Performability Studies of Automated Manufacturing Systems with Multiple Part Types

N. Viswanadham, *Fellow, IEEE,* Krishna R. Pattipati, *Fellow, IEEE,* and V. Gopalakrishna

Abstract—**In this paper, we consider the transient performance analysis of failure-prone manufacturing systems producing multiple part types. We decompose the exact monolithic model into (a) a slower time scale structure state process modeling the failure and repair and (b) a faster time scale performance model describing the part processing and the material movement. We combine the solution of these two models to show that the accumulated reward over a given time interval is a solution of a set of forward or adjoint multidimensional linear hyperbolic partial differential equations. This result generalizes the existing results on composite performance-dependability analysis of manufacturing systems. We also present efficient numerical methods for computing the distribution of the cumulative operational time, and the mean and variance of the cumulative production over a given time interval. Further, we bring out the significance of these results in the manufacturing systems context through several examples.**

I. Introduction

A. Manufacturing Systems

MODERN MANUFACTURING systems are interconnections of subsystems such as numerically controlled machines, assembly stations, automated guided vehicles (AGV's), robots, conveyors and computer control systems. These components are typically prone to *failures*. Usually, manual or automatic *repair* facilities are available to restore failed subsystems to the normal level of operation. Sometimes, *reconfiguration* of system resources and layout may be done, in order to cope with failures of subsystems. Failures, together with repairs and reconfigurations, constitute three basic events that we need to model for computing the performance of manufacturing systems in the presence of failures and repairs.

Production rate (the number of parts produced per unit time) and manufacturing lead time (the amount of time the workpiece resides on the factory floor) are two important performance measures in a manufacturing system. Most of the literature on performance analysis of manufacturing systems deals with computation of average performance measures of failure-free systems. However, there is recent awareness regarding the pitfalls that arise in just dealing with the mean production rates and neglecting the variance of production [9] and also the need to take into account failure-repair behavior of the system [24], [25]. In this paper, we conduct composite performance-dependability analysis of manufacturing systems producing multiple part types to determine the distribution and moments of performance measures such as the accumulated production over a time interval using Markov reward models.

In this paper, we consider the transient performance analysis of failure-prone manufacturing systems producing multiple part types. We decompose the exact monolithic model into: (a) a slower time scale structure state process modeling the failure and repair events, and (b) a faster time scale performance model describing the part processing and the material movement events. We combine the solution of these two models to show that the accumulated reward over a given time interval is a solution of a set of forward or adjoint multidimensional linear hyperbolic partial differential equations. We also derive the differential equations satisfied by the nth moment of each of the part types. These results generalize the existing results on composite performance-dependability analysis of manufacturing systems. We illustrate all the results through examples.

The main contributions of this paper are the following:

- Composite performance-reliability analysis for systems producing multiple part types.
- Derivation of multidimensional hyperbolic partial differential equations for the distribution of accumulated production over $[0, t]$.
- Efficient numerical method for solving the first and second moments of the distribution of accumulated production over $[0, t]$ for each part type.
- Several analytical and numerical examples of manufacturing systems producing multiple part types.

B. Earlier Work

There are several papers in the literature dealing with performance analysis of automated manufacturing systems. Markov chains constitute the most fundamental analytical tool [23], with queuing networks [4] and stochastic Petri nets [22] serving as higher level models. Discrete event simulation is another popular tool. Literature dealing with combined performance-reliability (performability) analysis is relatively sparse. Also, most authors focus on determining the mean values. There is a recent awareness in manufacturing literature regarding the pitfalls that arise in just dealing with first moments of the distribution and neglecting the higher ones [9].

Literature on performability modeling of fault-tolerant computer systems is abundant. Beaudry [1] introduced perfor-

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N. Viswanadham is with the Department of Computer, Science, and Automation, Indian Institute of Science, Bangalore 560 012, India.

K. R. Pattipati is with the Department of Electrical and Systems Engineering, University of Connecticut, Storrs, CT 06269 USA.

V. Gopalakrishna is with Integra Micro Systems Pvt. Ltd., Bangalore 560 001, India.

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mance related reliability for gracefully degrading fault-tolerant systems. Meyer [15] used the term *performability* for the first time and conducted early research work on this topic. Goyal *et al.* [10] have given an overview of availability modeling in fault-tolerant computer systems. Donatiello and Iyer [8] have presented a recursive algorithm to compute the moments of the cumulative distribution of performability. More recently, several authors have devised techniques to compute distribution or moments of performability for repairable systems. These include Iyer *et al.* [11], Smith *et al.* and, Silva and Gail [6]. In a series of papers, Pattipati *et al.* ([16], [19], [17]) generalized the results to time nonhomogeneous Markov chains.

Uniformization [5]–[7] is used with considerable success as a computational method for solving Markov chains with large state spaces. In [6], a generalized algorithm for the evaluation of performability distribution with exponential computational complexity in the cardinality of the reward set is presented. Donatiello and Grassi [7] presented algorithms with polynomial complexity for the evaluation of performability. Pattipati *et al.* [18] derived similar algorithms based on uniformization for the evaluation of distribution of cumulative operational time based on the hyperbolic PDE's associated with performability distribution function. Efficient numerical methods for computing the moments of performability distribution were given in [19].

Significant results in performability evaluation of manufacturing systems producing single part type have been obtained by Viswanadham *et al.* [24] and [25]. In [24], the notion of performability is introduced and the measure is computed for a variety of manufacturing systems. In [25], the main contribution is to develop results on performability for cellular manufacturing systems producing single part type.

C. Outline of the Paper

In Section II of this paper, we introduce the reliability and availability measures for manufacturing systems producing multiple part types. Next, we formulate the notion of performability of a manufacturing system using the concepts of structure state process and accumulated rewards. An example of a flexible machine cell illustrates the definitions in this section. In Section II-D we prove the main results of this paper, namely the partial differential equations satisfied by the performability distribution function. In Section II-E, we derive the moment equations.

In Section III, we consider two important building blocks of an AMS, namely a machine center processing multiple part types and a flexible cell processing two part types and obtain analytically the performability function. Also, we present an example of a cell manufacturing three part types and compute analytically the mean, second moment and the coefficient of correlation between part types.

In Section IV, we deal with the numerical issues concerned with the computation of cumulative availability and present two examples. In Section V, we discuss the numerical issues related to the computation of the moments of performability. We present a number of examples in each of these sections.

Discrete time versions of these results are reported in [13] and [14].

II. Dependability Model

Subsystems of a manufacturing system such as a machine or an AGV can be in two states: up (*properly functioning*) or down (*not properly functioning*). We say that a given manufacturing system is properly functioning if it is able to satisfy a given level of performance, i.e., if the throughput exceeds a given lower bound, or the lead time is below a given upper bound, the average machine utilization is above a given threshold, and so on.

It is possible to evaluate performance in the presence of failures and repairs by considering an exact monolithic stochastic model. But such an approach suffers from two problems: largeness and stiffness. These would pose problems of ill conditioning and numerical instability. An alternative to the monolithic approach is time-scale decomposition and we follow this approach here. It involves decomposing the exact model into (i) a *structure state process (SSP)*, a model to describe the system evolution as influenced only by failures and repairs, and (ii) a *performance model* that would describe the system performance (throughput, response time, etc.) in each state of the structure state model. The basis for this approach is the natural difference in the time scales of frequencies of the failure and repair events and of the performance events. The failure and repair activity durations are orders of magnitudes higher than the processing and material movements times. Hence, it can be assumed that the performance model reaches steady state between changes in system structure due to failures and repairs. Performability evaluation will then involve weighting the quasi steady state performance measures by the structure state probabilities. This leads to a natural hierarchy of models: a higher level dependability model and a set of lower level performance models.

The four modeling tools, *viz.*, Discrete event simulation, Markov chains, Queuing networks and Petri nets could be used for analysis at both the levels. A performance model of a manufacturing system is basically a stochastic model to compute its quantitative performance or rewards in terms of the throughput, lead time and others. The higher level dependability model is used to compute the performability distribution and its moments using the reward information.

In this section, we shall provide several definitions to describe the behavior of a manufacturing system, taking into account failures and repairs.

Definition 1: A manufacturing system is *degradable* if on the occurrence of a failure, it is operable at a reduced level of performance, and *nondegradable* if the system continues to operate, producing the same level of performance, in the presence of subsystem failures.

Since manufacturing systems are generally made up of expensive subsystems, only little excess capacity is available. This combined with the flexibility will provide redundancy and a certain degree of fault-tolerance. Thus manufacturing systems can be classified as *fault-tolerant, degradable systems.*

Definition 2: Given a manufacturing system, its *structure state* is a vector whose components describe the condition of the subsystems of the manufacturing system due to failures, repairs, and reconfigurations.

The structure state of the system evolves with time. The dynamics of the state transitions is captured via the structure state process defined below.

Definition 3: Let $Z(u)$ be the structure state of a manufacturing system at time $u \geq 0$. Then the family of random variables $\{Z(u): u \geq 0\}$ is called the *structure state process* (SSP) of the manufacturing system.

Let S be the state space of the structure state process of cardinality N. For each part type $j \in \{1, 2, \cdots, P\}$, S can be partitioned into two disjoint sets $S_{O,j}$ and $S_{F,j}$: $S_{O,j}$ is the set of operational states in which part type j can be produced and $S_{F,j}$ is the set of failed or nonoperational states in which part type j cannot be produced. We note that

$$S_{O,j} \cup S_{F,j} = S$$
$$S_{O,j} \cap S_{F,j} = \phi; \quad j = 1, 2, \cdots, P.$$

Flexibility implies that for some $i, j \in \{1, 2, \cdots, P\}$

$$S_{O,i} \cap S_{O,j} \neq \phi$$
$$S_{O,i} \cap S_{F,j} \neq \phi.$$

The last equation indicates the fact that in an operational state several part types could be produced and that there could be states in which only a subset of part types are produced.

Example 1(a): Consider a flexible cell with two identical machines $M_1$and M_2 and an AGV. Let the states of M_1, M_2, and AGV be designated as 1 when they are up and 0 if they are down. The state space of the structure state process has six states:

$$S = \{(21), (11), (01), (20), (10), (00)\}.$$

We assume that no work is done when AGV fails or when both machines fail. Thus work progresses only in (21) and (11) states. If the failure and repair times are assumed exponential, then S is a Markov chain.

Suppose we produce two part types 1 and 2 at this cell according to the following scheduling policy: (i) Produce both 1 and 2 if both machines are up. (ii) Produce 1 only if only one machine is up. Then we have

$$S_{O,1} = \{(21), (11)\}$$
$$S_{F,1} = \{(01), (20), (10), (00)\}$$
$$S_{O,2} = \{(21)\}$$
$$S_{F,2} = \{(11), (01), (20), (10), (00)\}.$$

A. Markov Dependability Models

Suppose that the structure state process $\{Z(u), u \geq 0\}$ represented by S is a homogeneous finite-state continuous time Markov chain with infinitesimal generator matrix Q. Let $p_i(t)$ be the unconditional probability that the Markov chain is in state i at time t. Then $\mathbf{p}(t)$ represents the state probability vector of the Markov chain, given by

$$\dot{\mathbf{p}}(t) = \mathbf{p}(t)Q$$
$$\mathbf{p}(0) = \mathbf{p}_0$$

where $\mathbf{p}_0$ is the initial probability vector.

1) Repairable Systems: When the system is repairable in every state, the Markov chain is likely to be irreducible. If the system is not repairable in some states, then the Markov chain will have absorbing states. The steady-state probability vector $\boldsymbol{\pi}$ exists if the Markov chain is irreducible and positive recurrent, and satisfies

$$\boldsymbol{\pi} Q = 0$$
$$\sum_{i \in S} \pi_i = 1.$$

Define $\mathbf{L}(t) = \int_0^t \mathbf{p}(u)\, du$. Then $\mathbf{L}(t)$ satisfies the differential equation

$$\dot{\mathbf{L}}(t) = \mathbf{L}(t)Q + \mathbf{p}_0$$
$$\mathbf{L}(0) = \mathbf{0}$$

where $\mathbf{0}$ represents $1 \times N$ row vector of zeros.

2) Nonrepairable Systems: For nonrepairable systems, the Markov chain model of the SSP will exhibit absorbing states that represent complete system failure. Here, the mean time to total system failure is of interest. To compute this, define

$$\eta_i = \lim_{t \to \infty} L_i(t) = \int_0^\infty p_i(u)\, du, \quad i \in S$$

then η_i represents the mean time spent by the Markov chain in state i until absorption. Also $\eta_i = 0$ for $i \in S_a$, and is finite for $i \in S_b$, where S_a and S_b are the sets of absorbing and transient states. Note that $S = S_a \cup S_b$. To compute η_i from Q, form a new matrix Q_b of size $|S_b| \times |S_b|$, where $|S_b|$ denotes the cardinality of the set S_b by restricting Q to only the set of transient states. Then the row vector $\boldsymbol{\eta} = [\eta_1, \eta_2, \cdots, \eta_{|S_b|}]$ satisfies the equation:

$$\boldsymbol{\eta} Q_b = -\mathbf{p}_t(0)$$

where $\mathbf{p}_t(0)$ is the initial probability vector restricted to the transient states. The mean time to absorption is then seen to be $\sum_{i \in S_b} \eta_i$.

B. Dependability Measures

We now define two important measures: reliability and availability. For this, we define the indicator random variables for each $j = 1, 2, \cdots, P$

$$I_j(u) = \begin{cases} 1 & \text{if } Z(u) \in S_{O,j} \\ 0 & \text{if } Z(u) \in S_{F,j}. \end{cases}$$

Also, define

$$I(u) = \begin{cases} 1 & \text{if } I_j(u) = 1 \quad \text{for some } j = 1, 2, \cdots, P \\ 0 & \text{otherwise.} \end{cases}$$

Definition 4: The probability of producing the ith part type, $R_i(t)$ is given by

$$R_i(t) = P\{I_i(u) = 1, \quad \forall u \in [0, t]\}.$$

Further the probability of system functioning properly is given by

$$R(t) = P\{I(u) = 1, \quad \forall u \in [0, t]\}.$$

Example 1(b): For the system in Example 1(a) with no repair, we have

$$R_1(t) = p_{21}(t) + p_{11}(t)$$
$$R_2(t) = p_{21}(t)$$
$$R(t) = p_{21}(t) + p_{11}(t)$$

where $p_{ij}(t)$ is the probability that we can find the system in state (ij).

1) Single Part Type Case: Consider a manufacturing system producing only a single homogeneous product. Then, we have

Definition 5: The *point availability* or *instantaneous availability* $\mathrm{PAV}(u)$ of a system, at time $u \geq 0$, is the probability that the system is properly functioning at time u.

The *cumulative operational time*, $O(t)$ of the system over $[0, t]$ is the amount of time the system is operational over $[0, t]$.

The *steady-state availability*, A, of a system is the limiting value of availability $\mathrm{PAV}(u)$ as $u \to \infty$. Thus,

$$A = \lim_{u\to\infty} \mathrm{PAV}(u).$$

Note that $A = 0$ for nonrepairable systems. For repairable systems, A is the fraction of time the system functions properly.

It can be easily seen that point availability $\mathrm{PAV}(t)$ reflects the state of the system at a given point of time; $O(t)$ is a much more useful measure. It is easy to see that the above measures are related by

$$\mathrm{PAV}(u) = P\{I(u) = 1\} = E[I(u)]$$

$$O(t) = \int_0^t I(u)\, du$$

and

$$E[O(t)] = \int_0^t \mathrm{PAV}(u)\, du.$$

2) Multiple Part Type Case: We now consider manufacturing systems producing multiple part types and define the availability measures with respect to a typical part type j.

Definition 6: The *point availability*$(\mathrm{PAV})_j(u)$ is the probability that the system is functioning at time u so that part type j could be produced.

$$(\mathrm{PAV})_j(u) = P\{I_j(u) = 1\} = E[I_j(u)].$$

The *cumulative operation time* for part type j, $O_j(t)$, over $[0, t]$ is amount of time part type j could be produced i.e.,

$$O_j(t) = \int_0^t I_j(u)\, du$$

and

$$E[O_j(t)] = \int_0^t (\mathrm{PAV})_j(u)\, du.$$

The *steady state availability* of part type j, A_j, is the fraction of time that the system is available to produce part type j i.e.,

$$A_j = \lim_{u\to\infty} (\mathrm{PAV})_j(u).$$

Note that $A_j = 0$ for nonrepairable systems. When the structure state process is described by a continuous-time Markov chain, we have for a given part type j, the following expressions

$$(\mathrm{PAV})_j(t) = \sum_{i\in S_{O,j}} p_i(t);$$

$$A_j = \sum_{i\in S_{O,j}} \pi_i;$$

$$E[O_j(t)] = \sum_{i\in S_{O,j}} L_i(t)$$

We can write down the expressions for mean time to absorption for various part types. As is clear from Example 1(a), the transient state for one part type may be the absorbing state for another; hence, the mean time to absorption is different for each part type. Let $S_{a,j}$ and $S_{t,j}$ be the absorbing and transient states for part type j such that $S_{a,j} \cup S_{t,j} = S$. We compute the vector $\boldsymbol{\eta}_j = [\eta_{j1}\eta_{j2} \cdots \eta_{j|S_{tj}|}]$ using equation

$$\boldsymbol{\eta}_j Q_{tj} = -\mathbf{p}_{jt}(0).$$

C. Markov Reward Models

1) Systems Producing Single Part Type: In this section, we present composite measures that combine both performance and reliability aspects, using the notions of structure state process and accumulated rewards. Let $\{Z(u){:}\ u \geq 0\}$ be the structure state process of a manufacturing system. In each structure state, the system can be associated with a performance measure which may be lead time, throughput, work in progress, machine utilization, etc. In the most general case, the chosen performance index is a random variable. Our discussion in this paper centers around throughput or lead time because these are important and encompass other performance measures.

Definition 7: Let $\{Z(u), u \geq 0\}$ be the structure state process of a manufacturing system. Let S be the state space. Let r_i be the reward in structure state i, which is assumed to be nonnegative. Then we have

Instantaneous reward at time t

$$X(t) = r_{Z(t)} = r_i \quad \text{if} \quad Z(t) = i$$

Accumulated reward over time interval $[0, t]$

$$Y(t) = \int_0^t X(u)\, du = \int_0^t r_{Z(u)}\, du.$$

Using the above definitions we can compute the expected values of instantaneous reward and accumulated reward and also the steady-state values of these expected rewards.

Expected instantaneous reward

$$E[X(t)] = \sum_{i\in S_O} r_i p_i(t).$$

Expected accumulated reward

$$E[Y(t)] = \sum_{i\in S_O} r_i L_i(t).$$

Steady-state expected instantaneous reward

$$\lim_{t\to\infty} E[X(t)] = \sum_{i\in S_O} r_i \pi_i.$$

Expected accumulated reward until absorption

$$\lim_{t\to\infty} E[Y(t)] = \sum_{i\in S_b} r_i \eta_i.$$

2) Systems Producing Multiple Part Types: We now extend the discussion on instantaneous and accumulated rewards to the case of systems with multiple part types.

Definition 8: Let $\{Z(u), u \geq 0\}$ be the structure state process of a manufacturing system with associated state space S. Let r_{ij} be the reward in structure state i for part type j, and $\mathbf{r}_i$ be the row vector $[r_{i1}, \cdots, r_{iP}]$. Then we have

Instantaneous reward at time $t$for part type j

$$X_j(t) = r_{Z(t)j} = r_{ij} \quad \text{if} \quad Z(t) = i; \quad j = 1, \cdots, P.$$

The *instantaneous reward vector* at time t, $X(t)$, is defined as

$$\mathbf{X}(t) = [X_1(t), X_2(t), \cdots, X_P(t)].$$

Also

$$\mathbf{X}(t) = \mathbf{r}_i \quad \text{if} \quad Z(t) = i.$$

Accumulated reward over time interval $[0, t]$ for part type j

$$Y_j(t) = \int_0^t X_j(u)\, du = \int_0^t r_{Z(u)j}\, du.$$

Accumulated reward vector at time t, $\mathbf{Y}(t)$, is defined as

$$\mathbf{Y}(t) = [Y_1(t), Y_2(t), \cdots, Y_P(t)] = \int_0^t \mathbf{r}_{z(u)}\, du$$

since r_{ij}'s are nonnegative, $\{Y_j(u); \quad j = 1, 2, \cdots, P\}$ are nonnegative monotonically nondecreasing function of time t.

We can now write down the expressions for expected values of instantaneous and cumulative rewards and also their steady state values.

Expected Instantaneous reward at time t for part type j,

$$E[X_j(t)] = \sum_{i \in S_{O,j}} r_{ij} p_i(t).$$

Expected Instantaneous reward vector at time t,

$$E[\mathbf{X}(t)] = \sum_{i \in S_O} \mathbf{r}_i p_i(t).$$

Expected accumulated reward over $[0, t]$ for part type j,

$$E[Y_j(t)] = \sum_{i \in S_{O,j}} r_{ij} L_i(t).$$

Expected accumulated reward vector over $[0, t]$,

$$E[\mathbf{Y}(t)] = \sum_{i \in S_O} \mathbf{r}_i L_i(t).$$

Steady state expected instantaneous reward vector,

$$\lim_{t \to \infty} E[\mathbf{X}(t)] = \sum_{i \in S_O} \mathbf{r}_i \pi_i.$$

Steady state expected accumulated reward vector until absorption,

$$\lim_{t \to \infty} E[\mathbf{Y}(t)] = \sum_{i \in S_b} \mathbf{r}_i \eta_i.$$

Example 1(c): Consider the cell with two identical machine centers and AGV in Example 1(a). The AGV transports workpieces between the pallet pool and the machine center. Let λ_a and λ_m be the exponential failure rates of AGV and the machines. We assume that no repair is possible and the system is down when AGV is down or both machines are down. The structure state process is given by $\{(2, 1), (1, 1), F\}$. The transition diagram is shown in Fig. 1. From Example 1(a) it is clear that $F = \{(01), (20), (10), (00)\}$.

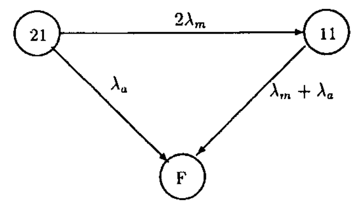

Fig. 1. Structure state process of Example 1(c).

Suppose our scheduling policy is such that we produce both products 1 and 2 in state (21) i.e., when both machines are up and we produce 1 only in (11). The infinitesimal generator of the SSP is given by

$$Q = \begin{bmatrix} -(2\lambda_m + \lambda_a) & 2\lambda_m & \lambda_a \\ 0 & -(\lambda_m + \lambda_a) & (\lambda_m + \lambda_a) \\ 0 & 0 & 0 \end{bmatrix}$$

and the reward vectors in each structure state are

$$\begin{aligned} \mathbf{r}_{21} &= [r_1 \quad r_2]; \\ \mathbf{r}_{11} &= [r_1 \quad 0]; \\ \mathbf{r}_F &= [0 \quad 0]. \end{aligned}$$

Assuming that the initial state of the system is (21), we have

$$\begin{aligned} E[X_1] &= r_1[2e^{-(\lambda_m+\lambda_a)t} - e^{-(2\lambda_m+\lambda_a)t}] \\ E[X_2] &= 2r_2[e^{-(\lambda_m+\lambda_a)t} - e^{-(-2\lambda_m+\lambda_a)t}] \\ E[Y_1(t)] &= r_1\left[\frac{3\lambda_m + \lambda_a}{(2\lambda_m + \lambda_a)(\lambda_m + \lambda_a)} - \frac{2}{(\lambda_m + \lambda_a)}\right. \\ &\qquad \left. \cdot\, e^{-(\lambda_m+\lambda_a)t} + \frac{1}{(2\lambda_m + \lambda_a)} e^{-(2\lambda_m+\lambda_a)t}\right] \\ E[Y_2(t)] &= r_2\left[\frac{2\lambda_m}{(\lambda_m + \lambda_a)(2\lambda_m + \lambda_a)} - \frac{2e^{-(\lambda_m+\lambda_a)t}}{(\lambda_m + \lambda_a)}\right. \\ &\qquad \left. + \frac{2e^{-(2\lambda_m+\lambda_a)t}}{(2\lambda_m + \lambda_a)}\right]. \end{aligned}$$

D. Distribution of $\mathbf{Y}(t)$

Variability is an inherent characteristic of manufacturing systems. Therefore, it is not sufficient to compute the mean accumulated reward $E[\mathbf{Y}(t)]$; one needs to evaluate higher moments of $\mathbf{Y}(t)$ and/or its distribution in order to ascertain the probability of meeting a target product mix. Our aim here is to characterize the distribution of $\mathbf{Y}(t)$, i.e., to compute $F_{\mathbf{Y}}(\mathbf{y}, t) = \text{Prob}\{\mathbf{Y}(t) \leq \mathbf{y}\}$, and its moments $E[Y_j^n(t)]$, $E[Y_j(t)Y_k(t)]$, for $n \geq 1$ and $j, k = 1, 2, \cdots, P$.

Consider an AMS producing P part types with structure state process described by $\{Z(u), u \geq 0\}$. The P-dimensional accumulated reward vector $\mathbf{Y}(t)$ is defined as earlier by

$$\mathbf{Y}(t) = \int_0^t \mathbf{r}_{z(u)}\, du$$

where $\mathbf{r}_i$ is the $1 \times P$ reward vector in the ith state. Now, consider the composite process $\{Z(u), \mathbf{Y}(u), u \geq 0\}$. The following lemma follows from the definition of a Markov process [20].

Lemma 1: The composite process $\{Z(u), \mathbf{Y}(u), u \geq 0\}$is Markov. More specifically,

$$\begin{aligned}\Pr\{Z(t) = j, \mathbf{Y}(t) \leq \mathbf{Y}_t \mid Z(s) = i,\\ \mathbf{Y}(s) = \mathbf{y}_s, \quad [Z(u), \mathbf{Y}(u), u \in [0, s)]\}\\ = \Pr\{Z(t) = j, \quad \mathbf{Y}(t) \leq \mathbf{y}_t \mid Z_s = i, \quad \mathbf{Y}(s) = \mathbf{y}_s\}\\ = \Pr\{Z(t) = j, \quad \mathbf{Y}(t) - \mathbf{Y}(s) \leq \mathbf{y}_t - \mathbf{y}_s \mid Z_s = i\}.\end{aligned}$$

Furthermore, the composite Markov process $\{Z(u), \mathbf{Y}(u), u \geq 0\}$ is homogeneous if $\{Z(u), \quad u \geq 0\}$ is homogeneous and the reward rates $\mathbf{r}_{z(u)}$ are time-invariant.

The evolution of the composite process $\{Z(u), \mathbf{Y}(u), u \geq 0\}$ is completely described by the $N \times N$ matrix $F(\mathbf{y}, t)$ defined by

$$F_{ij}(\mathbf{y}, t) = \Pr\{Z(t) = j, \quad \mathbf{Y}(t) \leq \mathbf{y} \mid Z(0) = i\}.$$

Evidently, $F_{ij}(\mathbf{y}, t)$ is the cumulative distribution function of $\{\mathbf{Y}(t) \leq \mathbf{y}, Z(t) = j\}$ given that the system starts in the initial structure state i. We note that since $r_{ij} \geq 0$, for any $y_j < 0$, $F(\mathbf{y}, t) = 0$ and also that $F(\mathbf{y}, t)$ has finite number of discontinuities and is differentiable. Let $W(\mathbf{y}, t) = [w_{ij}(\mathbf{y}, t)]$ be the matrix of joint density functions defined by

$$w_{ij}(\mathbf{y}, t) = \frac{\partial^P F_{ij}(\mathbf{y}, t)}{\partial y_1 \, \partial y_2 \cdots \partial y_P} = f_{Y(t), Z(t) \mid Z(0)}(\mathbf{y}, j \mid i).$$

Conversely it is true that

$$\begin{aligned}F_{ij}(\mathbf{y}, t) &= \int_{y_1} \int_{y_2} \cdots \int_{y_P} w_{ij}(\mathbf{y}, t) \, dy_1 \, dy_2 \cdots dy_p\\ &= \int_{y_1} \int_{y_2} \cdots \int_{y_P}\\ &\quad \cdot f_{\mathbf{Y}(t), Z(t) \mid Z(0)}(\mathbf{y}, j \mid i) \, dy_1 \, dy_2 \cdots dy_p.\end{aligned}$$

We can see that $w_{ij}(\mathbf{y}, t)$ and $f_{\mathbf{Y}(t), Z(t) \mid Z(0)}(\mathbf{y}, j \mid i)$ are the joint conditional density functions of $\{\mathbf{Y}(t), Z(t) = j\}$ given that $Z(0) = i$. We find it convenient to use both w_{ij} and f in the proofs of the Theorems.

Our aim here is to characterize the distribution of $\mathbf{Y}(t)$, i.e., to compute $F_{\mathbf{Y}}(t) = \text{Prob}\{\mathbf{Y}(t) \leq \mathbf{y}\}$. We can compute $F_{\mathbf{Y}}(t)$ from $F(\mathbf{y}, t)$ in the following way. We first define two N vectors $\boldsymbol{\alpha}(\mathbf{y}, t)$ and $\boldsymbol{\beta}(\mathbf{y}, t)$. The $1 \times N$ row vector $\boldsymbol{\alpha}(\mathbf{y}, t)$ is defined by

$$\boldsymbol{\alpha}(\mathbf{y}, t) = [\alpha_1(\mathbf{y}, t) \cdots \alpha_N(\mathbf{y}, t)]$$

where

$$\alpha_j(\mathbf{y}, t) = \Pr\{\mathbf{Y}(t) \leq \mathbf{y}, \, Z(t) = j\}; \quad j = 1, 2, \cdots, N.$$

It can be easily seen that

$$\alpha(\mathbf{y}, t) = \mathbf{p}(0) F(\mathbf{y}, t)$$

where $\mathbf{p}(0)$ is the initial probability vector $(1 \times N)$.

We define the column vector $\boldsymbol{\beta}(\mathbf{y}, t)$as

$$\begin{aligned}\boldsymbol{\beta}(\mathbf{y}, t) &= [\beta_1(\mathbf{y}, t), \cdots, \beta_N(\mathbf{y}, t)]^T\\ \beta_i(\mathbf{y}, t) &= \text{Prob}\{\mathbf{Y}(t) \leq \mathbf{y} \mid Z(0) = i\}.\end{aligned}$$

It can be easily seen that

$$\boldsymbol{\beta}(\mathbf{y}, t) = F(\mathbf{y}, t)\mathbf{e}.$$

Finally, we have

$$\begin{aligned}F_{\mathbf{Y}}(t) = \text{Prob}\{\mathbf{Y}(t) \leq \mathbf{y}\} &= \mathbf{p}(0) F(\mathbf{y}, t)\mathbf{e}\\ = \boldsymbol{\alpha}(\mathbf{y}, t)\mathbf{e} \qquad &= \mathbf{p}(0)\boldsymbol{\beta}(\mathbf{y}, t).\end{aligned}$$

We now proceed to prove several results useful in computing $F_{\mathbf{Y}}(t)$.

Lemma 2: The Matrix Chapman–Kolmogorov equation of the composite process $\{Z(u), \mathbf{Y}(u), \quad u \geq 0\}$ is given by

$$W(\mathbf{y}, t) = \int_{\tilde{\mathbf{y}}} W(\tilde{\mathbf{y}}, \sigma) W(\mathbf{y} - \tilde{\mathbf{y}}, t - \sigma) \, d\tilde{\mathbf{y}}.$$

Proof: From the Total probability Theorem, we have

$$\begin{aligned}w_{ij}(\mathbf{y}, t) &= f_{\mathbf{Y}(t), Z(t) \mid Z(0)}(\mathbf{y}, j \mid i)\\ &= \int_{\tilde{\mathbf{y}}} \sum_{k=1}^{N} f_{\mathbf{Y}(t), Z(t), \mathbf{Y}(\sigma), Z(\sigma) \mid Z(0)}(\mathbf{y}, j, \tilde{\mathbf{y}}, k \mid i) d\tilde{\mathbf{y}}\\ &= \int_{\tilde{\mathbf{y}}} \sum_{k=1}^{N} f_{\mathbf{Y}(t), Z(t) \mid \mathbf{Y}(\sigma), Z(\sigma), Z(0)}(\mathbf{y}, j \mid \tilde{\mathbf{y}}, k, i)\\ &\quad \cdot f_{\mathbf{Y}(\sigma), Z(\sigma) \mid Z(0)}(\tilde{\mathbf{y}}, k \mid i)\\ &= \int_{\tilde{\mathbf{y}}} \sum_{k=1}^{N} f_{\mathbf{Y}(t) - \mathbf{Y}(\sigma), Z(t) \mid Z(\sigma)}(\mathbf{y} - \tilde{\mathbf{y}}, j \mid k)\\ &\quad \cdot f_{\mathbf{Y}(\sigma), Z(\sigma) \mid Z(0)}(\tilde{\mathbf{y}}, k \mid i)\\ &= \int_{\tilde{\mathbf{y}}} \sum_{k=1}^{N} w_{ik}(\tilde{\mathbf{y}}, \sigma) w_{kj}(\mathbf{y} - \tilde{\mathbf{y}}, t - \sigma) \, d\tilde{\mathbf{y}}.\end{aligned}$$

□

Lemma 3:

$$\begin{aligned}F_{ij}(\mathbf{y}, t) &= \left[\prod_{m=1}^{P} U(y_m - r_{im} t)\right] e^{-\lambda_i t} \delta_{ij}\\ &\quad + \int_0^t \lambda_i e^{-\lambda_i \sigma} \sum_{k=1}^{N} P_{ik} F_{kj}(\mathbf{y} - \mathbf{r}_i \sigma, t - \sigma) \, d\sigma\end{aligned}$$

where $U(x) = 1$ if $x \geq 0$ and zero otherwise.

Proof: Suppose the system starts at time 0 in state i and is in state j at time t. Further, let $\mathbf{Y}(t) \leq \mathbf{y}$. This can happen in two ways:

1) System is in state i during $[0, t]$ earning a reward $r_{im} t$; $m = 1, 2, \cdots, P$ during $[0, t]$. We have the probability that the system stays in i during $[0, t]$ as

$$\text{Prob}\{Z(\tau) = i, \quad \tau \in [0, t] \mid Z(0) = i\} = e^{-\lambda_i t}.$$

Also

$$\text{Prob}\{\mathbf{Y}(t) \leq \mathbf{y} \mid Z(\tau) = i, \quad \tau \in [0, t]\} = \prod_{m=1}^{P} U(y_m - r_{im} t).$$

Combining the above two equations, we get

$$\text{Prob}\{\mathbf{Y}(t) \leq \mathbf{y},\ Z(\tau) = i, \quad \tau \in [0,\, t] \,|\, Z(0) = i\} = \prod_j U_j(y_j - r_{ij}t)e^{-\lambda_i t}.$$

2) System is in state i during $[0,\, \sigma)$ and transits to state k at time σ. In this case, we have

$$\Pr\{Z(\tau) = i, \quad \tau \in [0,\sigma) \quad \text{and} \quad Z(t) = k \,|\, Z(0) = i,\ t \in [\sigma,\, \sigma + d\sigma)\} = \lambda_i e^{-\lambda_i \sigma} P_{ik}\, d\sigma.$$

In addition,

$$\begin{aligned}\Pr\{\mathbf{Y}(t) \leq \mathbf{y}, Z(t) = j \,|\, Z(\tau) = i,\ &\tau \in [0,\sigma);\quad Z(\sigma) = k\}\\ &= \Pr\{Z(t) = j,\ \mathbf{Y}(t) - \mathbf{Y}(\sigma) \leq (\mathbf{y} - \mathbf{r}_i\sigma) \,|\, Z(\sigma) = k\}\\ &= F_{kj}(\mathbf{y} - \mathbf{r}_i\sigma,\, t).\end{aligned}$$

Since k can take any value between 1 and N and σ between $[0,\, t]$, we average the probability over $[0,\, t]$ and sum over 1 to N.
The result is obtained by realizing that $F_{ij}(\mathbf{y},\, t)$ is the sum of the two terms in (i) and (ii). □

E. Partial Differential Equations for $F(\mathbf{y},\, t)$

We now derive the forward and adjoint partial differential equations (PDE's) that characterize the evolution of the composite process $\{Z(u),\, \mathbf{Y}(u), \quad u \geq 0\}$.

Theorem 1: The joint distribution matrix $F(\mathbf{y},\, t)$ satisfies the following linear hyperbolic partial differential equation

$$\begin{aligned}\frac{\partial F(\mathbf{y},\, t)}{\partial t} &= -\sum_{j=1}^{P} \frac{\partial F(\mathbf{y},\, t)}{\partial y_j} R_j + F(\mathbf{y},\, t)Q;\\ F(\mathbf{0},\, t) &= 0, \quad t \geq 0\end{aligned}$$

and

$$F(\mathbf{y}, 0) = \prod_{p=1}^{P} U(y_p) I_N$$

where R_j is a diagonal matrix with elements $r_{1j},\, r_{2j},\, \cdots,\, r_{Nj}$ and I_N is a $N \times N$ identity matrix.

Proof: See Appendix A.

Theorem 2: The matrix $F(\mathbf{y},\, t)$ satisfies the adjoint partial differential equation

$$\begin{aligned}\frac{\partial F(\mathbf{y},\, t)}{\partial t} &= -\sum_{j=1}^{P} R_j \frac{\partial F(\mathbf{y},\, t)}{\partial \mathbf{y}_j} + QF(\mathbf{y},\, t);\\ F(\mathbf{0},\, t) &= 0;\\ F(\mathbf{y}, 0) &= \prod_{p=1}^{P} U(y_p) I_N.\end{aligned}$$

Proof: Can be obtained along the same lines as Theorem 1.

Corollary 1: For non-repairable systems, Q is lower triangular. Consequently, it is true that $F(\mathbf{y},\, t)$ is also lower triangular.

Corollary 2: $\boldsymbol{\alpha}(\mathbf{y},\, t)$ satisfies the following PDE

$$\begin{aligned}\frac{\boldsymbol{\alpha}(\mathbf{y},\, t)}{\partial t} &= -\sum_{j=1}^{P} \frac{\partial \boldsymbol{\alpha}(\mathbf{y},\, t)}{\partial \mathbf{y}_j} R_j + \boldsymbol{\alpha}(\mathbf{y},\, t)Q;\\ \boldsymbol{\alpha}(\mathbf{0},\, t) &= 0;\\ \boldsymbol{\alpha}(\mathbf{y}, 0) &= \mathbf{p}(0)\left[\prod_{p=1}^{P} U(y_p)\right] I_N.\end{aligned}$$

Proof: Premultiplying PDE in Theorem 1 by $\mathbf{p}(0)$ the result follows. □

As we noted earlier $F_{\mathbf{Y}}(\mathbf{y},\, t) = \boldsymbol{\alpha}(\mathbf{y},\, t)\mathbf{e}$. If we are given the initial probability distribution $\mathbf{p}(0)$, then by solving $\boldsymbol{\alpha}$ for various "t," we can compute the distribution of cumulative production at various times.

Corollary 3: $\boldsymbol{\beta}(\mathbf{y},\, t)$ satisfies the adjoint PDE

$$\begin{aligned}\frac{\partial \boldsymbol{\beta}(\mathbf{y},\, t)}{\partial t} &= -\sum_{j=1}^{P} R_j \frac{\partial \boldsymbol{\beta}(\mathbf{y},\, t)}{\partial \mathbf{y}_j} + Q\boldsymbol{\beta}(\mathbf{y},\, t);\\ \boldsymbol{\beta}(\mathbf{0},\, t) &= 0;\\ \boldsymbol{\beta}(\mathbf{y}, 0) &= \left[\prod_{p=1}^{P} U(y_p)\right] \mathbf{e}.\end{aligned}$$

Once $\boldsymbol{\beta}(\mathbf{y},\, t)$ is computed, we can determine $F_{\mathbf{Y}}(\mathbf{y},\, t)$ by using the fact that $F_{\mathbf{Y}}(t) = \mathbf{p}(0)\boldsymbol{\beta}(\mathbf{y},\, t)$. We can obtain $F_{\mathbf{Y}}(\mathbf{y},\, t)$ by solving $\boldsymbol{\beta}(\mathbf{y},\, t)$, if it is desired to compute $F_{\mathbf{Y}}(\mathbf{y},\, t)$ for various initial configurations.

Theorem 3: Suppose we associate the variable $s,\, \mu_j, \quad j = 1,\, 2,\, \cdots,\, P$ with $t,\, y_j, \quad j = 1,\, 2,\, \cdots,\, P$, then the multidimensional Laplace transforms of $W(\mathbf{y},\, t)$, $F(\mathbf{y},\, t)$, $\boldsymbol{\alpha}(\mathbf{y},\, t)$ and $\boldsymbol{\beta}(\mathbf{y},\, t)$ are given by

$$\begin{aligned}W(s,\, \boldsymbol{\mu}) &= \left[sI + \sum_{j=1}^{P} \mu_j R_j - Q\right]^{-1}\\ F(s,\, \boldsymbol{\mu}) &= \frac{\left[sI + \sum_{j=1}^{P} \mu_j R_j - Q\right]^{-1}}{\left(\prod_{j=1}^{P} \mu_j\right)}\\ \boldsymbol{\alpha}(s,\, \boldsymbol{\mu}) &= \mathbf{p}(0)F(s,\, \boldsymbol{\mu})\\ \boldsymbol{\beta}(s,\, \boldsymbol{\mu}) &= F(s,\, \boldsymbol{\mu})\mathbf{e}.\end{aligned}$$

F. Moment Recursions

We have derived in the previous section the PDE's satisfied by $F(\mathbf{y},\, t)$. We now derive recursive ordinary differential equations describing the evolution of the moments of Performability. Specifically, the determination of nth moment of performability will be in terms of $(n-1)$th moment. Formally, let $\mathbf{m}_n^j(t)$ be the column vector denoting the nth conditional moment vector for part type j, defined by

$$\begin{aligned} m^j_{n,i}(t) &= E[\mathbf{Y}^n_j(t) \mid Z(0) = i]; \quad i = 1, 2, \cdots, N; \\ &\qquad j = 1, 2, \cdots, P \\ &= \int_0^\infty y^n_j W(\mathbf{y}, t)\mathbf{e}\, dy_j; \quad j = 1, 2, \cdots, P \\ \mathbf{m}^j_n(t) &= [m^j_{n,1}, m_{n,2}, \cdots, m^j_{n,N}]. \end{aligned}$$

It is clear that $\mathbf{m}^j_0(t) = \mathbf{e}$; $\mathbf{m}^j_n(0) = \mathbf{0}$.

Also the nth moment of cumulative performability for part type j is given by $E[(\mathbf{Y}(t))^n] = \mathbf{p}(0)\mathbf{m}^j_n(t)$

Theorem 4: The moment vector $\mathbf{m}^j_n(t)$, $n \geq 1$, $j = 1, 2, \cdots, P$ evolves according to the equation

$$\frac{d\mathbf{m}^j_n(t)}{dt} = Q\mathbf{m}^j_n(t) + nR_j\mathbf{m}^j_{n-1}(t); \quad \mathbf{m}^j_0(t) = \mathbf{e}$$

Proof: This can be easily proved by first noting that

$$\mathbf{M}^j_n(s) = (-1)^n \frac{\partial^n}{\partial \boldsymbol{\mu}^n_j} \cdot \left(sI + \sum_{i=1}^{P} \boldsymbol{\mu}_i R_i - Q \right)^{-1} \mathbf{e} \Bigg|_{\substack{\boldsymbol{\mu}_i = 0 \\ i = 1, 2, \cdots, P}}$$

where $\mathbf{M}^j_n(s)$ is the Laplace transform of $\mathbf{m}^j_n(t)$.

Using the fact that $(sI - Q)^{-1}\mathbf{e} = (1/s)\mathbf{e}$, we obtain

$$\mathbf{M}^j_n(s) = \frac{n!}{s} [(sI - Q)^{-1}R_j]^n \mathbf{e}$$

which implies the recursion of the Theorem. □

We can also obtain the conditional variance for part type j, $j = 1, 2, \cdots, P$ and for each initial state i, $i = 1, 2, \cdots, N$ denoted by

$$\sum_{i,i}^{jj} = m^j_{2,i}(t) - [m^j_{1,i}(t)]^2; \quad i = 1, 2, \cdots, N; \quad j = 1, 2, \cdots, P$$

Theorem 5: The pairwise joint conditional moments of part types $j$and k, given that the system is in state i initially, denoted by $m^{jk}_{2,i}(t)$ and defined by

$$m^{jk}_{2,i}(t) = E[Y_j Y_k \mid Z(0) = i]; \quad i = 1, 2, \cdots, N$$

satisfy the following equations

$$\begin{aligned} \frac{d\mathbf{m}^{jk}_2(t)}{dt} &= Q\mathbf{m}^{jk}_2(t) + R_j\mathbf{m}^k_1(t) + R_k\mathbf{m}^j_1(t); \\ \mathbf{m}^{jk}_2(0) &= \mathbf{0} \end{aligned}$$

where $m^{jk}_2 = [m^{jk}_{2,1} \cdots m^{jk}_{2,N}]^T$.

Proof: The result can be proved by differentiating $W(s, \boldsymbol{\mu})$ w.r.t. μ_1 and μ_2 and setting $\mu_1 = \mu_2 = \cdots = \mu_P = 0$ and simplifying the resulting expressions. □

III. Illustrative Examples

Example 2: Consider an NC machine producing two part types, i.e., $P = 2$. The machine can be in two states: UP(1) or Down(0). When the machine is up, it produces simultaneously both the part-types at a rate r_1/hr and r_2/hr. The system is initially working and failure rate is λ, i.e., $\mathbf{p}(0) = [0\ 1]$. We have

$$\begin{aligned} \text{Prob}\{\mathbf{Y}(t) \leq y\} &= \mathbf{p}(0)F(\mathbf{y}, t)\mathbf{e} \\ &= F_{21}(\mathbf{y}, t) + F_{22}(\mathbf{y}, t). \end{aligned}$$

Also the Q matrix is given by

$$Q = \begin{bmatrix} 0 & 0 \\ \lambda & -\lambda \end{bmatrix}.$$

From Theorem 1, we have

$$\begin{aligned} \frac{\partial F_{22}}{\partial t} &= -\frac{\partial F_{22}}{\partial y_1} r_1 - \frac{\partial F_{22}}{\partial y_2} r_2 - \lambda F_{22} \\ \frac{\partial F_{21}}{\partial t} &= \lambda F_{22}. \end{aligned}$$

The above equations yield

$$\begin{aligned} F_{22}(y_1, y_2, t) &= e^{-\lambda t}U(y_1 - r_1 t)U(y_2 - r_2 t) \\ &= \lambda \int_0^t e^{-\lambda t}U(y_1 - r_1 t)U(y_2 - r_2 t)\, dt \\ &= \lambda \int_0^{\min[t, (y_1/r_1), (y_2/r_2)]} e^{-\lambda t}\, dt \\ F_{21}(y_1, y_2, t) &= 1 - e^{-\lambda \min[t, (y_1/r_1), (y_2/r_2)]}. \end{aligned}$$

Hence,

$$\begin{aligned} P(Y_1 \leq y_1, Y_2 \leq y_2, t) =& 1 - e^{-\lambda \min[t, (y_1/r_1), (y_2/r_2)]} \\ &+ e^{-\lambda t}U(y_1 - r_1 t)U(y_2 - r_2 t). \end{aligned}$$

The marginal distributions of Y_1 and Y_2 are given by

$$\begin{aligned} P(Y_1 \leq y_1, t) &= 1 - e^{-\lambda \min[t, (y_1/r_1)]} + e^{-\lambda t}U(y_1 - r_1 t) \\ P(Y_2 \leq y_2, t) &= 1 - e^{-\lambda \min[t, (y_2/r_2)]} + e^{-\lambda t}U(y_2 - r_2 t). \end{aligned}$$

We can also compute the moments as follows

$$\begin{aligned} m^1_1(t) &= [m^1_{1,1} \quad m^1_{1,2}]^T \\ \frac{dm^1_1(t)}{dt} &= Qm^1_1(t) + R_1\mathbf{e}; \\ m^1_1(0) &= \mathbf{0}. \end{aligned}$$

We have $m^1_{1,1}(t) = 0$, $m^1_{1,2}(t) = r_1/\lambda(1 - e^{-\lambda t})$.

Similarly we have

$$\begin{aligned} m^2_{1,1}(t) &= 0, \\ m^2_{1,2}(t) &= \frac{r_2}{\lambda}(1 - e^{-\lambda t}). \end{aligned}$$

Also for the second moment we have

$$\begin{aligned} m^j_{2,1} &= 0; \quad j = 1, 2 \\ m^1_{2,2} &= \frac{2r_1^2}{\lambda^2}(1 - e^{-\lambda t} - \lambda t e^{-\lambda t}) \\ m^2_{2,2} &= \frac{2r_2^2}{\lambda^2}(1 - e^{-\lambda t} - \lambda t e^{-\lambda t}) \\ \sigma_1^2 &= m^1_{2,2} - (m^1_{1,2})^2 \\ &= \frac{r_1^2}{\lambda^2}(1 - e^{-2\lambda t} - 2\lambda t e^{-\lambda t}) \end{aligned}$$

TABLE I
PROCESSING TIMES FOR PART TYPES

Part Type (No. of fixtures)	Mean machining time in minutes	
	Operation 1	Operation 2
P1(3)	10	-
P2(3)	-	8
P3(4)	5	10

TABLE II
REWARDS ASSOCIATED WITH PART TYPES

State	(11)	(10)	(01)	(00)
Part Type	Reward (parts per hour)			
P1	4.5961	6	0	0
P2	3.9903	0	7.5	0
P3	2.8078	0	0	0

and

$$\sigma_2^2 = \frac{r_2^2}{\lambda^2}(1 - e^{-2\lambda t} - 2\lambda t e^{-\lambda t}).$$

Let us now turn our attention to the second moments $m_{1,1}^{12}(t)$ and $m_{1,2}^{12}(t)$. From the differential equation in Theorem 5, we can show that $m_{1,1}^{12}(t) = 0$ and

$$\frac{dm_{1,2}^{12}(t)}{dt} = -\lambda m_{1,2}^{12}(t) + 2r_1 r_2 \frac{(1 - e^{-\lambda t})}{\lambda}.$$

Solving the above equation, we get

$$\begin{aligned} m_{1,2}^{12}(t) &= \frac{2r_1 r_2}{\lambda^2}(1 - e^{-\lambda t} - \lambda t e^{-\lambda t}) \\ \mathrm{Cov}\,(1, 2) &= m_{1,2}^{12} - m_{1,2}^{1} m_{1,2}^{2} \\ &= \frac{r_1 r_2}{\lambda^2}(1 - 2\lambda t e^{-\lambda t} - e^{-2\lambda t}). \end{aligned}$$

The correlation coefficient $\rho(1, 2)$ is

$$\rho(1, 2) = \frac{\mathrm{Cov}\,(1, 2)}{\sigma_1 \sigma_2} = 1.$$

Hence the products 1 and 2 are linearly correlated.

Example 1(d): Consider the flexible cell in Example 1(a) manufacturing two products A and B. Let λ_a and λ_m be the exponential failure rates of AGV and machines. The state diagram is shown in Fig. 1. We use the result of Corollary 3 to determine the distribution of performability. We note that $R_1 = \mathrm{diag}\,\{r_A, r_B, 0\}$ and $R_2 = \mathrm{diag}\,\{r_A, 0, 0\}$.

We have the following equations from the multi-dimensional Laplace Transform equations

$$\boldsymbol{\beta}(\mu_1, \mu_2, s) = \frac{[sI + \mu_1 R_1 + \mu_2 R_2 - Q]^{-1}\mathbf{e}}{\mu_1 \mu_2}.$$

Let $q_1(s) = [\mu_1 \mu_2 (s + \mu_1 r_A + \lambda_m + \lambda_a)]^{-1}$, and $q_2(s) = [s + \mu_1 r_A + \mu_2 r_B + 2\lambda_m + \lambda_a]^{-1}$. Expanding the above equation

$$\begin{aligned} \beta_3(\mu_1, \mu_2, s) &= \frac{1}{\mu_1 \mu_2 s} \\ \beta_2(\mu_1, \mu_2, s) &= q_1(s)\left[1 + \frac{(\lambda_m + \lambda_a)}{s}\right] \\ \beta_1(\mu_1, \mu_2, s) &= q_2(s)\left[\frac{2\lambda_m (s + \lambda_m + \lambda_a) q_1(s)}{s}\right. \\ &\qquad \left. + \frac{\lambda_a}{\mu_1 \mu_2 s} + \frac{1}{\mu_1 \mu_2}\right]. \end{aligned}$$

Time domain results can be obtained by inversion.

Example 3(a): Consider a flexible cell with two machines M1 and M2 and producing three part types P1, P2, and P3. The machines are assumed to be failure prone and no repair is possible. Table I gives the mean processing requirements on each machine for each part type and these times are assumed to be exponentially distributed.

We assume that the failure rates of machines M1 and M2 are λ_1 and λ_2 respectively. The structure state process S of the underlying reliability model containing four states given by

$$S = \{(11), (10), (01), (00)\}$$

where the tuple (i, j) denotes the state of the machines M1 and M2 respectively. The state of machine is represented by 1 if it is up and 0 if it is down. In each of these structure states, the performance model is solved using MVA analysis of the underlying queuing model [12]. Table II summarizes the rewards for each part type in each of the structure states.

The composite performability model is solved analytically and the equations for the mean and second moments of the performability distribution are obtained.

$$\begin{aligned} m_1^1(t) &= \frac{4.5961}{(\lambda_1 + \lambda_2)}[1 - e^{-(\lambda_1 + \lambda_2)t}] + \frac{6.0}{\lambda_1(\lambda_1 + \lambda_2)} \\ &\quad \cdot [\lambda_2 - (\lambda_1 + \lambda_2)e^{-\lambda_1 t} + \lambda_1 e^{(\lambda_1 + \lambda_2)t}] \\ m_1^2(t) &= \frac{3.5961}{(\lambda_1 + \lambda_2)}[1 - e^{-(\lambda_1 + \lambda_2)t}] + \frac{7.5}{\lambda_2(\lambda_1 + \lambda_2)} \\ &\quad \cdot [\lambda_1 - (\lambda_1 + \lambda_2)e^{-\lambda_2 t} + \lambda_2 e^{(\lambda_1 + \lambda_2)t}] \\ m_1^3(t) &= \frac{2.8078}{(\lambda_1 + \lambda_2)}[1 - e^{-(\lambda_1 + \lambda_2)t}]. \end{aligned}$$

Similar expressions can be obtained for the second moments and cross moments analytically. To study the variation of the correlation coefficient, we have plotted the behavior of this function with respect to various parameters. Fig. 2 plots the variation of the correlation coefficient between part types as a function of time for $\lambda_1 = 0.02$ and $\lambda_2 = 0.01$.

Typically, correlation is an expression of relationship. Perfect correlation is indicated by a correlation coefficient 1.0; no relationship between phenomena is indicated by a correlation coefficient of 0; and a perfect negative relationship (in the sense that one variable increases as the other decreases) is indicated by a correlation coefficient of -1.0 [21]. In Fig. 2, we observe that the correlation between part types 1–3 and 2–3 is generally positive, except at the beginning where the correlation between 2–3 is negative. This can attributed to the fact that when parts of type 3 are produced, both the machines M1 and M2 are up and they also produce parts P1 and P2. Fig. 3 plots the variation of the correlation

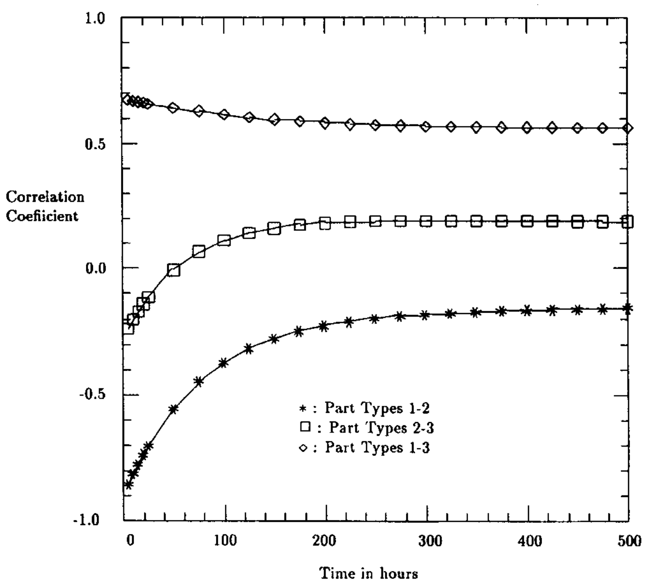

Fig. 2. Variation of correlation coefficient with time $\lambda_1 = 0.02, \lambda_2 = 0.01$.

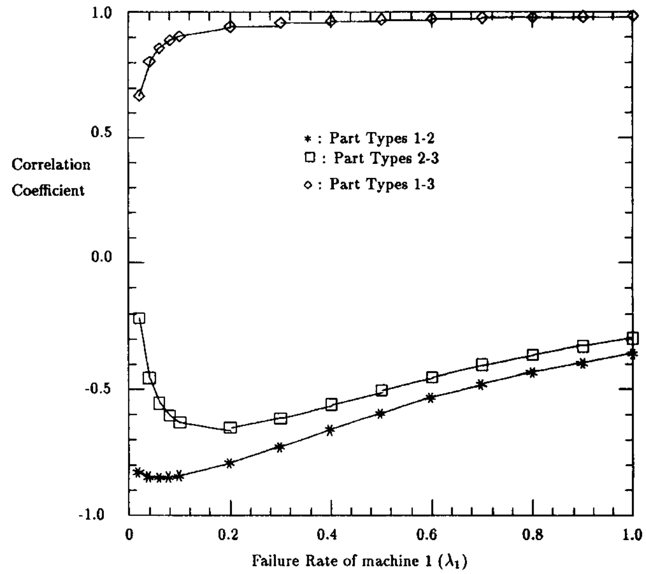

Fig. 4. Variation of correlation coefficient with failure rate λ_1 at $t = 8$ h.

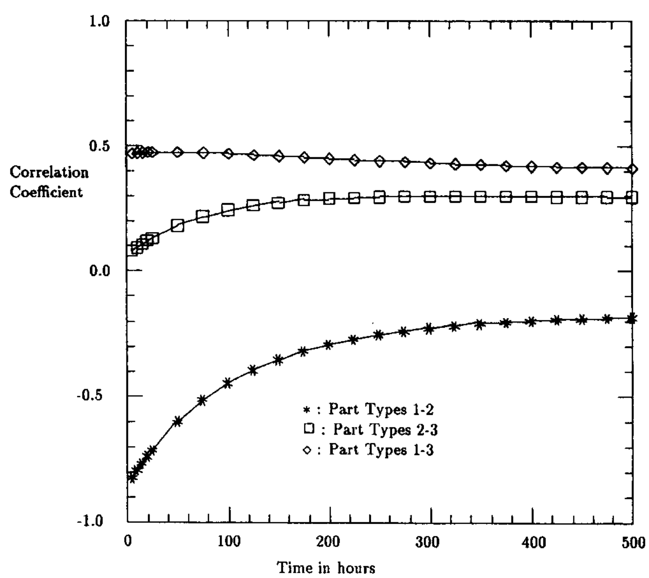

Fig. 3. Variation of correlation coefficient with time $\lambda_1 = \lambda_2 = 0.01$.

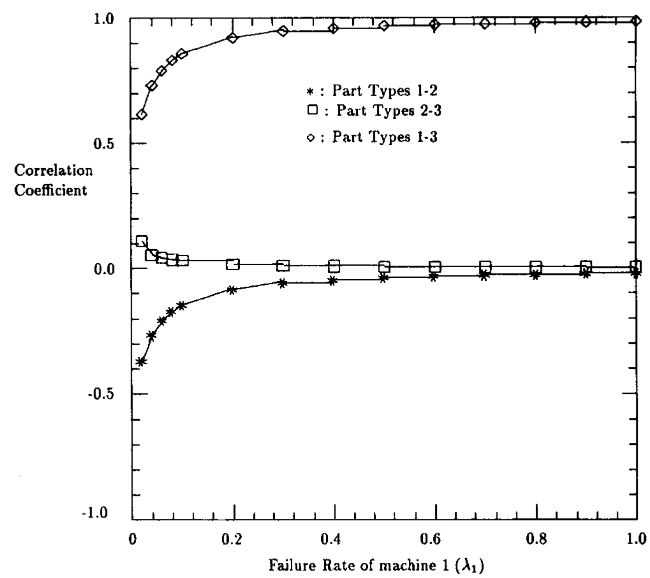

Fig. 5. Variation of correlation coefficient with failure rate λ_1 at $t = 100$ h.

coefficient with time for $\lambda_1 = \lambda_2 = 0.01$. We observe that the correlation between part types 1–3 has decreased and the correlation between part types 2–3 has increased with the reduction of failure rate of machine 1. Figs. 4 and 5 plot the variation of correlation coefficient at 8 and 100 h of production as a function of failure rate λ_1. In both cases the correlation between part types 1–3 is very close to 1 for higher λ_1, showing a definite failure of machine 1 and thus lower production of both part types 1 and 3. The correlation between part types 1–2 and 2–3 approaches 0 in case of 100-h production, due to a definite trend of lower production of part types 1 and 2 and relatively higher production of part 2.

IV. Distribution of Cumulative Operational Time

The goal of traditional reliability analysis is to find the distribution of availability or cumulative operation time. This measure is important in its own right during the system design phase to assess the influence of failure and repair rates on system availability. Here we are concerned with computing the distribution of cumulative operation time for systems producing multiple part types. The results in this section are thus a generalization of the single part type results presented in [7]. We use the results from [18] to compute recursively the distribution of $O_j(t)$, the cumulative operation time for part type j.

A. Numerical Methods

In Section II-D, we defined the column vector $\boldsymbol{\beta}(o_j, t)$ as

$$\boldsymbol{\beta}(o_j, t) = [\beta_1(o_j, t), \cdots, \beta_N(o_j, t)]^T$$
$$\beta_i(o_j, t) = \text{Prob}\{\mathbf{O}_j(t) \le o_j \mid Z(0) = i\}.$$

Let, for a given part type j,

$$\beta_i(o_j, t; n) = \Pr\{\mathbf{O}_j(t) \le o_j; \; n \text{ transitions in } (0, t) \mid z(0) = i\}; \quad i = 1, 2, \cdots, N.$$

Then,

$$\boldsymbol{\beta}(o_j, t) = \sum_{n=0}^{\infty} \boldsymbol{\beta}(o_j, t; n).$$

We state the following theorem in the case of availability computation.

Theorem 6: For an availability problem and for a given part type j, $\beta_i(o_j, t; n)$ can be recursively computed as

$$\beta_i(o_j, t; n) = \begin{cases} U(o_j - t) + \sum_{n=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^n}{n!} \left[\sum_{k=1}^{n} \binom{n}{k} \left(\frac{o_j}{t}\right)^k \cdot \left(1 - \frac{o_j}{t}\right)^{n-k} d_{n,i,j}(k-1)\right] \cdot U(t - o_j); \; i \in S_{0,j} \\ U(o_j - t) + \sum_{n=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^n}{n!} \left[\sum_{k=0}^{n} \binom{n}{k} \left(\frac{o_j}{t}\right)^k \cdot \left(1 - \frac{o_j}{t}\right)^{n-k} d_{n,i,j}(k)\right] \cdot U(t - o_j); \quad i \in S_{F,j} \end{cases}$$

where $d_{n,i,j}(k)$ are given by the recurrence equations

$$\begin{aligned} d_{n,i,j}(0) &= \sum_{p \in S_{F,j}} P_{ij} d_{n-1,p,j}(0), \\ d_{n,i,j}(k) &= \sum_{p \in S_{0,j}} P_{ij} d_{n-1,p,j}(k-1) \\ &\quad + \sum_{j \in S_{F,j}} P_{ij} d_{n-1,p,j}(k); \quad k \ne n, 0, \\ d_{n,i,j}(n) &= \sum_{p \in S_{0,j}} P_{ij} d_{n-1,p,j}(n-1) \\ &\quad + \sum_{j \in S_{F,j}} P_{ij} d_{n-1,p,j}(n-1) \\ &= 1. \end{aligned}$$

Proof: See [18].

Corollary 4: For an availability problem, the kth moment of the cumulative availability distribution for a given part type j, is given by

$$m_{k,i}^{j}(t) = \int_0^{\infty} k o_j^{k-1}[1 - \beta_i(o_j, t)]\, do_j = t^k \left\{1 - \sum_{n=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^n}{n!} \cdot \left[\sum_{l=0}^{n} \frac{\binom{k+l-1}{l}}{\binom{n+k}{k}} d_{n,i,j}(l)\right]\right\}.$$

TABLE III
AVERAGE MACHINING TIME IN MTS

Part Type	*M1*	*M2*
Part P1	10	-
Part P2	-	8
Part P3	5	10

B. Examples

Example 3(b): Consider a flexible manufacturing cell with two identical machines M1 and M2 producing three part types P1, P2 and P3. Part P1 is produced on machine M1 and part P2 on M2. Part P3 however requires both M1 and M2. The average processing times required for each part type are given in Table III.

We assume that both the machines are prone to failure and the failure rates are assumed to be 0.02 and 0.01, respectively for M1 and M2. We consider two alternate schemes of operation. Under scheme A, once a machine fails no repair is possible. In scheme B, we assume the availability of a single repair facility and the failed machine can be repaired at a rate of 0.5 per hour. Also we assume that the repair is undertaken on a nonpreemptive basis and the repair of machine M2 has higher priority over that of M1.

The availability model of the above cell is a continuous time Markov chain and the distribution of cumulative operational time can be obtained using the results of Theorem above. Fig. 6 shows the distribution of cumulative operational time $O_j(t)$ for various part types (j) under scheme A, over a period of 100 hours of operation of the cell. It can be seen from the distribution function that $\Pr[O_3(t) > 80] = 0.1$ in contrast to $\Pr[O_2(t) > 80] = 0.45$. Fig. 7 shows the same distributions under scheme B, when repair is possible. In this case the above figures are very close to 0.99 for both part types 2 and 3.

Example 4(a): We consider a cellular manufacturing system, consisting of identical cells 1 and 2, linked by a conveyor. Each cell contains two identical machines and a robot for material handling. The cell is operational if conveyor, robot and either of the machines in the cell are up. The overall system is operational if either of the cells is operational. Two part types 1 and 2 are produced in this system. Part type 1 is produced in cell 1 and requires 0.167 h of average processing time. Part type 2 is produced in cell 2 and requires 0.25 h of average processing time. We assume that all the subsystems are prone to failure and the failure and repair times are assumed to be exponentially distributed. A pool of one or

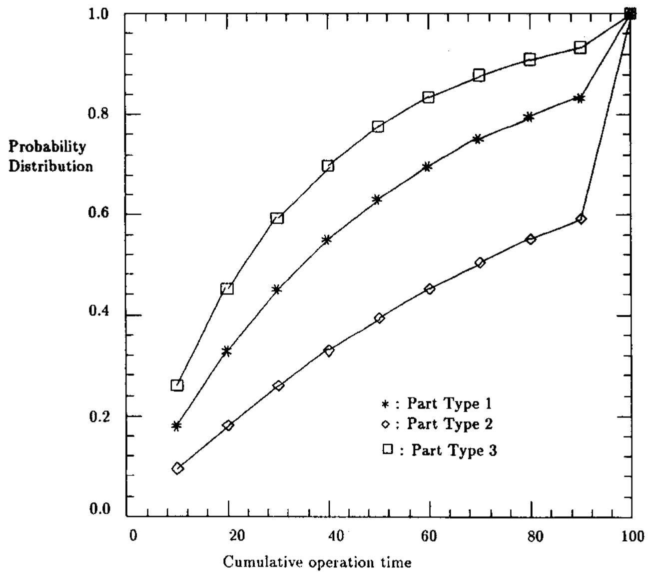

Fig. 6. Distribution of cumulative operation time for cell with repair.

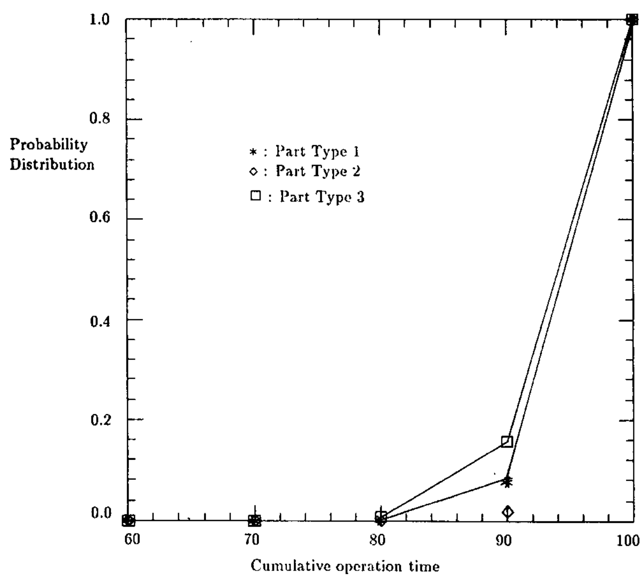

Fig. 7. Distribution of cumulative operation time for cell with repair.

TABLE IV
FAILURE AND REPAIR TIMES OF THE SUBSYSTEMS OF CMS

Equipment	MHS	Cell 1			Cell 2		
	Conveyor	R1	M_1	M_2	R2	M_1	M_2
MTTF	480	96	72	72	64	72	72
MTTR	40	15	12	12	15	12	12

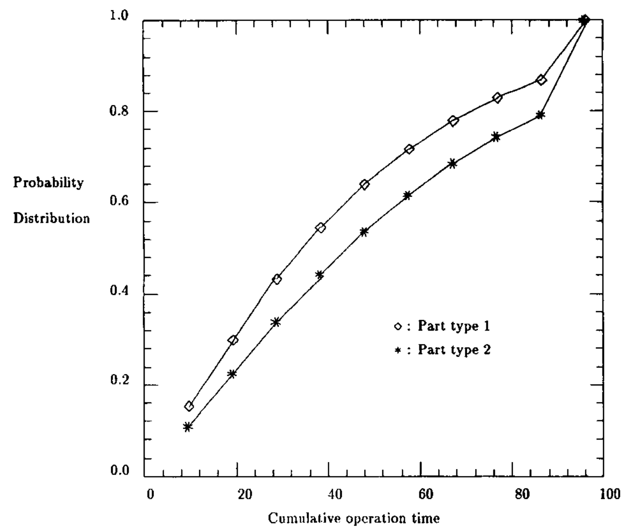

Fig. 8. Distribution of cumulative operation time for CMS without repair.

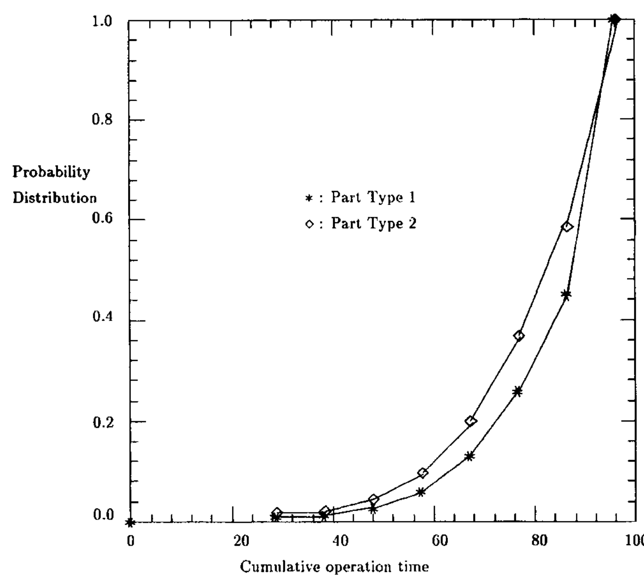

Fig. 9. Distribution of cumulative operation time for CMS with repair.

more equally capable repairmen is assumed to be available to repair the failed subsystems. A repairman can work on only one failed subsystem at a time, and a failed subsystem can use the services of only one repairman. When repairman has a choice in taking up the equipment for repair, he employs the following *nonpreemptive priority* rule.

- MHS: priority level 3 (highest priority)
- Robots: priority level 2
- Other machines: priority level 1 (lowest priority)

Among failed units with same priority level, each failed unit is chosen with equal probability. When number of repairmen is zero, this corresponds to a system without repair.

The failure and repair times (in hours) of the individual components of the system are given in Table IV.

Figs. 8 and 9 show the distributions of cumulative operational time of the cellular manufacturing system for part types 1 and 2 over an operational period of 96 h, with number of repairmen 0 and 3, respectively.

The effect of repair on the availability of the system is evident from the graphs. Consider the values of $\Pr(O_j(t) > 80)$ for both part types with and without repair. The probability is higher for part type 1 in both cases as the cell 1, where part type 1 is manufactured, is more reliable.

V. MOMENTS OF PERFORMABILITY

A. Numerical Methods

Although the performability function satisfies nice partial differential equation, obtaining stable numerical solution is a difficult task. In this section, we concentrate on developing numerical methods based on uniformization for solving the moment equations.

B. Computation of the First Moment

We now consider a numerical method for obtaining $\mathbf{m}_1^j(t)$, the first moment for part type j, using the differential equation

$$\frac{d\mathbf{m}_1^j(t)}{dt} = Q\mathbf{m}_1^j(t) + \mathbf{r}_j$$

where $\mathbf{r}_j = R_j\mathbf{e}$ is a $N \times 1$ column vector.

Theorem 7:

$$\mathbf{m}_1^j(t) = \sum_{m=0}^{\infty} \delta_m(t)\boldsymbol{\sigma}_m^j(t) \tag{1}$$

where

$$\begin{aligned} \boldsymbol{\sigma}_m^j &= P\boldsymbol{\sigma}_{m-1}^j + \frac{\mathbf{r}_j}{\lambda}; \\ \boldsymbol{\sigma}_0^j &= \frac{\mathbf{r}_j}{\lambda} \end{aligned} \tag{2}$$

$$\begin{aligned} \delta_m(t) &= \delta_{m-1}(t)\frac{\lambda t}{(m+1)}; \\ \delta_0 &= \lambda t e^{-\lambda t} \end{aligned} \tag{3}$$

Proof: The solution to the above structure equation is well known and is given by

$$\mathbf{m}_1^j(t) = \int_0^t e^{Q\tau}\mathbf{r}_j\, d\tau.$$

Let P be a matrix defined by $(Q/\lambda) + I$, where $\lambda \geq \max(|q_{ii}|)$. This implies that

$$e^{Qt} = \sum_{n=0}^{\infty} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!} P^n.$$

Now $\mathbf{m}_1^j(t)$ can be expressed as

$$\begin{aligned} \mathbf{m}_1^j(t) &= \sum_{n=0}^{\infty} \frac{P^n\mathbf{r}_j}{n!} \int_0^t \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!}\, d\tau \\ &= \frac{1}{\lambda}\sum_{m=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^{m+1}}{(m+1)!}\left[\sum_{n=0}^{m} P^n\mathbf{r}_j\right] \\ &= \sum_{m=0}^{\infty} \delta_m(t)\boldsymbol{\sigma}_m^j(t) \end{aligned}$$

where

$$\boldsymbol{\sigma}_m^j = \sum_{n=0}^{m} \frac{P^n\mathbf{r}_j}{\lambda}$$

and

$$\delta_m = \frac{e^{-\lambda t}(\lambda t)^{m+1}}{(m+1)!}.$$

It is easy to see that both $\boldsymbol{\sigma}_m^j$ and δ_m can be recursively computed using (1)–(3).

C. Computation of the Second Moment

Consider the differential equation for the second moment $\mathbf{m}_2^j(t)$,

$$\frac{d\mathbf{m}_2^j(t)}{dt} = Q\mathbf{m}_2^j(t) + R_j\mathbf{m}_1^j(t). \tag{4}$$

In the following theorem we present a recursive solution for $\mathbf{m}_2^j$.

Theorem 8: Define the recursion

$$\begin{aligned} S_2^j(t,k) = S_2^j(t,k-1) &+ 2\sum_{m=0}^{k-1} \cdot \left[\boldsymbol{\pi}_k\left(\frac{R_j}{\lambda}\right)\boldsymbol{\sigma}_m^j + \boldsymbol{\pi}_m\left(\frac{R_j}{\lambda}\right)\boldsymbol{\sigma}_k^j\right]\delta_{k+m+1}(t) \\ &+ 2\boldsymbol{\pi}_k\left(\frac{R_j}{\lambda}\right)\boldsymbol{\sigma}_k^j\delta_{2k+1}(t) \end{aligned}$$

where

$$\boldsymbol{\pi}_n = \boldsymbol{\pi}_{n-1}P$$

and

$$\boldsymbol{\pi}_0 = \mathbf{p}_0. \tag{5}$$

Let

$$S_2^j(t,\, 0) = 2\boldsymbol{\pi}_0\left(\frac{R_j}{\lambda}\right)\boldsymbol{\sigma}_0^j\,\delta_1 \tag{6}$$

and $\boldsymbol{\sigma}_m^j(t)$ and $\delta_m(t)$ are as defined in Theorem 7. Then, we have

$$S_2^j(t) = \mathbf{p}_0\mathbf{m}_2^j(t) = \lim_{k\to\infty} S_2^j(t,k).$$

Proof: From (4), we have

$$\begin{aligned} \mathbf{m}_2^j(t) &= \int_0^t e^{Q(t-\tau)}R_j\mathbf{m}_1^j(\tau)\, d\tau \\ &= \frac{2}{\lambda}\int_0^t \sum_{n=0}^{\infty} \frac{e^{-\lambda(t-\tau)}[\lambda(t-\tau)]^n}{n!} \\ &\quad \cdot P^nR_j\sum_{m=0}^{\infty}\frac{e^{-\lambda\tau}(\lambda\tau)^{m+1}}{(m+1)!}\left[\sum_{q=0}^{m} P^q\mathbf{r}_j\right] \\ &= \frac{2}{\lambda}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{q=0}^{m} P^nR_jP^q\mathbf{r}_j\int_0^t \\ &\quad \cdot \frac{e^{-\lambda(t-\tau)}[\lambda(t-\tau)]^n}{n!}\,\frac{e^{-\lambda\tau}(\lambda\tau)^{m+1}}{(m+1)!}\, d\tau \\ &= \frac{2}{\lambda^2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{q=0}^{m} \\ &\quad \cdot P^nR_jP^q\mathbf{r}_je^{-\lambda t}\frac{(\lambda t)^{n+m+2}}{(n+m+2)!} \\ &= \frac{2}{\lambda^2}\sum_{n=0}^{\infty}P^n\sum_{m=0}^{\infty}R_je^{-\lambda t}\frac{(\lambda t)^{n+m+2}}{(n+m+2)!}\sum_{q=0}^{m}P^q\mathbf{r}_j. \end{aligned}$$

From the above equation, we have

$$S_2^j(t) = \mathbf{p}_0\mathbf{m}_2^j(t) = \frac{2}{\lambda}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\boldsymbol{\pi}_nR_j\boldsymbol{\sigma}_m\,\delta_{n+m+1}$$

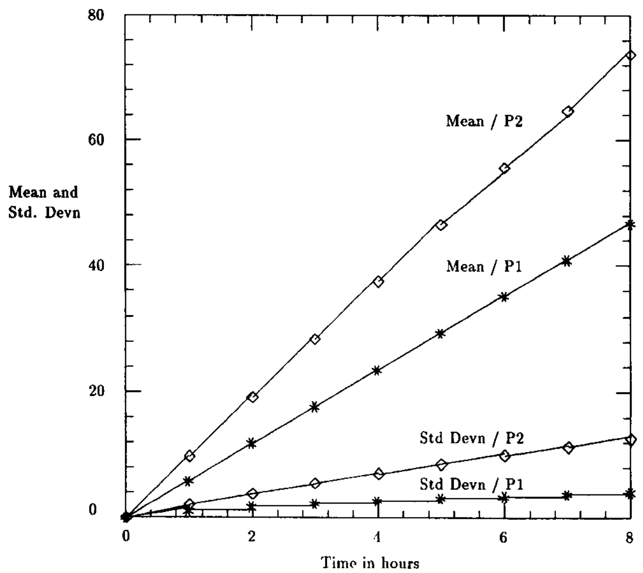

Fig. 10. Variation of mean and std. deviation for cell with repair.

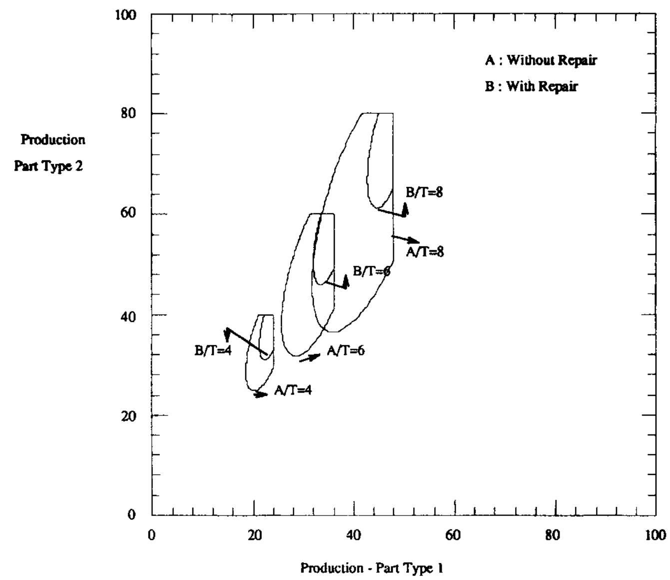

Fig. 11. Confidence regions for cell with repair.

where $\boldsymbol{\pi}_n = \boldsymbol{\pi}_{n-1}P$ and $\boldsymbol{\pi}_0 = \mathrm{p}_0$Let

$$S_2^j(t, k) = \frac{2}{\lambda}\sum_{n=0}^{k}\sum_{m=0}^{k}\boldsymbol{\pi}_n R_j \boldsymbol{\sigma}_m \delta_{n+m+1}.$$

Clearly, $S_2^j(t, k) \to S_2^j(t)$ as $k \to \infty$. We can now recursively obtain $S_2^j(t, k)$ toward this end.

$$\begin{aligned}
S_2^j(t,k) &= 2\sum_{n=0}^{k}\sum_{m=0}^{k}\boldsymbol{\pi}_n\left(\frac{R_j}{\lambda}\right)\boldsymbol{\sigma}_m\,\delta_{n+m+1}\\
&= S_2^j(t, k-1) + 2\sum_{m=0}^{k}\boldsymbol{\pi}_k\left(\frac{R_j}{\lambda}\right)\boldsymbol{\sigma}_m\,\delta_{k+m+1}\\
&\quad + 2\sum_{m=0}^{k-1}\boldsymbol{\pi}_m\left(\frac{R_j}{\lambda}\right)\boldsymbol{\sigma}_k\,\delta_{k+m+1}\\
&= S_2^j(t,k-1) + 2\sum_{m=0}^{k-1}\\
&\quad \cdot\left[\boldsymbol{\pi}_k\left(\frac{R_j}{\lambda}\right)\boldsymbol{\sigma}_m + \boldsymbol{\pi}_m\left(\frac{R_j}{\lambda}\right)\boldsymbol{\sigma}_k\right]\delta_{k+m+1}\\
&\quad + 2\boldsymbol{\pi}_k\left(\frac{R_j}{\lambda}\right)\boldsymbol{\sigma}_k\,\delta_{2k+1}.
\end{aligned}$$

Thus, the theorem.

D. Examples

The methodology of Section II has been applied to a variety of manufacturing applications with number of states N ranging from 2–600. In this section, we present two examples to illustrate the approach.

Example 1(e): Consider Example 1(b) described in Section III. For computational experiments, we consider two alternate schemes of operation.

Scheme A: The cell is operational only if at least one of the two machines and AGV are operational and none of the components can be repaired once they fail. The rate of failure of machines (λ_m) is identical and is equal to 0.02 per hour. The rate of failure of AGV (λ_a) is assumed to be 0.025 per hour. The reward vector for part type 1 is [6.0 6.0 0.0] and part type 2 is [10.0 0.0 0.0].

Scheme B: In this scheme, the machines and the AGV are assumed to be repairable. We assume the availability of one repair facility common to both the machines and AGV. The machines can be repaired at a rate (μ_m) equal to 0.5 per hour and the AGV repair rate (μ_a) is assumed to be 1.0 per hour. We also assume that the AGV and machines can fail when the other components are undergoing repair and that the repair facility gives higher priority to the repair of AGV over the machines.

The first two moments are calculated for the cell operating under both the schemes over a mission time (T) of 8 h (1 shift). Fig. 10 shows the variation of mean and standard deviation for part types 1 and 2 over the 8-h time period under scheme B. We observe that all these variables increase with time and this trend continues for larger mission times. In contrast, they tend to saturate for larger machine times under scheme A. In particular, note that the coefficient of variation is between 0.09–0.18, a result consistent with that in [9]. Fig. 11 shows the one sigma confidence regions considering the production of part types 1 and 2 as a bivariate normal distribution. The plot shows the confidence regions for schemes A and B and for mission times of 4, 6, and 8 h. We observe that the size of the ellipses increases with mission time and also that the ellipses are smaller and well separated under scheme B. Thus, we can meet the production schedules with more confidence under scheme B.

Example 4(b): We consider a cellular manufacturing system, discussed in Example 4(a). We assume that the number of repairmen is 0 i.e., no repair is possible. We modeled the above system using Generalized Stochastic Petri Nets (GSPN's) and obtained the Q matrix from the tangible states of GSPN, which number 128. Since the part types are produced in different cells, the production rate in each state of GSPN is obtained

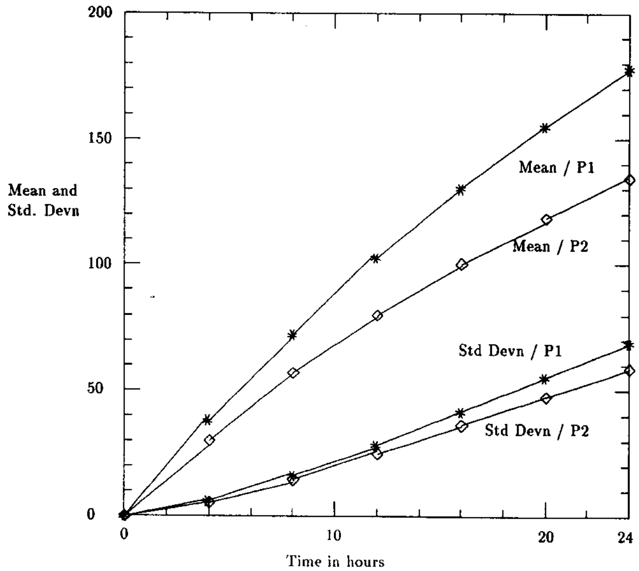

Fig. 12. Variation of mean and standard deviation for cellular manufacturing system without repair.

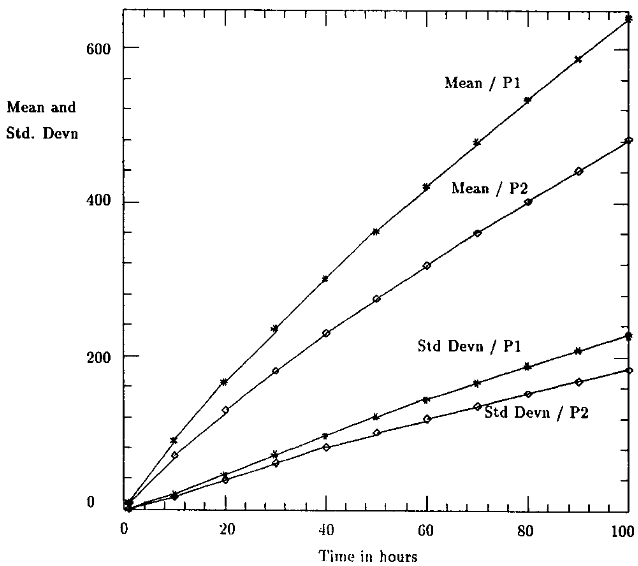

Fig. 13. Variation of mean and standard deviation for cellular manufacturing system with repair (1 repairman).

from the throughput capacity of each cell in that state. Fig. 12 shows the plots of the mean and variance of cumulative production for part types 1 and 2 over a time period of 24 h. Note that the production rate of part type 1 varies between 38–86% even though the mean is 62%. The production rate of part type 2 shows a similar behavior, thus justifying the relevance of variance. Fig. 13 shows the variation of the mean and variance of cumulative production with one repairman. The coefficient of variation i.e., the ratio of standard deviation to mean, reduces in this case, thus reducing the variability.

Example 5: In this example, we consider a generalized flexible manufacturing system producing two part types P1 and P2. The schematic of the system is shown in Fig. 14. The system contains four machines and an AGV for transporting the workpieces to the machines. Each part type requires three

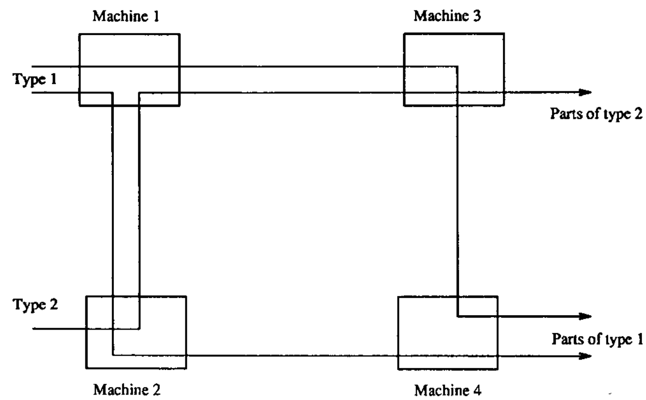

Fig. 14. Schematic of flexible manufacturing system producing 2 part types.

TABLE V
OPERATIONS FOR PART TYPES

Part Type	Operation 1	Operation 2	Operation 3
P1	M1	M2/M3	M4
P2	M2	M1	M3

TABLE VI
PROCESSING TIMES FOR PART TYPES

Part Type	*Mean machining time in minutes*		
	Operation 1	Operation 2	Operation 3
P1	10	5 on M2 8 on M3	12
P2	8	15	5

operations and the routing table in Table V shows the machines and time requirement for various part types.

We assume that the processing times on each machine are exponentially distributed and the mean machining times are given in Table VI. AGV transports the work pieces after each operation to the next selected machine. We assume that the time taken for an AGV operation is exponentially distributed with a mean of 1 min. Also, we assume that the four fixtures each are available for part types P1 and P2.

This generalized FMS is modeled using stochastic Petri nets and the performance measures such as throughputs for each part type are obtained, considering that there are no failures during the operation. The failures and repairs of individual components are given in Table VII and are modeled using a separate reliability model. A centralized repair facility is assumed for repairing the failed components on a nonpreemptive basis. The repair of AGV has the highest priority and repair of all other machines has equal priority at this facility. The repair facility is assumed to be manned by n repairmen, and n is varied from 1 to 3 to study the behavior of the system with better repair facilities. The performability model of this system is obtained by substituting the rewards in each of the structure states of the reliability model, from the performance model. The moments of the performability distribution are obtained for each part type, over an operational period of 100 h.

We study the variation of mean throughput and its variability under different possible design options. The design choices,

TABLE VII
FAILURE AND REPAIR RATES OF THE COMPONENTS OF FMS

Component	Failure Rate	Repair Rate
$M1$	0.005	0.5
$M2$	0.01	1.0
$M3$	0.1	1.0
$M4$	0.01	1.0
AGV	0.002	0.25

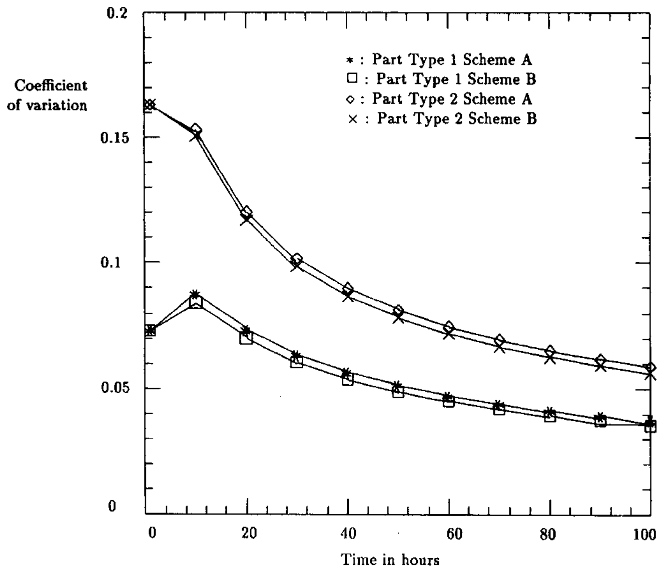

Fig. 15. Variation of coefficient of variation for GFMS.

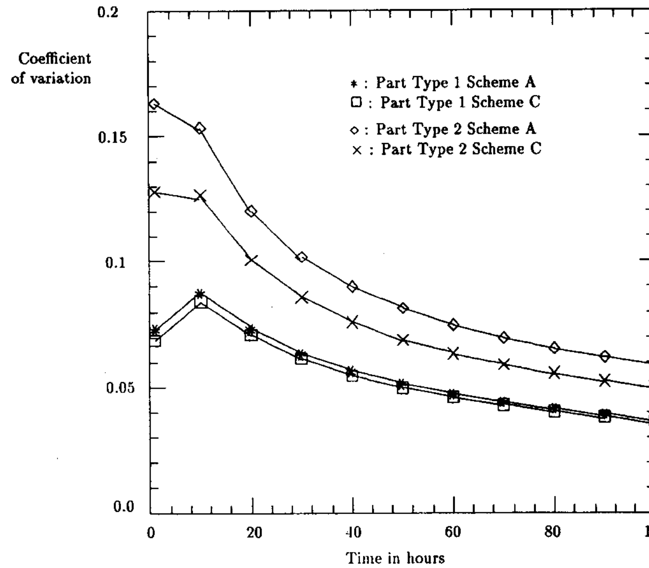

Fig. 16. Variation of coefficient of variation for GFMS.

TABLE VIII
MEAN THROUGHPUTS IN PARTS PER HOUR

Scheme	Part 1	Part 2
A	312.3477	154.5891
B	313.1952	155.1238
C	309.5294	162.1136

we consider are

1) a repair facility with one repairman
2) a repair facility with three repairmen
3) a repair facility with one repairman and machine M3 replaced with a more reliable machine (failure rate changed to 0.05).

The mean throughput of parts under these different schemes are listed in Table VIII. We can see that there is not much difference in the variation of throughputs. In fact, the maximum variation is about 1.16% for part type 1 and 4.86% for part type 2. The coefficient of variation, defined as the ratio of the standard deviation to the mean, provides an estimate of the variation of the mean output. Fig. 15 shows the variation of the coefficient of variation for part types 1 and 2 under the schemes A and B. Here also, we find that the variability decreases marginally for both part types when we increase the repairmen from 1 to 3. Fig. 16 shows the coefficient of variation for part types 1 and 2 under schemes A and C. We can observe that the decrease in the coefficient of variation for part type 2 is much higher under scheme C, in comparison to the schemes A and B. It can be concluded that scheme C is more suitable in case we wish to keep the variability of the mean production minimum.

VI. CONCLUSION

This paper showed that the probability distribution of cumulative production (reward) of a manufacturing system producing several part types satisfies a linear hyperbolic partial differential equation. We also derived the moment recursions, which are in the form of ordinary differential equations. Examples illustrate the efficacy of the theory in the manufacturing system context. There are several problems that need attention (i) efficient numerical methods for solving the PDE's, (ii) significance of the results for some real world systems, and (iii) extension of analysis to discrete time models. Some of these issues are currently under investigation by the authors.

Traditionally, reliability of production is maintained by addition of buffers at various machine centers or at a central place. The modeling methodologies presented are applicable for buffered systems but would lead to Markov models with larger state spaces. An interesting study would be to design the "optimal" combination of buffer space, repair strategies and the scheduling policy. In actual practice there are several preventive maintenance policies that are followed. Performability evaluation of each of these schemes would enable us to rank order these policies [3]. Also Integrated Production-Inventory systems is an interesting subject for further study [2].

APPENDIX

A. Proof of Theorem 1

From Lemma 2, we have

$$w_{ij}(\mathbf{y}, t) = \int_{\tilde{\mathbf{y}}} \sum_{k=1}^{N} w_{ik}(\tilde{\mathbf{y}}, \sigma) w_{kj}(\mathbf{y} - \tilde{\mathbf{y}}, t - \sigma)\, d\tilde{\mathbf{y}}.$$

Substituting $\sigma = t - h$ and integrating both sides with respect to $\mathbf{y}$, we get

$$F_{ij}(\mathbf{y}, t) = \int_{\tilde{\mathbf{y}}} \sum_{k=1}^{N} F_{kj}(\mathbf{y} - \tilde{\mathbf{y}}, h) w_{ik}(\tilde{\mathbf{y}}, t - h)\, d\tilde{\mathbf{y}}.$$

From Lemma 3, we have

$$\begin{aligned} F_{kj}(y - \tilde{\mathbf{y}}, h) = &U(\mathbf{y} - \tilde{\mathbf{y}} - r_k h) e^{-\lambda_k h} \delta_{kj} + \int_0^h \lambda_k e^{-\lambda_k \tau} \\ &\cdot \sum_{m=1}^{N} P_{km} F_{mj}(\mathbf{y} - \tilde{\mathbf{y}} - r_k \tau, h - \tau)\, d\tau. \end{aligned}$$

From the above equations we have

$$F_{ij}(\mathbf{y}, t) = I_1 + I_2$$

where

$$\begin{aligned} I_1 &= \int_{\tilde{\mathbf{y}}} \sum_{k=1}^{N} U(\mathbf{y} - \tilde{\mathbf{y}} - r_k h) w_{ik}(\tilde{\mathbf{y}}, t - h) e^{-\lambda_k h} \delta_{kj}\, d\tilde{\mathbf{y}} \\ &= \int_{\tilde{\mathbf{y}}} U(\mathbf{y} - \tilde{\mathbf{y}} - r_j h) w_{ij}(\tilde{\mathbf{y}}, t - h) e^{-\lambda_j h}\, d\tilde{\mathbf{y}} \\ &= \int_0^{\mathbf{y} - \mathbf{r}_j h} w_{ij}(\tilde{\mathbf{y}}, t - h) e^{-\lambda_j h}\, d\tilde{\mathbf{y}} \\ &= F_{ij}(\mathbf{y} - \mathbf{r}_j h, t - h) e^{-\lambda_j h} \end{aligned}$$

and

$$\begin{aligned} I_2 = &\sum_{k=1}^{N} \int_0^h \lambda_k e^{-\lambda_k \tau} \sum_{m=1}^{N} P_{km} \int_{\tilde{\mathbf{y}}} \\ &\cdot F_{mj}(\mathbf{y} - \tilde{\mathbf{y}} - \mathbf{r}_k \tau, h - \tau) w_{ik}(\tilde{\mathbf{y}}, t - h)\, d\tilde{\mathbf{y}}. \end{aligned}$$

Consider the first order expansion of $F_{ij}(\mathbf{y}, t) - I_1$,

$$\begin{aligned} F_{ij}(\mathbf{y}, t) - I_1 &= F_{ij}(\mathbf{y}, t) - F_{ij}(\mathbf{y} - \mathbf{r}_j h, t - h) e^{-\lambda_j h} \\ &= F_{ij}(\mathbf{y}, t) - h \Bigg[F_{ij}(\mathbf{y}, t - h) \\ &\quad + \frac{\partial F_{ij}(\mathbf{y}, t - h)}{\partial \mathbf{y}} \mathbf{r}_j^T + \cdots \Bigg] \\ &\quad \cdot [1 - \lambda_j h + \cdots] \end{aligned}$$

$$\begin{aligned} \lim_{h \to 0} \frac{F_{ij}(\mathbf{y}, t) - I_1}{h} &= \lim_{h \to 0} \frac{F_{ij}(\mathbf{y}, t) - F_{ij}(\mathbf{y}, t - h)}{h} \\ &\quad + \lambda_j F_{ij}(\mathbf{y}, t) + \frac{\partial F_{ij}(\mathbf{y}, t)}{\partial \mathbf{y}} \mathbf{r}_j^T \\ &= \frac{\partial F_{ij}(\mathbf{y}, t)}{\partial t} + \frac{\partial F_{ij}(\mathbf{y}, t)}{\partial \mathbf{y}} \mathbf{r}_j^T \\ &\quad + \lambda_j F_{ij}(\mathbf{y}, t) \end{aligned}$$

$$\begin{aligned} \lim_{h \to 0} \frac{I_2}{h} &= \sum_{k=1}^{N} \lambda_k \sum_{m=1}^{N} P_{km} \int_{\tilde{\mathbf{y}}} F_{mj}(\mathbf{y} - \tilde{\mathbf{y}}, 0) w_{ik}(\tilde{\mathbf{y}}, t)\, d\tilde{\mathbf{y}} \\ &= \sum_{k=1}^{N} \lambda_k P_{kj} \int_0^y w_{ik}(\tilde{\mathbf{y}}, t)\, d\tilde{\mathbf{y}} \\ &= \sum_{k=1}^{N} \lambda_k P_{kj} F_{ik}(\mathbf{y}, t). \end{aligned}$$

In deriving the above expression, we have used the fact that

$$F_{mj}(\mathbf{y} - \tilde{\mathbf{y}}, 0) = \prod_{p=1}^{P} U(y_p - \tilde{y_p})\, \delta_{mj}.$$

Now, from the equality

$$\lim_{h \to 0} \frac{F_{ij}(\mathbf{y}, t) - I_1}{h} = \lim_{h \to 0} \frac{I_2}{h}.$$

We get

$$\begin{aligned} \frac{\partial F_{ij}(\mathbf{y}, t)}{\partial t} = &-\frac{\partial F_{ij}(\mathbf{y}, t)}{\partial \mathbf{y}} \mathbf{r}_j^T - \lambda_j F_{ij}(\mathbf{y}, t) \\ &+ \sum_{k=1}^{N} \lambda_k P_{kj} F_{ik}(\mathbf{y}, t). \end{aligned}$$

In the matrix form we obtain,

$$\frac{\partial F(\mathbf{y}, t)}{\partial t} = -\sum_{j=1}^{P} \frac{\partial F(\mathbf{y}, t)}{\partial y_j} R_j + F(\mathbf{y}, t) \Lambda [P - I]$$

where $\Lambda = \text{diag}\{\lambda_1 \cdots \lambda_N\}$. Since $Q = \Lambda[P - I]$, the theorem follows. □

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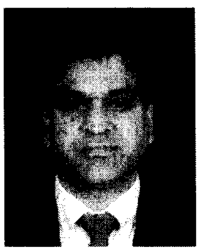
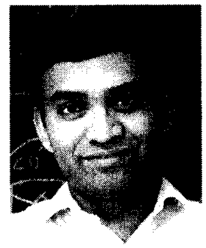


N. Viswanadham (SM'86–F'93) received the Ph.D. degree in 1970 from the Indian Institute of Science (IISc), Bangalore, India.

While he has held visiting appointments at several North American universities, he is currently a TataChem Professor and chair of the Department of Computer Science and Automation, at the IISc. He was a GE Research Fellow at the corporate research center during 1989. His current research interests are in the areas of modeling, control and management of competitive manufacturing systems and software quality and reliability. He is the author of several journal articles and conference papers. He is a joint author of two textbooks: *Reliability in Computer and Control Systems*, (North-Holland, 1987) and *Performance Modeling of Automated Manufacturing Systems*, (Prentice-Hall, 1992). He is co-editor of four other books. He currently is editor of *Sadhana: Academy Proceedings in Engineering Sciences.*

Dr. Viswanadham is an Associate Editor of the journals *Journal of Franklin Institute*, *Journal of Manufacturing Systems*, *International Journal on Information Technology*, *Intelligent and Robotic Systems*, IEEE TRANSACTIONS ON ROBOTICS AND AUTOMATION. He was Associate Editor-at-Large for the IEEE TRANSACTIONS ON AUTOMATIC CONTROL, 1990 and 1991. He is a fellow of the Indian National Science Academy, the Indian Academy of Sciences, and the Indian National Academy of Engineering.



Krishna R. Pattipati (M'79–SM'92–F'95) received the B.Tech. degree in electrical engineering with highest honors from the Indian Institute of Technology, Kharagpur, in 1975, and the M.S. and Ph.D. degrees in systems engineering from the University of Connecticut in 1977 and 1980, respectively.

He was employed by Alphatech Inc., Burlington, MA, from 1980 to 1986, where he supervised and performed research on human decision modeling, multitarget tracking, queuing networks, automated testing, and large-scale mixed-integer optimization. Since September 1986, he has been with the University of Connecticut, Storrs, where he is a Professor of Electrical and Systems Engineering. He is also president of QUALTECH Systems Inc., Storrs, CT, a small business specializing in software tools and solutions for System Testability, Maintainability and quality control. He has served as consultant to Alphatech Inc. and the IBM Thomas J. Watson Research Center.

Dr. Pattipati was selected by IEEE Systems, Man, and Cybernetics (SMC) Society as the Outstanding Young Engineer of 1984, and received the Centennial Key to the Future award. He won the best technical paper awards at the 1985, 1990, and 1994 IEEE AUTOTEST Conferences. In 1995, he was elected a Fellow of the IEEE for his contributions to Discrete Optimization Algorithms for Large-Scale Systems and Team-Decisionmaking. He has served as the Vice-Chairman for invited sessions of the IEEE International Conference on SMC Boston, MA, 1989. He is currently serving as an Associate Editor of the IEEE TRANSACTIONS ON SYSTEMS, MAN, AND CYBERNETICS.

V. Gopalakrishna obtained the B.E., M.E., and Ph.D. degrees from the Indian Institute of Science, Bangalore, India.

He was a Systems Analyst at the National Informatics Center, New Delhi during 1977–1979. He worked as a Research Engineer at the Canada Center for Remote Sensing in Ottawa, Canada during 1981–1983. He is the founding Director of Integra Micro Systems, a leading company, specializing in the development of software tools and business solutions. He is author of several papers in the areas of databases, manufacturing systems and performance evaluation. His current research interests include software productivity tools, performance evaluation, and workflow automation.