Decentralized Control of Interconnected Dynamical Systems

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***Abstract*—In this paper we consider the decentralized stabilization problem for a class of large systems formed by the dynamic interconnection of several multivariable systems. For this structured class of systems, we establish the conditions under which the interconnected system is controllable and observable. We then simplify and interpret these conditions to obtain simple sufficient conditions that guarantee controllability and observability in terms of the subsystem and interconnection subsystem coefficient matrices. Also, the conditions under which stabilization using decentralized feedback is possible are explicitly stated. We then simplify these to obtain sufficient conditions at the subsystem level. These conditions imply that if the interaction subsystems are stable and, in addition, certain mild restrictions on the subsystems and the interconnections hold, then the large system is stabilizable with decentralized feedback. Finally, we state the sufficient conditions for stabilizing this class of systems via local state feedback.**

I. Introduction

There has been a great deal of interest in the area of decentralized control of large scale interconnected systems [1]–[4]. This is a direct result of the need to analyze large scale technological systems like power systems [5], computer communication networks [6], transportation systems and process control systems [1], for stabilizability under constrained feedback. This paper examines the stabilizability under local feedback of a specially structured class of interconnected systems which appear naturally in many practical situations. These systems are those formed by a dynamic interconnection of several subsystems.

Previous results in the literature on stabilization and regulation via decentralized feedback were mainly concerned with either interconnected systems with constant (static) interconnections or with large multivariable systems. For systems with constant interactions, Sezer and Hussein [4], Davison [10], Saeks [11], deal with the question of decentralized stabilization. We may mention the excellent development of the corresponding results for large multivariable systems given by Corfmat and Morse [7] and Wang and Davison [8]. Chan and Desoer consider a class of dynamically interacting interconnected systems for certain stability studies using summing node and column subsystem notions [3], [9].

Our aim here is to develop simple sufficient conditions under which a class of large systems with dynamic interactions is jointly controllable and observable and is also stabilizable by means of decentralized feedback. These conditions are given in terms of the subsystem coefficient matrices and are easy to check. This paper is organized as follows. In Section II, we describe two interconnected system structures dealt with in this paper and formulate the main problems of interest. Some preliminary results required in the later development are stated in Section III. In Section IV we develop the necessary and sufficient conditions under which the large system is controllable, and weaken these to obtain simpler sufficient conditions which require less computational effort. The corresponding results on decentralized stabilization are presented in Section V. Section VI deals with the design of stabilizing controllers based on local state feedback.

Notation

$A(s)$ denotes a polynomial matrix, i.e., a matrix with polynomial entries, $M(s)\overset{S}{\sim}N(s)$ denotes that $M(s)$ is Smith (form) equivalent to $N(s)$, $M(s)\sim N(s)$ denotes $M(s)$ and $N(s)$ are equivalent up to elementary operations. With a set $K\triangleq\{1,2,\cdots,k\}$, we have $\mathbf{K}\triangleq$ power set of K (modulo K), i.e., the set of all proper subsets of K. $\mathbf{K}(i)$ denotes the set of all proper subsets of K containing i. q denotes any proper subset $\{n_1,n_2,\cdots,n_q\}$ of K, i.e., $q\in\mathbf{K}$. Given any matrix M, M_q denotes the submatrix of M associated with $q\in\mathbf{K}$. M_{K-q} denotes the submatrix of M associated with the proper subset $K/q\triangleq\{n_{q-1},n_{q+2},\cdots,n_j,\cdots,n_k;\ n_j\notin q\}$, also called the complement of q. $\sigma(A)$ denotes the set of eigenvalues of a matrix A. Block diag. (F_i) denotes a matrix with block diagonal elements F_i.

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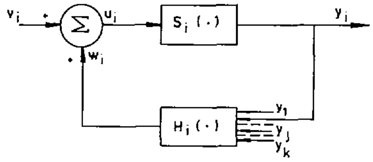

Fig. 1. The ith subsystem of Structure I.

II. Problem Formulation

In this section, we present two interconnected system structures, wherein the subsystems interact with each other through dynamic output feedback. We also state the decentralized control problems treated in this paper relating to these structures.

Structure I:

Consider the large system formed by the interconnection of the k subsystems described by

$$S_i:\ \dot{x}_i=A_ix_i+B_iu_i;\qquad i=1,2,\cdots,k \tag{1a}$$

$$y_i=C_ix_i \tag{1b}$$

and the interaction subsystems given by

$$H_i:\ \dot{z}_i=M_iz_i+\sum_{j=1}^{k}L_{ij}y_j \tag{1c}$$

$$w_i=N_iz_i+\sum_{j=1}^{k}P_{ij}y_j,\qquad i=1,2,\cdots,k \tag{1d}$$

according to the interconnection rule

$$u_i=v_i+w_i. \tag{1e}$$

We assume that (C_i,A_i,B_i) is a controllable and observable triple, $i=1,2,\cdots,k$, where $x_i\in R^{n_i}$ are the states of the subsystems, $u_i\in R^{m_i}$, $y_i\in R^{p_i}$ are the corresponding inputs and outputs, $z_i\in R^{a_i}$, $w_i\in R^{m_i}$ are the states and outputs of the ith interaction subsystem, respectively. $v_i\in R^{m_i}$ is the external input to the ith subsystem, and the coefficient matrices are of compatible dimensions.

Schematically the ith subsystem would look as shown in Fig. 1. Such structures arise in practical systems such as a countercurrent heat exchanger [1].

Structure II:

Another large system, with a more detailed dynamic interconnection structure, consists of k subsystems having the state-space description

$$S_i:\ \dot{x}_i=A_ix_i+B_iu_i;\qquad i=1,2,\cdots,k \tag{2a}$$

$$y_i=C_ix_i \tag{2b}$$

with interaction subsystems given by

$$H_{ij}:\ \dot{z}_{ij}=M_{ij}z_{ij}+L_{ij}y_j;\qquad i,j=1,2,\cdots,k \tag{2c}$$

$$w_{ij}=N_{ij}z_{ij}+P_{ij}y_j \tag{2d}$$

and interconnected according to

$$u_i=v_i+\sum_{j=1}^{k}w_{ij}. \tag{2e}$$

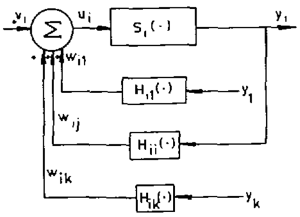

Fig. 2. The ith subsystem of Structure II.

Here also, we assume that each (C_i, A_i, B_i), $i=1,2,\cdots,k$, is a controllable, observable triple. Fig. 2 shows a schematic for the ith subsystem of this structure.

The following questions concerning the above systems are of interest.

i) Under what conditions are the composite systems described by (1) and (2) controllable and observable?

ii) What are the conditions on the subsystem and interconnection parameters to guarantee stable fixed modes, i.e., stabilization using decentralized dynamic output feedback?

iii) Under what conditions can local state feedback of the type $u_i = F_i x_i + v_i$ stabilize the composite system?

We shall provide answers to these questions in the following sections. We note that ii) and iii) are distinct problems. Even in the centralized case, there are stronger conditions required for stabilization with dynamic output feedback (controllability and observability of the system) than for stabilization with state feedback (controllability of the system). As we shall see later, the conditions corresponding to cases ii) and iii), in the decentralized context, are also different.

III. Preliminary Results

In this section, we summarize the definitions and results that are needed in the development to follow. Here we consider large multivariable systems which are strongly connected [7]. The results obtained can be easily extended to include nonstrongly connected systems. A strongly connected system is one in which, after local feedback, a transfer path exists from every input channel to every output channel. That is, every node is connected to every other node in the graph of (1). This has been shown [7] equivalent to the requirement that all the complementary subsystems (see Definition 2 below) have nonzero transfer function matrices.

We now define the remnant polynomial which is closely related to the fixed modes, and which we use frequently in the ensuing discussions.

Definition 1 (Remnant polynomial): The remnant polynomial $t(C, A, B)$ of the triple (C, A, B) is defined as the product of the first n invariant polynomials of the system matrix

$$\begin{bmatrix} sI-A & B \\ C & 0 \end{bmatrix}.$$

Definition 2 (Complementary subsystems): The triples (C_{K-q}, A, B_q) with $q \in \boldsymbol{K}$, are called the complementary subsystems of the system (C, A, B). Here $B_q = (B_{i_1}, \cdots, B_{i_q})$ and $C_{K-q} = (C_{j_1}, \cdots, C_{j_{k-q}})$ with $i_j \in q$ and $j_i \in K - q$.

Definition 3: A triple (C, A, B) is called complete if its transfer matrix $C(sI-A)^{-1}B$ is nonzero, and $t(C, A, B) = 1$.

Now, we state the following results due to Corfmat and Morse. Proposition 1 establishes conditions under which a multivariable system can be made single channel controllable whereas Proposition 2 deals with the decentralized stabilization problem.

Proposition 1 [7]: Consider the k-channel multivariable system

$$\mathbb{S}: \dot{x} = \hat{A}x + \sum_{i=1}^{k} \hat{B}_i u_i$$

$$y_i = \hat{C}_i x, \qquad i=1,2,\cdots,k$$

with the decentralized feedback

$$u_i = F_i y_i, \qquad i=1,2,\cdots,k.$$

Then, for almost any F_d, and for any $j=1,2,\cdots,k$,

i) $(\hat{A} + \hat{B}F_d\hat{C}, \hat{B}_j)$ is a controllable pair if and only if $(\hat{A}, \hat{B})$ is a controllable pair and each complementary subsystem $(\hat{C}_{K-q}, \hat{A}, \hat{B}_q)$ containing input channel j is complete, i.e., $\hat{C}_{K-q}(sI-\hat{A})^{-1}\hat{B}_q \not\equiv 0$ and $t(\hat{C}_{K-q}, \hat{A}, \hat{B}_q) = 1$, for all $q \in \boldsymbol{K}(j)$.

ii) Further, $(\hat{A} + \hat{B}F_d\hat{C}, \hat{B}_j)$ is stablizable if and only if $(\hat{A}, \hat{B})$ is stabilizable and if $\hat{C}_{K-q}(sI-\hat{A})^{-1}\hat{B}_q \not\equiv 0$ and $t(\hat{C}_{K-q}, \hat{A}, \hat{B}_q)$ is stable for all $q \in \boldsymbol{K}(j)$, where $\hat{B} = [\hat{B}_1, \hat{B}_2, \cdots, \hat{B}_k]$; $\hat{C} = [\hat{C}_1', \hat{C}_2', \cdots, \hat{C}_k']'$; $F_d =$ Block diag. $(F_1, F_2, \cdots, F_k)$.

Proposition 2 [7]: Consider the strongly connected system $\mathbb{S}$ in Proposition 1 under decentralized feedback $u_i = F_i y_i$, $i=1,2,\cdots,k$. Then the following hold.

i) The eigenspectrum of $(\hat{C}_j, \hat{A} + \hat{B}F_d\hat{C}, \hat{B}_j)$ is freely assignable, if and only if $(\hat{A}, \hat{B})$ is a controllable pair, $(\hat{C}, \hat{A})$ is an observable pair, and the remnant polynomials corresponding to each of the complementary subsystems are unity, i.e., $t(\hat{C}_{K-q}, \hat{A}, \hat{B}_q) = 1$, for all $q \in \boldsymbol{K}$.

ii) The triple $(\hat{C}_j, \hat{A} + \hat{B}F_d\hat{C}, \hat{B}_j)$ is stabilizable if and only if $(\hat{A}, \hat{B})$ is stabilizable, $(\hat{C}, \hat{A})$ is detectable and $t(\hat{C}_{K-q}, \hat{A}, \hat{B}_q)$ is stable, for all $q \in \boldsymbol{K}$.

Remark 1: Proposition 2 indicates clearly the role of the remnant polynomials of the complementary subsystems in proving the stability under decentralized feedback. It is evidently sufficient to prove that each of the remnant polynomials is stable, to prove the stabilizability under decentralized dynamic output feedback of the system.

IV. Controllability and Observability

A. Controllability

Here, we develop the conditions under which the large interconnected system (Structure I) described by (1a)–(1e) is controllable. The corresponding results for Structure II follow similarly and are summarized towards the end of the section.

To proceed, we rewrite (1) as

$$\dot{\bar{x}} = \begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A+BPC & BN \\ LC & M \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} v$$

$$= \bar{A}\bar{x} + \bar{B}v \tag{3a}$$

$$y = [C \quad 0] \begin{bmatrix} x \\ z \end{bmatrix} = \bar{C}\bar{x} \tag{3b}$$

where

$$z = (z_1, \cdots, z_k)', \quad x = (x_1, \cdots, x_k)' \tag{4a}$$

and for $i, j = 1, 2, \cdots, k$

$A =$ Block diag. (A_i), $B =$ Block diag. (B_i),

$C =$ Block diag. (C_i), (4b)

$P = [P_{ij}]$, $N =$ Block diag. (N_i), $L = [L_{ij}]$,

$M =$ Block diag. (M_i).

Also, let $n = \sum n_i$, $a = \sum a_i$, $p = \sum p_i$, $m = \sum m_i$. (4c)

In what follows, we use the term joint controllability of Structure I to imply controllability of the pair $(\bar{A}, \bar{B})$, i.e., the interconnected system (1) is jointly controllable if and only if [12]

$$\operatorname{rank}(sI - \bar{A} \quad \bar{B}) = n + a, \qquad \text{for all } s. \tag{5}$$

More explicitly, using (3), (5) can be written as

$$\operatorname{rank}\bar{U}(s) = \operatorname{rank}\begin{bmatrix} sI-A-BPC & -BN & B \\ -LC & sI-M & 0 \end{bmatrix} = n+a, \qquad \text{for all } s. \tag{6}$$

We then have the following.

Lemma 1: The system described by (3) and (4) is jointly controllable if and only if

i) (A, B) is a controllable pair

ii) $\operatorname{rank} U(s) = \operatorname{rank}\begin{bmatrix} sI-A & B & 0 \\ -LC & 0 & sI-M \end{bmatrix} = n + a$, for all $s \in \sigma(M)$

where $\sigma(M)$ is the set of eigenvalues of M.

Proof: It can be easily seen that $\bar{U}(s)$ is column equivalent to $U(s)$, and thus, rank $\bar{U}(s)=$ rank $U(s)$, for all s.

Sufficiency: Suppose (A,B) is a controllable pair, then rank$(sI-A\ B)=n$, for all s. For $s\notin\sigma(M)$, $(sI-M)$ is nonsingular and hence rank $U(s)=n+a$. Thus, if rank $U(s)=n+a$ for all $s\in\sigma(M)$, then rank $U(s)=n+a$ for all s, i.e., $(\bar{A},\bar{B})$ is a controllable pair.

Necessity: The necessity of condition ii), i.e., rank $U(s)=n+a$ for $s\in\sigma(M)$ is obvious and that of i) is clear from the fact that if i) does not hold, then ii) cannot be true. □

The necessary and sufficient conditions given in Lemma 1 are computationally burdensome. The following theorem provides elegant sufficient conditions, while its corollary (Corollary 1) provides simple sufficient conditions which require computations on the subsystem matrices only.

Theorem 1: The interconnected system described by (3) and (4) (Structure I) is controllable if

i) (A,B) is a controllable pair

ii) (M,L) is a controllable pair

iii) $$\operatorname{rank}\begin{bmatrix} sI-A & B\\ C & 0\end{bmatrix}=n+p, \qquad \text{for all } s\in\sigma(M). \tag{7}$$

Proof: In view of Lemma 1, the theorem is proved if we can show that ii) and iii) together imply that rank $U(s)=n+a$, $s\in\sigma(M)$.

To this end, note that $U(s)$ can be rewritten as

$$U(s)=\begin{bmatrix} I_n & 0 & 0\\ 0 & -L & sI-M\end{bmatrix}\begin{bmatrix} sI-A & B & 0\\ C & 0 & 0\\ 0 & 0 & I_a\end{bmatrix}$$
$$\triangleq U_1(s)U_2(s).$$

For $s\in\sigma(M)$, conditions ii) and iii) imply rank $U_1(s)=n+a$, rank $U_2(s)=n+p+a$, respectively.

Using Sylvester's inequality, we now have

$$n+a\geq \operatorname{rank} U(s)\geq n+a+(n+p+a)-(n+p+a),$$

i.e.,

$$\operatorname{rank} U(s)=n+a, \qquad \text{for all } s\in\sigma(M).$$ □

We can easily obtain the following corollary, by making use of the block diagonal structure of A, B, and C given in (4b). Note that this corollary provides conditions which are given in terms of the subsystem matrices, thus providing computational advantage.

Corollary 1: The interconnected system described by (3) and (4) (Structure I) is jointly controllable if

i) (A_i,B_i) is a controllable pair, $i=1,2,\cdots,k$

ii) $(M_i,L_{i,i})$ is a controllable pair, $i=1,2,\cdots,k$

iii) $$\operatorname{rank}\begin{bmatrix} sI-A_i & B_i\\ C_i & 0\end{bmatrix}=n_i+p_i \qquad \text{for all } s\in\sigma(M),\ i=1,2,\cdots,k.$$

The sufficient conditions provided by Theorem 1 and Corollary 1 are interesting. In addition to controllability of the subsystems, joint controllability of (3) requires that the eigenvalues of the interaction subsystems H_i [(1c)–(1d)] should not coincide with the invariant zeros of the subsystems (1a) and (1b). If, however, all the eigenvalues of $\sigma(M)$ are stable, then we see that any uncontrollable modes that may be present, are stable. We have thus obtained a sufficient condition for stabilizability, which is given in the following corollary.

Corollary 2: The interconnected system in Corollary 1 is stabilizable, if, for $i=1,2,\cdots,k$, (A_i,B_i) is a controllable pair, and $\sigma(M)$ is stable.

The results for observability can be obtained easily by dualizing the above results.

We summarize the joint controllability results for Structure II below.

Theorem 2: The joint controllability of (2), i.e., Structure II, holds if, with $\tilde{L}_i\triangleq[L'_{1i},L'_{2i},\cdots,L'_{ki}]'$, $\tilde{M}_i\triangleq$ Block diag.$(M_{1i},M_{2i},\cdots,M_{ki})$, for $i=1,2,\cdots,k$:

i) (A_i,B_i) is a controllable pair

ii) $(\tilde{M}_i,\tilde{L}_i)$ is a controllable pair

iii) $$\operatorname{rank}\begin{bmatrix} sI-A_i & B_i\\ C_i & 0\end{bmatrix}=n_i+p_i, \qquad \text{for all } s\in\sigma(M).$$

B. Structural Extensions

The results in Section IV-A on the controllability and stabilizability of Structure I were derived assuming that the coefficient matrices in (1) are all fixed. It is possible to derive an alternate set of necessary and sufficient conditions for joint controllability and to weaken them to provide simpler sufficient conditions, by using Proposition 1. These results are valid for almost any (arbitrarily structured) L, whereas those presented earlier (Theorem 1 and Corollaries 1 and 2) are valid for a given L. The special case when L is block diagonal is also treated. Specifically, we prove the following results.

Theorem 3: Consider the system described by (3) and (4) (Structure I). Let (A,B) be controllable and let $C(sI-A)^{-1}B\not\equiv 0$. For almost any L, (5) holds, i.e., $(\bar{A},\bar{B})$ is controllable if and only if

$$\operatorname{rank} T(s)=\operatorname{rank}\left[\begin{array}{cc|c} sI-A & B & 0\\ C & 0 & 0\\ \hline 0 & 0 & sI-M\end{array}\right]\geq n+a, \qquad \text{for all } s. \tag{8}$$

Proof: We first note that $U(s)$ can be written as

$$U(s)=\left[\begin{bmatrix} sI-A & 0\\ 0 & sI-M\end{bmatrix}+\begin{bmatrix}0\\ I\end{bmatrix}L[C\ \ 0]\ \Big|\ \begin{bmatrix}B\\ 0\end{bmatrix}\right].$$

Also, the pair $\left[\begin{bmatrix} A & 0\\ 0 & M\end{bmatrix},\begin{bmatrix} B & 0\\ 0 & I\end{bmatrix}\right]$ is controllable.(Note that (A,B) is controllable by assumption.) The transfer function matrix corresponding to the triple $[C\ \ 0]$, $\begin{bmatrix} A & 0\\ 0 & M\end{bmatrix}$, $\begin{bmatrix} B & 0\\ 0 & I\end{bmatrix}$ is $C(sI-A)^{-1}B$ which by assumption is $\not\equiv 0$. Further L is an unconstrained matrix. Now applying Proposition 1 to our problem with $k=2$ and with the identifications

$$\hat{A}=\begin{bmatrix} A & 0\\ 0 & M\end{bmatrix},\quad \hat{B}_1=\begin{bmatrix} B\\ 0\end{bmatrix},\quad \hat{B}_2=\begin{bmatrix} 0\\ I\end{bmatrix},$$
$$F_2=L,\quad \hat{C}_2=[C\ \ 0]$$

we get the result that rank $U(s)=n+a$, for all s, if and only if condition (8) of Theorem 3 holds. □

We now state a theorem which gives a condition which is equivalent to that of Theorem 3, but is easier to test.

Theorem 4: The system considered in Theorem 3 is controllable for almost any L, if and only if

$$\operatorname{rank}\begin{bmatrix} sI-A & B\\ C & 0\end{bmatrix}\geq n+a-\operatorname{rank}(sI-M), \qquad \text{for all } s\in\sigma(M). \tag{9}$$

Proof: Suppose (9) holds; then it can be easily seen that for all $s\in\sigma(M)$,

$$\operatorname{rank}\begin{bmatrix} sI-A & B & 0\\ C & 0 & 0\\ 0 & 0 & sI-M\end{bmatrix}\geq n+a. \tag{10}$$

Again, for $s\notin\sigma(M)$, (10) still holds since $(sI-M)$ is nonsingular, and $\operatorname{rank}\begin{bmatrix} sI-A & B\\ C & 0\end{bmatrix}\geq n$, for all s, due to the controllability of (A,B). Thus, we have rank $T(s)\geq n+a$, for all s. This proves that (9) implies (8).

We prove the converse by contradiction. Suppose (9) does not hold, i.e., $\operatorname{rank}\begin{bmatrix} sI-A & B\\ C & 0\end{bmatrix}<n+a-\operatorname{rank}(sI-M)$ for some $s\in\sigma(M)$, then, from the block diagonal structure of $T(s)$, we get, rank $T(s)<n+a$, for some $s\in\sigma(M)$, which contradicts (8). □

As before, it would be of interest to investigate whether (9) could be simplified to provide simple conditions at the subsystem level. The following corollary provides precisely these conditions utilizing the block diagonal structure of the matrices A, B, C, and M.

Corollary 3: The joint controllability of $(\bar{A},\bar{B})$ holds for almost any L of Structure I if for $i=1,2,\cdots,k$

i) (A_i, B_i) is a controllable pair

ii) $\text{rank}\begin{bmatrix} sI-A_i & B_i \\ C_i & 0 \end{bmatrix} \geq n_i + a_i - \text{rank}(sI - M_i),$

for all $s \in \sigma(M_i)$.

Remark 2: Note that in checking for joint controllability in Theorem 4 and Corollary 3, it is sufficient to check the conditions therein only at $s \in \sigma(M)$, i.e., at a finite number of points.

Remark 3: For each i, we check the conditions of Corollary 1 at all $s \in \sigma(M)$, whereas the conditions of Corollary 3 needs to be checked only at $s \in \sigma(M_i)$. Furthermore, even at $s \in \sigma(M_i)$, the condition of Corollaries 1 and 3 are identical only when $a_i - \text{rank}(sI - M_i) = p_i$.

Corollary 4: $(\bar{A}, \bar{B})$ of Structure I is a stabilizable pair for almost any (arbitrarily structured) L, if for $i = 1, 2, \cdots, k$, conditions (i) of Corollary 3 holds and $\sigma(M_i)$ is stable.

It is important to note that Theorems 3 and 4 and Corollaries 3 and 4 are valid only for an L that is arbitrarily structured, i.e., no element of L is constrained to have a fixed value. However, for the special case where L is constrained to be block diagonal, with the block diagonal elements being (arbitrary and) unconstrained, necessary and sufficient conditions can be obtained, using Proposition 1. We then have the following.

Theorem 5: The system described by (3) and (4) (Structure I) is jointly controllable, i.e., (6) holds, for almost any L with a block diagonal structure if and only if, for $i = 1, 2, \cdots, k$:

i) (A_i, B_i) is a controllable pair

ii) $\text{rank}\begin{bmatrix} sI-A_i & B_i \\ C_i & 0 \end{bmatrix} \geq n_i + a_i - \text{rank}(sI - M_i), \quad \text{for all } s \in \sigma(M_i).$

Proof: We prove this theorem using Proposition 1 for the case of $(k+1)$ channels. We also need to make the identifications:

$$\hat{B}_{k+1} = \begin{bmatrix} B \\ 0 \end{bmatrix},$$

$$\hat{B}_i = \begin{bmatrix} 0 \\ I_{a_i} \\ 0 \end{bmatrix},$$

$$\hat{C}_i = [0 \quad C_i \quad 0], \qquad i = 1, 2, \cdots, k,$$

$$\hat{A} = \begin{bmatrix} A & 0 \\ 0 & M \end{bmatrix}. \tag{11}$$

Now, applying Proposition 1, with $j = k+1$, we have the following necessary and sufficient conditions for joint controllability to hold, i.e., for (5) to hold.

a) $\hat{C}_{K+1-q}[sI - \hat{A}]^{-1}\hat{B}_q \not\equiv 0$

b) $t(\hat{C}_{K+1-q}, \hat{A}, \hat{B}_q) = 1 \qquad \text{for all } q \in \boldsymbol{K+1}(k+1)$

Simplifying these conditions, using the diagonal structure of A, B, C, and M, gives the result. □

In addition to being interesting in its own right, Theorem 5 is useful as its direct dual gives us the conditions for joint observability. In contrast, the block diagonal structure of N makes impossible a direct dualization of Theorems 3 and 4.

Finally, we have the following corollary to Theorem 5.

Corollary 5: The system described by (3) and (4) (Structure I) is stabilizable for almost any block diagonal L, if condition i) of Theorem 5 holds and $\sigma(M_i)$ is stable, for $i = 1, 2, \cdots, k$.

The results for Structure II follow similarly.

V. Decentralized Stabilization of Interconnected Systems

A. Decentralized Stabilization

In this section, we obtain the main results of this paper concerning the decentralized stabilization of the interconnected system (Structure I). Proposition 2, applied to Structure I, described by (3) and (4), helps us obtain the conditions which guarantee the stabilizability of the dynamically interconnected system under decentralized feedback. We then simplify and interpret these conditions to obtain simple sufficient conditions which are easy to test.

In the light of Remark 1, it is enough to investigate whether the remnant polynomials of the complementary subsystems of the interconnected system (3) and (4) are stable. To this end, we identify the ith channel of our dynamically interconnected system with the input and output groups of the ith local subsystem. To use Proposition 2, we partition the vectors and matrices of (3) and (4) for Structure I, in the complementary subsystem format:

$$x = \begin{bmatrix} x_q \\ x_{K-q} \end{bmatrix}, \quad u = \begin{bmatrix} u_q \\ u_{K-q} \end{bmatrix}, \quad z = \begin{bmatrix} z_q \\ z_{K-q} \end{bmatrix}, \quad v = \begin{bmatrix} v_q \\ v_{K-q} \end{bmatrix},$$

$B_q = \text{Block diag.}(B_{i_1}, \cdots, B_{i_q})$, $C_{K-q} = \text{Block diag.}(C_{j_1}, \cdots, C_{j_{k-q}})$ with $i_j \in q, j_i \in K - q$.

$$A = \begin{bmatrix} A_q & 0 \\ 0 & A_{K-q} \end{bmatrix}, \quad B = \begin{bmatrix} B_q & 0 \\ 0 & B_{K-q} \end{bmatrix} = \begin{bmatrix} \bar{\bar{B}}_q & \bar{\bar{B}}_{K-q} \end{bmatrix},$$

$$N = \begin{bmatrix} N_q & 0 \\ 0 & N_{K-q} \end{bmatrix}, \quad M = \begin{bmatrix} M_q & 0 \\ 0 & M_{K-q} \end{bmatrix},$$

$$P = \begin{bmatrix} P_{q,q} & P_{q,K-q} \\ P_{K-q,q} & P_{K-q,K-q} \end{bmatrix}, \quad L = \begin{bmatrix} L_{q,q} & L_{q,K-q} \\ L_{K-q,q} & L_{K-q,K-q} \end{bmatrix}$$

$$C = \begin{bmatrix} C_q & 0 \\ 0 & C_{K-q} \end{bmatrix} = \begin{bmatrix} \bar{\bar{C}}_q \\ \bar{\bar{C}}_{K-q} \end{bmatrix}. \tag{12a}$$

We also make the identifications from (3),

$$\bar{A} = \begin{bmatrix} A + BPC & BN \\ LC & M \end{bmatrix},$$

$$\bar{B}_q = \begin{bmatrix} B_q \\ 0 \end{bmatrix} = \begin{bmatrix} B_q \\ 0 \\ 0 \end{bmatrix}; \quad \bar{B}_{K-q} = \begin{bmatrix} \bar{\bar{B}}_{K-q} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ B_{K-q} \\ 0 \end{bmatrix}$$

$$\bar{C}_q = \begin{bmatrix} \bar{\bar{C}}_q & 0 \end{bmatrix} = [C_q \quad 0 \quad 0], \quad \bar{C}_{K-q} = \begin{bmatrix} \bar{\bar{C}}_{K-q} & 0 \end{bmatrix} = [0 \quad C_{K-q} \quad 0] \tag{12b}$$

so that the remnant polynomial $t(\bar{C}_{K-q}, \bar{A}, \bar{B}_q)$ $(q \in \boldsymbol{K})$ of the interconnected system (3) can be computed and simplified in terms of the parameters of the subsystems (1a)–(1d). More specifically, for all $q \in \boldsymbol{K}$, $t(\bar{C}_{K-q}, \bar{A}, \bar{B}_q)$ now equals the product of the first $(n+a)$ invariant polynomials of $R_q(s)$ defined below. Now,

$$R_q(s) \triangleq \begin{matrix} & n+a & m_q \\ n+a & \multirow{2}{*}{} & \\ p_{K-q} & & \end{matrix} \begin{bmatrix} sI - \bar{A} & \bar{B}_q \\ \bar{C}_{K-q} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} sI - (A_q + B_q P_{q,q} C_q) & -B_q P_{q,K-q} C_{K-q} & -B_q N_q & 0 & B_q \\ -B_{K-q} P_{K-q,q} C_q & sI - (A_{K-q} + B_{K-q} P_{K-q,K-q} C_{K-q}) & 0 & -B_{K-q} N_{K-q} & 0 \\ -L_{q,q} C_q & -L_{q,K-q} C_{K-q} & sI - M_q & 0 & 0 \\ -L_{K-q,q} C_q & -L_{K-q,K-q} C_{K-q} & 0 & sI - M_{K-q} & 0 \\ 0 & C_{K-q} & 0 & 0 & 0 \end{bmatrix} \tag{13}$$

where $m_q \triangleq \sum_{j \in q} m_j$, $p_{K-q} \triangleq \sum_{j \in K-q} p_j$.

If, for a given q, $\text{rank}\, R_q(s) \geq n + a$, for all s, i.e., the remnant polynomial equals unity, there are no fixed modes due to $R_q(s)$ (i.e., due to $t(\bar{C}_{K-q}, \bar{A}, \bar{B}_q)$). If the remnant polynomial $t(\bar{C}_{K-q}, \bar{A}, \bar{B}_q)$ is stable, then it contributes only stable fixed modes. Starting with (13), we simplify $R_q(s)$ to obtain convenient conditions guaranteeing stable fixed modes.

$$R_q(s) \overset{S}{\sim} \begin{array}{c} \\ n_q \\ a_q \\ n_{K-q} \\ p_{K-q} \\ a_{K-q} \end{array} \begin{array}{c} \begin{array}{ccc|cc} n_q & m_q & a_q & n_{K-q} & a_{K-q} \end{array} \\ \left[\begin{array}{ccc|cc} sI-A_q & B_q & 0 & 0 & 0 \\ -L_{q,q}C_q & 0 & sI-M_q & 0 & 0 \\ \hline -B_{K-q}P_{K-q,q}C_q & 0 & 0 & sI-A_{K-q} & -B_{K-q}N_{K-q} \\ 0 & 0 & 0 & C_{K-q} & 0 \\ -L_{K-q,q}C_q & 0 & 0 & 0 & sI-M_{K-q} \end{array}\right] \end{array}$$

$$\triangleq \begin{array}{c} \\ n_q+a_q \\ n_{K-q}+a_{K-q}+p_{K-q} \end{array} \begin{array}{c} \begin{array}{cc} n_q+a_q+m_q & n_{K-q}+a_{K-q} \end{array} \\ \begin{bmatrix} T_{1q}(s) & 0 \\ T_{21q}(s) & T_{2q}(s) \end{bmatrix} \end{array}.$$

We can now use the triangular structure of the above matrix to obtain our main result. If $T_{1q}(s)$ has rank $n_q + a_q$ (for all s) and $T_{2q}(s)=$ rank $n_{K-q} + a_{K-q}$ (for all s), then rank $R_q(s) = n + \underline{a}$ (for all s) and there will be no contribution to the fixed modes from $t(\bar{C}_{K-q}, \bar{A}, \bar{B}_q)$, for this particular q. If either $T_{1q}(s)$ or $T_{2q}(s)$ drops rank, $\sigma(M_q)$ and/or $\sigma(M_{K-q})$ are the only modes at which the ranks could drop, for $q \in \boldsymbol{K}$ (assuming (A_q, B_q) controllable and (C_{K-q}, A_{K-q}) observable, which is true from the controllability and observability of the subsystems). Hence, the set of fixed modes can only be a subset of the set of eigenvalues of M. We now use Theorem 1 and its dual to establish the conditions under which $T_{1q}(s)$ and $T_{2q}(s)$ have the requisite ranks or have stable invariant polynomials. The preceding arguments constructively establish the following theorem.

Theorem 6: Let the large system described by (3) and (4) be strongly connected. Then complete eigenvalue assignment can be achieved through decentralized dynamic feedback if, for all $q \in \boldsymbol{K}$,

i) $(M_q, L_{q,q})$ is a controllable pair

ii) (N_{K-q}, M_{K-q}) is an observable pair

iii) $$\operatorname{rank}\begin{bmatrix} sI-A_q & B_q \\ C_q & 0 \end{bmatrix} = n_q + p_q, \qquad \text{for all } s \in \sigma(M_q)$$

iv) $$\operatorname{rank}\begin{bmatrix} sI-A_{K-q} & B_{K-q} \\ C_{K-q} & 0 \end{bmatrix} = n_{K-q} + m_{K-q},$$

$$\text{for all } s \in \sigma(M_{K-q}). \quad \square$$

Note that conditions iii) and iv) of Theorem 6 imply that $m_i = p_i$, for $i = 1,2,\cdots,k$. To see this, use the diagonal structure of A_q, B_q, and C_q in iii) and note that iii) holds if and only if

$$\operatorname{rank} W_i(s) = \operatorname{rank}\begin{bmatrix} sI-A_i & B_i \\ C_i & 0 \end{bmatrix} = n_i + p_i,$$

$$\text{for all } s \in \sigma(M),\ i = 1,2,\cdots,k. \quad (14a)$$

Similarly, condition iv) holds if and only if,

$$\operatorname{rank} W_i(s) = n_i + m_i, \qquad \text{for all } s \in \sigma(M);\ i = 1,2,\cdots,k \quad (14b)$$

(14a) and (14b) together require the strong structural conditions $m_i = p_i$, $i = 1,2,\cdots,k$. In general, this would not hold, i.e., one of the conditions (14a) or (14b) would not be satisfied. The unassignable part of the spectrum of (3) under decentralized feedback will then consist of a subset of $\sigma(M_i)$, $i = 1,2,\cdots,k$.

In view of the above discussion, we shall be concerned in the further analysis only with the stabilization of (3) and (4), using decentralized feedback, and try to provide simpler sufficient conditions to achieve this goal. We note the following simplifications.

i) From the structure of M_q and $L_{q,q}$ (M_q is block diagonal), it is easy to see that $(M_i, L_{i,i})$ being a controllable pair for $i = 1,2,\cdots,k$, implies $(M_q, L_{q,q})$ is controllable.

ii) Also from the diagonal structure of N_q and M_q, we see that the observability of (N_i, M_i); $i = 1,2,\cdots,k$, implies that (N_q, M_q) is an observable pair.

Now, using the above facts along with Corollary 2 and its dual, we obtain the following elegant result, which gives sufficient conditions under which decentralized stabilization is possible.

Theorem 7: The strongly connected dynamically interconnected Structure I of (1) is stabilizable by decentralized dynamic feedback if, for all $i = 1,\cdots,k$.

i) (C_i, A_i, B_i) is a controllable, observable triple.

ii) $(N_i, M_i, L_{i,i})$ is a controllable, observable triple.

iii) $\sigma(M_i)$ is stable.

Intuitively, Theorem 7 is an interesting result, as only controllability and observability of the subsystems as also of the interconnection subsystems are being assumed, together with the requirement that the interconnection systems must have stable modes. Theorem 6 and the related simplifications imply that the transmission zeros of the subsystems must not coincide with the modes of the interconnections, if complete eigenvalue assignment is desired.

Before we conclude this subsection, we summarize the sufficient conditions for Structure II to be stabilizable with decentralized feedback.

Theorem 8: The strongly connected dynamically interconnected Structure II of (2) is stabilizable by decentralized dynamic feedback around the subsystems under the following sufficient conditions.

i) (C_i, A_i, B_i) is a controllable and observable triple, $i = 1,\cdots,k$.

ii) $(\tilde{M}_j, \tilde{L}_j)$ is a controllable pair, $j = 1,2,\cdots,k$.

iii) $(\hat{N}_j, \hat{M}_j)$ is an observable pair, $j = 1,2,\cdots,k$.

iv) $\sigma(M_{ij})$ are stable, $i, j = 1,2,\cdots,k, j \neq i$,

where $\tilde{M}_j$ and $\tilde{L}_j$ are as defined in Theorem 2 and $\hat{N}_j$ and $\hat{M}_j$ are defined by $\hat{N}_j \triangleq [N_{j1}, N_{j2},\cdots,N_{jk}]$; $\hat{M}_j \triangleq$ Block diag.(M_{ji}), $i = 1,2,\cdots,k$.

B. Structural Extensions

We now examine structural results that hold for almost any (arbitrarily structured) L and block diagonal N of Structure I. We now state and prove the main result on structural stabilizability under decentralized feedback.

Theorem 9: For almost any (arbitrarily structured) L and block diagonal N, of Structure I,

i) free spectrum assignment, using decentralized feedback, can be achieved if, for $i = 1,2,\cdots,k$,

a) (C_i, A_i, B_i) is a controllable, observable triple.

b) $$\operatorname{Rank}\begin{bmatrix} sI-A_i & B_i \\ C_i & 0 \end{bmatrix} \geq n_i + a_i - \operatorname{rank}(sI - M_i),$$

for all $s \in \sigma(M_i)$;

ii) stabilization can be achieved if condition i-a) holds and $\sigma(M_i)$ is stable, $i = 1,2,\cdots,k$.

The proof of this theorem is similar to that of Theorem 5, and is omitted.

VI. Stabilization Through Local State Feedback

In an interconnected system, it is sometimes possible to obtain the entire local state through measurements or by estimating the local states via observers. Then the question that arises is under what conditions we can stabilize (3) and (4) by using local state feedback controls

$$v_i = F_i x_i + w_i \qquad i = 1,2,\cdots,k. \quad (15)$$

We answer this question constructively below. For ease of presentation, we consider a two subsystem case. The k-subsystem results follow easily. Thus, with $k=2$, (3) and (15) result in

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} M_1 & 0 & L_{11}C_1 & L_{12}C_2 \\ 0 & M_2 & L_{21}C_1 & L_{22}C_2 \\ B_1N_1 & 0 & A_1+B_1P_{11}C_1+B_1F_1 & B_1P_{12}C_2 \\ 0 & B_2N_2 & B_2P_{21}C_1 & A_2+B_2P_{22}C_2+B_2F_2 \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \\ x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ B_1 & 0 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \quad (16)$$

i.e.,

$$\dot{\bar{z}} = \overline{W}\bar{z} + \overline{T}\bar{w} \quad (17)$$

with the appropriate identifications.

To obtain the conditions under which $\overline{W}$ can be made stable, as well as the feedback gains required to achieve this, we need the following preliminary result.

Lemma 2: Given the matrix

$$V = \begin{bmatrix} A_1 & B_1G_{12}C_2 \\ B_2G_{21}C_1 & A_2+B_2F_2 \end{bmatrix} \quad (18)$$

if

i) A_1 is stable,

ii) (A_2, B_2) is a controllable pair

it is possible to constructively choose an F_2 such that V is stable.

Proof: We omit the proof as it is very similar that of Lemma 2, Sezer and Huseyin [4]. We merely note that the constructive proof makes it possible to obtain explicit values of the feedback gain F_2 required to stabilize V. □

Now, we are in a position to prove the following result.

Theorem 10: The dynamically interconnected system (3) (Structure I) can be stabilized by local state feedback of the type (15), if, for $i = 1, 2, \cdots, k$,

i) M_i is stable;

ii) (A_i, B_i) is a controllable pair.

Proof: We prove the result for $k=2$, i.e., we prove the stabilizability of (16), through an appropriate choice of F_1, F_2. The k-subsystem results follow similiarly.

$$\text{Partition } \overline{W} \text{ as } \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}$$

with

$$W_{11} = \left[\begin{array}{c|c} \begin{matrix} M_1 & 0 \\ 0 & M_2 \end{matrix} & I\begin{bmatrix} L_{11} \\ L_{21} \end{bmatrix} C_1 \\ \hline B_1[N_1 \quad 0]I & \hat{A}_1 + B_1F_1 \end{array}\right] \quad (19)$$

where $\hat{A}_1 \triangleq A + B_1P_{11}C_1$.

From standard results in system theory, if (A_1, B_1) is a controllable pair, so is $(\hat{A}_1, B_1)$. It is now immediately obvious, applying Lemma 2, to (19), with appropriate identifications, that W_{11} is stable if

i) M_1, M_2 are stable. (20)

ii) (A_1, B_1) is a controllable pair (21)

and F_1 is chosen using the procedure described in the lemma.

Now, $\overline{W}$ of (17) can be partitioned as

$$\overline{W} = \left[\begin{array}{c|c} W_{11} & I\begin{bmatrix} L_{12} \\ L_{22} \\ B_1P_{12} \end{bmatrix} C_2 \\ \hline B_2[0 \quad N_2 \quad P_{21}C_1]I & \hat{A}_2 + B_2F_2 \end{array}\right]$$

with $\hat{A}_1 = A_2 + B_2P_{22}C_2$.

Once again, by Lemma 2, W is stable if (A_2, B_2) is a controllable pair, and F_2 is appropriately chosen, since W_{11} is stable (from (20) and (21) and our choice of F_1).

This gives us the theorem when $k=2$. The result for a general k follows similarly. □

VII. Conclusions

In this paper, we have considered the problem of stabilizing an interconnected system, and have obtained sufficient conditions for the stabilizability of certain practically important dynamic interconnection structures. There are other structures representative of practical situations and these could also be analyzed in a manner similar to the one given here. In fact, the multiarea load frequency control problem provides an example of another structure closely related to those considered in this paper.

While the sufficient conditions obtained are neat, it would be interesting to obtain possibly simple necessary and sufficient conditions for the system structures considered here. Also, in contrast to the local state feedback considered here for stabilization, the use of decentralized dynamic compensators is also an important problem awaiting solution.

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References

[1] M. G. Singh and A. Titli, *Systems: Decomposition, Optimization and Control.* Elmsford, NY: Pergamon, 1978.

[2] D. D. Siljak, *Large Scale Dynamic Systems.* New York: North Holland, 1978.

[3] W. S. Chan and C. A. Desoer, "Eigenvalue assignment and stabilization of interconnected systems using local feedbacks," *IEEE Trans. Automat. Contr.*, vol. AC-24, pp. 312–317, Apr. 1979.

[4] M. E. Sezer and O. Huseyin, "Stabilization of linear time invariant interconnected systems using local state feedback," *IEEE Trans. Syst., Man, Cybern.*, vol. SMC-8, pp. 751–756, Oct. 1978.

[5] O. I. Elgerd, *Electric Energy Systems Theory: An Introduction.* New York, McGraw-Hill, 1971.

[6] N. Viswanadham and V. V. S. Sarma, "Decentralized feedback routing in data communication networks," *Symp. Record, Data Communication and Computer Networks*, sponsored by IFIP and CSI, Feb. 1980.

[7] J. P. Corfmat and A. S. Morse, "Decentralized control of linear multivariable systems," *Automatica*, vol. 12, pp. 479–495, 1976.

[8] S. H. Wang and E. J. Davison, "On the stabilization of decentralized control systems," *IEEE Trans. Automat. Contr.*, vol. AC-18, pp. 473–478, Oct. 1973.

[9] F. M. Callier, W. S. Chan, and C. A. Desoer, "Input-output stability of interconnected systems using decompositions: An improved formulation," *IEEE Trans. Automat. Contr.*, vol. AC-23, pp. 150–163, Apr. 1978.

[10] E. J. Davison, "The robust decentralized control of a general servomechanism problem," *IEEE Trans. Automat. Contr.*, vol. AC-21, pp. 14–24, Feb. 1976.

[11] R. Sacks, "On the decentralized control of interconnected systems," *IEEE Trans. Automat. Contr.*, vol. AC-24, pp. 269–271, Apr. 1979.

[12] H. H. Rosenbrock, *State-Space and Multivariable Theory.* New York: Nelson, 1970.

[13] J. P. Corfmat and A. S. Morse, "Control of linear systems through specified input channels," *SIAM J. Contr.*, vol. 14, pp. 163–175, 1976.

Topological Optimization of Networks: A Nonlinear Mixed Integer Model Employing Generalized Benders Decomposition

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***Abstract*—A class of network topological optimization problems is formulated as a nonlinear mixed integer programming model, which can be used to design transportation and computer communication networks subject to a budget constraint. The approach proposed for selecting an optimal network consists of separating the continuous part of the model from the discrete part by generalized Benders decomposition. One then solves a sequence of master and subproblems. The subproblems of the minimal convex cost multicommodity flow type are used to generate cutting planes for choosing potential topologies by means of the master problems. Computational techniques suited to solving the master and subproblems are suggested, and very encouraging experimental results are reported.**

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