REFERENCES

[1] R. H. Bartels and G. W. Stewart, "Solution of the matrix equation $AX+XB=C$," *Commun. Ass. Comput. Mach.*, vol. 15, no. 9, pp. 820–826, 1972.
[2] R. Clough, "Analysis of structural vibrations and dynamic response," in *Recent Advances in Matrix Methods of Structural Analysis and Design.* University, Ala.: Univ. Alabama Press, 1970.
[3] J. Francis, "The *QR* transformation," *Comput. J.*, vol. 4, p. 265, 1961; also p. 332, 1962.
[4] F. R. Gantmacher, *The Theory of Matrices.* New York: Chelsea, 1959, vol. 1.
[5] S. C. Gupta, *Transform and State Variable Methods in Linear Systems.* New York: Wiley, 1966, p. 250.
[6] A. Jameson, "Solution of the equation $AX+XB=C$ by inversion of an $M \times M$ $N \times N$ matrix," *SIAM J. Appl. Math.*, vol. 16, pp. 1020–1023, 1968.
[7] B. C. Kuo, *Automatic Control Systems.* Englewood Cliffs, N. J.: Prentice-Hall, 1967, pp. 493–496.
[8] J. H. Seinfeld and J. Lapidus, *Numerical Solution of Ordinary Differential Equations.* New York: Academic, 1971, pp. 267–292.
[9] H. W. Turnbull and A. C. Aitken, *An Introduction to the Theory of Canonical Matrices.* New York: Dover, 1961, pp. 58–62.
[10] J. H. Wilkinson, *The Algebraic Eigenvalue Problem.* New York: Clarendon, 1970, p. 434.

Uniqueness of Minimal Partial Realizations

BRADLEY W. DICKINSON

***Abstract*—Two methods for determining whether all minimal partial realizations of a given finite sequence have isomorphic state spaces are described. In particular, uniqueness of the minimal polynomial of the minimal partial realizations is shown to be an equivalent condition.**

In the minimal partial realization problem, we are given a finite sequence of $p \times m$ matrices $\{T_i, 1 \leqslant i \leqslant N\}$ and we wish to find a triple of matrices $(A(N), B(N), C(N))$, where $A(N)$ is $n_N \times n_N$, $B(N)$ is $n_N \times m$, and C is $p \times n_N$, with n_N as small as possible, so that

$$C(N)A(N)^{i-1}B(N) = T_i, \qquad 1 \leqslant i \leqslant N.$$

Kalman introduced this problem [4] and the main existence theorem was also given by Tether [10]. Rissanen [6], [7] and Dickinson, Morf, and Kailath [3] have studied algorithmic aspects of the problem, especially its importance in efficiently solving the "total" minimal realization problem ($N = \infty$).

In a previous paper [2], a complete set of independent invariants for the minimal partial realization problem was given, and a corresponding set of canonical forms for the triple $(A(N), B(N), C(N))$ was derived. These canonical forms have the general structure of the *observer canonical forms* of Luenberger [5], which are a set of canonical forms for controllable and observable triples (A, B, C) with isomorphism of state spaces being the equivalence relation. In general, solutions to the minimal partial realization problem do not have isomorphic state spaces, and the canonical forms for this case are obtained by looking at the observer canonical forms under this more general equivalence relation.

The details of this approach are given in [2], and here we consider the problem of determining when the minimal partial realization is unique up to state-space isomorphism as is the case for the solution to the total minimal realization problem. From the discussion above, we see that in this case, the two notions of equivalence coincide and this provides one test for this form of uniqueness.

Proposition 1: The minimal partial realization of a sequence $\{T_i, 1 \leqslant i \leqslant N\}$ is unique up to isomorphism of state spaces if and only if there is precisely one solution of dimension n_N in observer canonical form.

The proof of this statement follows directly from the preceding discussion. Anderson, Brasch, and Lopresti [1] have obtained a result that is very similar in character to this one. The main problem with the result is the difficulty of showing that there is only one observer canonical form solution, although it is possible by using the constructive method in [2] for obtaining the invariants of the problem.

A much simpler condition can be obtained by using another complete invariant for state-space isomorphism of controllable and observable systems, namely $\varphi(z)$, the minimal polynomial of the state module, with degree $\varphi(z) = q$, and the terms $\{T_1, T_2, \cdots, T_q\}$. Although this well-known complete invariant is not very useful for obtaining canonical forms, it gives a nice condition or uniqueness for minimal partial realizations.

Proposition 2: The minimal partial realization of a sequence $\{T_i, 1 \leqslant i \leqslant N\}$ is unique up to isomorphism of state spaces if and only if every minimal partial realization has the same minimal polynomial.

Before giving the proof, we point out that this test is more in the spirit of Kalman's main theorem [4]; also see Tether [10]. In the case of scalar ($p = m = 1$) sequences, the result is almost obvious.

Proof: The minimal polynomial of a controllable and observable realization gives the linear recurrence relation of least order satisfied by the terms of the impulse response. Thus, realizations with isomorphic state spaces have the same minimal polynomial. However, the minimal polynomial for minimal Nth partial realizations can be unique only if its degree is less than $N+1$. If it is unique, then the entire impulse response is uniquely determined and all realizations have isomorphic state spaces by the usual "total" realization result.

Proposition 2 reduces the test of uniqueness to checking the rank of a particular matrix in order to guarantee a unique solution to a single linear equation. Furthermore, this rank is identified with the degree of the minimal polynomial, a new observation.

It is an interesting fact to note that the minimal polynomial specifies the complete set of invariant factors for the unique partial realization, but this is to be expected from the well-known recurrence for the terms in the impulse response. No easy way of computing the other invariant factors is suggested by this result. However, it is easy to extend the result to systems defined over some classes of commutative rings as discussed, for example, in Rouchaleau [8, cf. (3.1), (3.4)] and Rouchaleau and Wyman [9].

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REFERENCES

[1] B. M. Anderson, F. Brasch, and P. Lopresti, "The sequential construction of minimal partial realizations from finite input-output data," Dep. Elec. Eng., Northwestern Univ., Evanston, Ill., 1973.
[2] B. W. Dickinson, T. Kailath, and M. Morf, "Canonical matrix fraction and state-space descriptions for deterministic and stochastic linear systems," *IEEE Trans. Automat. Contr.*, vol. AC-19, pp. 656–667, Dec. 1974.
[3] B. W. Dickinson, M. Morf, and T. Kailath, "A minimal realization algorithm for matrix sequences," *IEEE Trans. Automat. Contr.*, vol. AC-19, pp. 31–38, Feb. 1974.
[4] R. E. Kalman, "On minimal partial realization of a linear input-output map," in *Aspects of Network and System Theory*, R. E. Kalman and N. DeClaris, Ed. New York: Holt, Rinehart, and Winston, 1971, pp. 385–407.
[5] D. G. Luenberger, "Canonical forms for linear multivariable systems," *IEEE Trans. Automat. Contr.*, vol. AC-12, pp. 290–293, June 1967.
[6] J. Rissanen, "Recursive identification of linear systems," *SIAM J. Contr.*, vol. 9, pp. 420–430, 1971.
[7] ——, "Realization of matrix sequences," IBM Tech. Rep., San Jose, Calif., 1972.
[8] Y. Rouchaleau, "Linear, discrete time, finite dimensional, dynamical systems over some classes of commutative rings," Ph.D. dissertation, Stanford Univ., Stanford, Calif., 1972.
[9] Y. Rouchaleau and B. Wyman, "Linear dynamical systems over integral domains," *J. Comp. Syst. Sci.*, vol. 9, pp. 129–142, Oct. 1974.
[10] A. J. Tether, "Construction of minimal linear state-variable models from finite input-output data," *IEEE Trans. Automat. Contr.*, vol. AC-15, pp. 427–436, Aug. 1970.

On Invariance of Degree of Controllability Under State Feedback

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***Abstract*—Attention is given to time varying multivariable systems. An algebraic proof is presented to show that the degree of controllability and certain indices useful in developing canonical forms for the stabilization of time varying multivariable systems are invariant under state variable feedback.**

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Consider the linear continuous system

$$\dot{x}(t)=A(t)x(t)+B(t)u(t) \tag{1}$$

$$y(t)=C(t)x(t) \tag{2}$$

where $x(t)$, $u(t)$, and $y(t)$ are state, input and output vectors of dimension n, m, and p, respectively. $A(t)$, $B(t)$, and $C(t)$ are matrices of compatible order and assumed to be differentiable the required number of times. Silverman and Meadows [1] have defined three types of controllability called uniform, total, and complete controllability in terms of the matrix $Q_n(t)$ defined by

$$Q_k(t)=[P_0(t),P_1(t),\cdots,P_{k-1}(t)] \tag{3}$$

where the matrices P_k, $k=0, 1, 2,\cdots$ are given by

$$P_0(t)=B(t); \qquad P_{k+1}(t)=A(t)P_k(t)-\dot{P}_k(t). \tag{4}$$

The object of this note is to show invariance of the degree of controllability for the class of time varying systems described in (1), under state feedback of the type

$$u(t)=F(t)x(t)+v(t) \tag{5}$$

where v is an $(m\times 1)$ reference input vector and $F(t)$ is an $m\times n$ matrix. For the case when $A(t)$, $B(t)$ are constant matrices, Brockett [2] has established the invariance of controllability under state feedback. Silverman and Anderson [3] have proved that uniform complete controllability [4] which is different from the above three types is preserved under feedback of type (5). Recently Chandrasekharan [5] considered the case when $A(t)$ and $B(t)$ are analytic and proved the invariance of uniform controllability.

Here, we present an algebraic proof showing the invariance of the degree of controllability under state feedback (5). The method of proof seems to be new even for time invariant systems. Also, the procedure explicitly shows the invariance of certain indices which are useful in developing canonical forms for multivariable systems [6]–[8].

Substituting (5) into (1) we get

$$\dot{x}(t)=[A(t)+B(t)F(t)]x(t)+B(t)v(t)$$

$$y(t)=C(t)x(t). \tag{6}$$

Now define the matrices P_k^c and Q_k^c for the closed-loop system (6) in a similar manner as in (3) and (4). Our aim here is to show that

$$Q_k^c(t)=Q_k(t)H_k(t), \qquad k=0,1,2,\cdots \tag{7}$$

where H_k, $k=0, 1, 2,\cdots$ is an upper triangular block matrix, nonsingular for all t, defined by

$$H_0=I_m$$

$$H_k(t)=\begin{bmatrix} J_{0,0} & J_{1,0}(t) & \cdots & J_{k-1,0}(t)\\ & J_{1,1} & \cdots & J_{k-1,1}(t)\\ & & \ddots & \vdots\\ & \mathbf{0} & & \vdots\\ & & & J_{k-1,k-1}\end{bmatrix} \tag{8}$$

and

$$J_{k,k}=I_m$$

$$J_{k,i}(t)=\sum_{j=0}^{k-1-i}(-1)^j\binom{j+i}{j}\frac{d^j}{dt^j}(F(t)P_{k-1-i-j}^c(t)), \qquad k>i. \tag{9}$$

In other words, the following theorem will be proved.

Theorem 1: $\operatorname{Rank} Q_j^c(t)=\operatorname{rank} Q_j(t)\triangleq d_j(t), \quad j\geqslant 0.$

Proof: First observe that by direct calculation and use of (9) the following identities are true where the time dependence notation is omitted for notational convenience.

$$P_0^c=P_0=B$$

$$P_1^c=(A+BF)P_0^c-\dot{P}_0^c=P_1+P_0(FP_0^c)=P_1+P_0J_{1,0}$$

$$P_2^c=(A+BF)P_1^c-\dot{P}_1^c=A(P_1+P_0(FP_0^c))-\dot{P}_1-\frac{d}{dt}(P_0FP_0^c)+P_0FP_1^c$$

$$=P_2+P_1(FP_0^c)+P_0\left[FP_1^c-\frac{d}{dt}(FP_0^c)\right]$$

$$=P_2+P_1J_{2,1}+P_0J_{2,0}$$

$$P_3^c=AP_2^c-\dot{P}_2^c+BFP_2^c=A\left[P_2+P_1(FP_0^c)+P_0\left(FP_1^c-\frac{d}{dt}(FP_0^c)\right)\right]$$

$$-\frac{d}{dt}\left\{P_2+P_1(FP_0^c)+P_0\left[FP_1^c-\frac{d}{dt}(FP_0^c)\right]\right\}$$

$$+P_0FP_2^c$$

$$=P_3+P_2(FP_0^c)+P_1\left[FP_1^c-2\frac{d}{dt}(FP_0^c)\right]$$

$$+P_0\left[FP_2^c-\frac{d}{dt}(FP_1^c)+\frac{d^2}{dt^2}(FP_0^c)\right]$$

$$=P_3+P_2J_{3,2}+P_1J_{3,1}+P_0J_{3,0}. \tag{10}$$

Proceeding this way, we get for any $k\geqslant 0$.

$$P_k^c=P_k+P_{k-1}J_{k,k-1}+P_{k-2}J_{k,k-2}+\cdots+P_0J_{k,0}. \tag{11}$$

From (10), (11), and (8) we note that (7) holds for all $k\geqslant 0$. Since H_k is nonsingular for all k and t, it can be concluded that rank $Q_k^c=$ rank $Q_k=d_k(t)$ for all t and k. Q.E.D.

It is known that [1], system (1) is uniformily (totally) controllable if and only if $Q_n(t)$ has rank n for all t (for almost all t). Also it is completely controllable if Q_n has rank n for some t. In view of Theorem 1 the following corrollary follows.

Corollary 1: $(A(t),B(t))$ is uniformily (totally) ⟨completely⟩ controllable if and only if $(A(t)+B(t)F(t),B(t))$ is uniformily (totally) ⟨completely⟩ controllable.

Observe that the indices $d_j(t)$, $j=0,1,\cdots,n$ may be functions of time. Systems for which these indices are constant integers for all t are termed index invariant systems [6], [7]. One can observe from Theorem 1 that the following corrollary is true.

Corollary 2: System (1) is index invariant if and only if the closed-loop system (6) is index invariant. Furthermore, the indices d_j, $j=0,1,\cdots,n$ are identical for both the systems.

Brunovsky [6], Morse and Silverman [7] developed canonical forms to stabilize system (1) which is index invariant and the indices d_j, $j=0,1,\cdots,n$ play an important role in determining the structure of the canonical form. A similar analysis was carried out for time invariant systems in [6] and [8].

Finally it may be noted that a similar analysis can be carried out to show the invariance of the degree of observability under output feedback in a straightforward manner.

REFERENCES

[1] L. M. Silverman and H. E. Meadows, "Degrees of controllability in time varying linear systems," in *Proc. Nat. Electronics Conf.*, vol. 21, pp. 689–693, 1965.
[2] R. W. Brockett, "Poles, zeros, and feedback: State space interpretation," *IEEE Trans. Automat. Contr.*, vol. AC-10, pp. 129–135, Apr. 1965.
[3] L. M. Silverman and B. D. O. Anderson, "Controllability, observability and stability of linear systems", *SIAM J. Contr.*, vol. 6, pp. 121–130, 1968.
[4] R. E. Kalman, "Contributions to the theory of optimal control," *Bol. Soc. Mat. Mexicana*, pp. 102–119, 1960.
[5] P. C. Chandrasekharan, "Invariant properties of state variable feedback," in *Proc. 2nd IFAC Symp. Multivariable Technical Control Systems*, Duesseldorf, 1971, vol. 1, paper 1.1.1.
[6] P. Brunovsky, "A classification of linear controllable systems," *Kybernetika*, vol. 3, 1970, pp. 173–187.
[7] A. S. Morse and L. M. Silverman, "Structure of index invariant systems," *SIAM J. Contr.*, vol. 11, no. 2, pp. 215–225, 1973.
[8] R. Yokoyama and E. Kinnen, "Phase variable canonical forms for multi-input multi-output systems," *Int. J. Contr.*, vol. 17, no. 6, pp. 1297–1312, 1973.

On the Similarity Between a Matrix and its Routh Canonical Form

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Abstract—**A necessary and sufficient condition for a nonderogatory matrix A to be similar to its Routh canonical form R is established in this note.**

I. Introduction

It was shown by Schwarz [6] that every nonderogatory matrix A can be transformed by similarity to the form

$$S=\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ -s_n & 0 & 1 & \cdots & 0 \\ 0 & -s_{n-1} & 0 & 1 & 0 \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ 0 & \cdots & \cdots & \cdots & \cdots & 1 \\ 0 & 0 & \cdots & 0 & -s_2 & -s_1 \end{pmatrix}.$$

The matrix S is called Schwarz matrix. It is also known [4] that a nonderogatory stability matrix A is similar to the real Routh matrix

$$R=\begin{pmatrix} b_1 & b_2 & 0 & \cdots & 0 \\ -b_2 & 0 & b_3 & 0 & 0 \\ 0 & -b_3 & 0 & b_4 & 0 \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ 0 & 0 & \cdots & 0 & b_n \\ 0 & 0 & \cdots & -b_n & 0 \end{pmatrix}$$

where $b_1, b_2, \cdots, b_n$ are nonzero and

$$b_1=-s_1 \text{ and } b_i^2=s_i, \qquad i=2,3,\cdots,n.$$

The aim of this correspondence is to present a more general result that follows.

II. Statement of the Main Result

Theorem 1: Let A be a nonderogatory matrix. Then a necessary and sufficient condition for A to be similar to R is that either A is a stability matrix or else has all its eigenvalues in right-half plane. When A is similar to R, eigenvalues are all in right- or left-half plane according as b_1 is positive or negative.

The following lemma plays the main role in the proof of Theorem 1.

Lemma 1: R is either a stability matrix or else has all its eigenvalues in right-half plane if and only if $b_1, b_2, \cdots, b_n$ are all nonzero.

The proof of this lemma is based upon a recent inertia theorem due to Chen and Wimmer.

III. An Inertia Theorem

The inertia of an arbitrary matrix A is defined to be a triplet ($\pi(A)$, $\nu(A)$, $\delta(A)$), where $\pi(A)$, $\nu(A)$, and $\delta(A)$ are, respectively, number of eigenvalues of A with positive, negative, and zero real parts. Inertia of A is denoted by $IN(A)$. The following inertia theorem has been recently proved by Chen [1] and independently by Wimmer [7].

Theorem 2: Let A be a $n\times n$ matrix and let there exist a Hermitian matrix H so that the matrix N given by

$$AH+HA^*=N$$

where $A^*=(\bar{A})^T$, is positive semidefinite and the rank of $[A, N]$ be n. Then $\delta(A)=0$ and $IN(A)=IN(H)$.

IV. Proof of the Lemma

Let $b_1, b_2, \cdots, b_n$ be all nonzero. Then it is easy to see that the diagonal matrix $D=dg(b_1, b_1, b_i, \cdots, b_1)$ is such that $RD+DR^T=P=dg(2b_1^2, 0, \cdots, 0)$ is positive semidefinite. Also, the rank of $[R, P]$ is n, because the $n\times n$ matrix formed by taking the 1st to $(n-1)$th column of R and the first column of P is clearly nonsingular having determinant $\pm 2b_1^2 b_2 b_3 \cdots b_n$. So, by Theorem 2, $\delta(R)=0$ and furthermore, $IN(R)=IN(D)$.

Thus, when b_1 is positive, R has all its eigenvalues in right-half plane and when b_1 is negative, R is a stability matrix.

Conversely, if any of b_i, $i=2,3,\cdots,n-1$ is zero, then

$$R=\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

where A_1 is a $(i-1)\times(i-1)$ matrix and A_2 is a real skew symmetric matrix of order $(n-i+1)$. Also, if $b_n=0$, R is singular and $b_1=0$ makes the whole matrix R real skew symmetric. Thus, vanishing any of b_1, $b_2, \cdots, b_n$ implies that $\delta(R)\neq 0$ contradicting the assumption that R has all its eigenvalues either in left- or right-half plane.

V. Proof of the Main Result

Necessity: Since two similar matrices have same eigenvalues, necessity follows immediately from the Lemma 1.

Sufficiency: Since A is nonderogatory, there exists a nonsingular T such that

$$TAT^{-1}=C=\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & 1 \\ a_1 & a_2 & \cdots & \cdots & a_n \end{pmatrix}$$

where C is the companion form of A. Let $\Delta_1, \Delta_2, \cdots, \Delta_n$ be the Hurwitz determinants of $\det(\lambda I-C)$ and let us choose

$$b_1=-\Delta_1,\ b_2^2=\frac{\Delta_2}{\Delta_1},\ b_3^2=\frac{\Delta_3}{\Delta_1\Delta_2}$$

$$b_r^2=\frac{\Delta_{r-3}\Delta_r}{\Delta_{r-2}\Delta_{r-1}}, \qquad (r=4,5,\cdots,n).$$

Then, in case A is a stability matrix, by Hurwitz's criterion of stability [2, p. 194], Δ_i, $i=1,\cdots,n$ are positive and thus, $b_1, b_2, \cdots, b_n$ are all real. In

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