani, A. (2008). *Dynamical processes on complex networks*. New York, NY: Cambridge University Press.

Bloch, F., & Jackson, M.O. (2007). The formation of networks with transfers among players. *Journal of Economic Theory*, *133*(1), 83–110.

Boorman, S.A. (1975). A combinatorial optimization model for transmission of job information through contact networks. *Bell Journal of Economics*, *6*(1), 216–249.

Boost-C++-Libraries (2012). *Boost C++ libraries*. Retrieved from http://www.boost.org/.

Borgs, C., Chayes, J.T., Ding, J., & Lucier, B. (2011). The hitchhiker's guide to affiliation networks: A game-theoretic approach. *Proceedings of the 2nd Symposium on Innovations in Computer Science (ICS)* (pp. 389–400). Beijing, China: Institute for Theoretical Computer Science (ITCS), Tsinghua University.

Brandes, U., & Erlebach, T. (2005). *Network analysis: methodological foundations* (volume 3418). Netherlands: Springer.

Brautbar, M., & Kearns, M. (2011). A clustering coefficient network formation game. *Proceedings of the 4th international conference on Algorithmic game theory*, SAGT'11 (pp. 224–235). Berlin, Heidelberg: Springer-Verlag.

Burt, R.S. (1992). *Structural holes: The social structure of competition* (volume 58). Cambridge, USA: Harvard University Press.

——— (2004). Structural holes and good ideas. *The American Journal of Sociology*, *110*(2), 349–399.

——— (2007). Secondhand brokerage: Evidence on the importance of local structure for managers, bankers, and analysts. *Academy of Management Journal*, *50*(1), 119–148.

Buskens, V., & Van De Rijt, A. (2008). Dynamics of networks if everyone strives for structural holes. *American Journal of Sociology*, *114*(2), 371–407.

Calvo-Armengol, A. (2004). Job contact networks. *Journal of Economic Theory*, *115*(1), 191–206.

Cooper, R.L. (1982). *Language spread: Studies in diffusion and social change*. Bloomington, IL and Washington, DC: Indiana University Press and Center for Applied Linguistics.

Corbo, J., & Parkes, D. (2005). The price of selfish behavior in bilateral network formation. *Proceedings of the 24th annual ACM symposium on Principles of distributed computing*, PODC ’05 (pp. 99–107) New York, NY: ACM.

Demange, G., & Wooders, M. (Eds) (2005). *Group formation in economics: Networks, clubs and coalitions*. Cambridge, UK: Cambridge University Press.

Doreian, P. (2006). Actor network utilities and network evolution. *Social Networks*, *28*(2), 137–164.

——— (2008a). Actor utilities, strategic action and network evolution. *Network Strategy, Advances in Strategic Management*, *25*(1), 247–271.

——— (2008b). A note on actor network utilities and network evolution. *Social Networks*, *30*(1), 104–106.

Dutta, B., Nouweland, A.v.d., & Tijs, S. (1998). Actor utilities, strategic action and network evolution. *International Journal of Game Theory*, *27*(2), 245–256.

Easley, D., & Kleinberg, J. (2010). *Networks, crowds, and markets: Reasoning about a highly connected world*. Cambridge, UK: Cambridge University Press.

Elias, J., Martignon, F., Avrachenkov, K., & Neglia, G. (2011). A game theoretic analysis of network design with socially-aware users. *Computer Networks*, *55*(1), 106–118.

Fabrikant, A., Luthra, A., Maneva, E., Papadimitriou, C.H., & Shenker, S. (2003). On a network creation game. *Proceedings of the 22nd Annual Symposium on Principles of Distributed Computing*, PODC ’03 (pp. 347–351). New York, NY: ACM.

Galeotti, A., Goyal, S., & Kamphorst, J. (2006). Network formation with heterogeneous playersâ. *Games and Economic Behavior*, *54*(2), 353–372.

Gilles, R.P., & Johnson, C. (2000). Spatial social networks. *Review of Economic Design*, *5*(3), 273–299.

Goyal, S., & Joshi, S. (2003). Networks of collaboration in oligopoly. *Games and Economic Behavior*, *43*(1), 57–85.

Goyal, S., & Vega-Redondo, F. (2007). Structural holes in social networks. *Journal of Economic Theory*, *137*(1), 460–492.

Goyal, S. (2007). *Connections: An introduction to the economics of networks*. Princeton, New Jersey, USA: Princeton University Press.

Hummon, N.P. (2000). Utility and dynamic social networks. *Social Networks*, *22*(3), 221–249.

Jackson, M., & Wolinsky, A. (1996). A strategic model of social and economic networks. *Journal of Economic Theory*, *71*(1), 44–74.

Jackson, M.O., & Dutta, B. (2000). The stability and efficiency of directed communication networks. *Review of Economic Design*, *5*(3), 251–272.

Jackson, M.O. & van den Nouweland, A. (2005). Strongly stable networks. *Games and Economic Behavior*, *51*(2), 420–444.

Jackson, M.O., & Watts, A. (2002). The evolution of social and economic networks. *Journal of Economic Theory*, *106*(2), 265–295.

Jackson, M.O. (2003). The stability and efficiency of economic and social networks. In B. Dutta, & M.O. Jackson (Eds), *Networks and Groups: Models of Strategic Formation*. Heidelberg: Springer–Verlag.

——— (2005). Allocation rules for network games. *Games and Economic Behavior*, *51*(1), 128–154.

——— (2008). *Social and economic networks*. Princeton, New Jersey, USA: Princeton University Press.

Kleinberg, J., Suri, S., Tardos, E., & Wexler, T. (2008). Strategic network formation with structural holes. *Proceedings of the 9th ACM conference on Electronic commerce*, EC ’08 (pp. 284–293). New York, NY: ACM.

Mehra, A., Kilduff, M., & Brass, D. J. (2001). The social networks of high and low self-monitors: Implications for workplace performance. *Administrative Science Quarterly*, *46*(1), 121–146.

Myerson, R.B. (1991). *Game theory: Analysis of conflict*. Cambridge, USA: Harvard University Press.

Narayanam, R., & Narahari, Y. (2011). Topologies of strategically formed social networks based on a generic value function: Allocation rule model. *Social Networks*, *33*(1), 56–69.

Newman, M.E.J. (2003). The structure and function of complex networks. *SIAM Review*, *45*(2), 58.

Newman, M.E., Strogatz, S.H., & Watts, D.J. (2001). Random graphs with arbitrary degree distributions and their applications. *Physical Review E - Statistical, Nonlinear and Soft Matter Physics*, *64*(2 Pt 2), 026118.

Newman, M., Barabasi, A.-L., & Watts, D.J. (2006). *The structure and dynamics of networks: (Princeton Studies in Complexity)*. Princeton, New Jersey, USA: Princeton University Press.

Nordhaus, E., & Stewart, B. (1963). Triangles in ordinary graph. *Canadian Journal of Mathematics*, *15*(1), 33–41.

Podolny, J.M. & Baron, J.N. (1997). Resources and relationships: Social networks and mobility in the workplace. *American Sociological Review*, *62*(5), 673–693.

Reka, A., & Barabási (2002). Statistical mechanics of complex networks. *Reviews of Modern Physics*, *74*(1), 47–97.

Rogers, E.M. (2003). *Diffusion of innovations* (volume 27). New York, USA: Free Press (Simon and Schuster).

Schelling, T.C. (1978). *Micromotives and macrobehavior*. New York, USA: W.W. Norton.

Slikker, M., & Nouweland, A. (2001). *Social and economic networks in cooperative game theory*. Theory and decision library: Game theory, mathematical programming, and operations research. Netherlands: Kluwer Academic Publishers.

Song, C., Havlin, S., & Makse, H.A. (2005). Self-similarity of complex networks. *Nature*, *433*(7024), 392–5.

Strang, D., & Soule, S.A. (1998). Diffusion in organizations and social movements: From hybrid corn to poison pills. *Annual Review of Sociology*, *24*(1), 265–290.

Strogatz, S.H. (2001). Exploring complex networks. *Nature*, *410*(6825), 268–76.

Turan., P. (1941). On an extremal problem in graph theory. *Matematikai es Fizikai Lapok*, *48*(137), 436–452.

Valente, T.W. (1995). *Network models of the diffusion of innovations* (volume 2). New York, USA: Hampton Press.

Wasserman, S., & Faust, K. (1994). *Social network analysis: Methods and applications* (volume 24). Cambridge, UK: Cambridge University Press.

Watts, D.J., & Strogatz, S.H. (1998). Collective dynamics of ‘small-world’ networks. *Nature*, *393*(6684), 440–442.

Xie, F., & Cui, W. (2008a). Cost range and the stable network structures. *Social Networks*, *30*(1), 100–101.

——— (2008b). A note on the paper ‘cost range and the stable network structures’. *Social Networks*, *30*(1), 102–103.