Topologies of Strategically Formed Social Networks Based on a Generic Value Function - Allocation Rule Model

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Abstract

A wide variety of game theoretic models have been proposed in the literature to explain social network formation. Topologies of networks formed under these models have been investigated, keeping in view two key properties, namely efficiency and stability. Our objective in this paper is to investigate the topologies of networks formed with a more generic model of network formation. Our model is based on a well known model, the value function - allocation rule model. We choose a specific value function and a generic allocation rule and derive several interesting topological results in the network formation context. A unique feature of our model is that it simultaneously captures several key determinants of network formation such as (i) benefits from immediate neighbors through links, (ii) costs of maintaining the links, (iii) benefits from non-neighboring nodes and decay of these benefits with distance, and (iv) intermediary benefits that arise from multi-step paths. Based on this versatile model of network formation, our study explores the structure of the networks that satisfy one or both of the properties, efficiency and pairwise stability. The following are our specific results: (1) We first show that the complete graph and the star graph are the only topologies possible for non-empty efficient networks; this result is independent of the allocation rule and corroborates the findings of more specific models in the literature. (2) We then derive the structure of pairwise stable networks and come up with topologies that are richer than what have been derived for extant models in the literature. (3) Next, under the proposed model, we state and prove a necessary and sufficient condition for any efficient network to be pairwise stable. (4) Finally, we study topologies of pairwise stable networks in some specific settings, leading to unravelling of more specific topological possibilities.

Keywords: Social networks, network formation, pairwise stability, efficiency, value function, allocation rule.

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1. Introduction

Examples of networks in the field that form as a result of autonomous agents seeking to fulfill their individual objectives include the Internet, peer-to-peer networks, and social networks. In this paper, we focus on social networks which are social structures comprised of individuals connected by one or more relationships (Wasserman and Faust , 1994). In this paper, graphs represent social networks with nodes indicating individuals and edges indicating the social interactions connecting them.

It is well known that social networks play an important role in spreading ideas and information (Boorman (1975), Schelling (1978), Cooper (1982), Rogers (1995), Valente (1995), Strang and Soule (1998), Calvo-Armengol (2004), Calvo-Armengol and Jackson (2004), Jackson and Yariv (2005), Jackson and Yariv (2006), Jackson and Rogers (2007), Leskovec et. al. (2007)). Individuals that disseminate information in social networks receive benefits and incur costs in terms of money, time, and effort as a consequence of the links with other individuals. As individuals incur costs, they act strategically while selecting their immediate neighbors. It is important to understand the effect of the strategic behavior of the individuals on the formation of social networks. Recently, many researchers have proposed several models of social network formation using game theoretic approaches (Jackson and Wolinsky (1996), Dutta et. al. (1998), Johnson and Gilles (2000), Dutta and Jackson (2000), Slikker and Nouweland (2001), Jackson and Watts (2002), Jackson (2005), Jackson and Nouweland (2005), Jackson (2003), Demange and Wooders (2005), Goyal (2007), Jackson (2008), Hummon (2000), Galeotti et. al. (2006), Doreian (2006), Buskens and Rijt (2008), Goyal and Vega-Redondo (2007), Kleinberg et. al. (2008), Doreian (2008), Doreian (2008a)). The crux of most of these studies is the underlying strategic form game (Myerson (1977)) where the players, strategies, and utilities are defined as follows: (i) the individuals are the players, (ii) the choice of neighbors is the strategy of each individual, and (iii) the utility of each individual depends on its neighborhood and the structure of the network. The main emphasis of these models (Jackson and Wolinsky (1996), Johnson and Gilles (2000), Goyal (2007), Jackson (2008), Hummon (2000), Galeotti et. al. (2006), Doreian (2006), Buskens and Rijt (2008), Goyal and Vega-Redondo (2007), Kleinberg et. al. (2008), Doreian (2008), Doreian (2008a)) is to study the stability and efficiency properties of the networks that emerge. Informally, a network is said to be stable if it is in some strategic equilibrium[2] (Myerson (1977)) and we call a network efficient if the sum of the utilities of the nodes in the network is maximal. Some of the studies Jackson and Wolinsky (1996), Johnson and Gilles (2000), Slikker and Nouweland (2001), Goyal (2007), Hummon (2000), Galeotti et. al. (2006), Doreian (2006), Buskens and Rijt (2008), Goyal and Vega-Redondo (2007), Kleinberg et. al. (2008), Doreian (2008), Doreian (2008a) yield precise predictions on the network topologies that result, if stability and efficiency are to be satisfied.

Our objective in this paper is to investigate the structure or topologies of social networks formed using a model of network formation. Our model is based on a well known model, the value function - allocation rule model that has been proposed by Jackson and Wolinsky (1996), which

[2]Various notions of stability notions are present in the literature such as pairwise stability (Jackson and Wolinsky (1996)), Nash stability (Goyal (2007)), unilateral stability (Buskens and Rijt (2008)).

is investigated later, among others, by Johnson and Gilles (2000), Dutta and Jackson (2000), Slikker and Nouweland (2001), Watts (2001), Jackson and Watts (2002), Jackson (2003), Jackson (2005), Galeotti et. al. (2006), Bloch (2007), Goyal (2007). We choose a specific value function that captures several important aspects of network formation and we choose a class of allocation rules satisfying a set of appropriate axioms. We work with a rich class of allocation rules rather than a single specific allocation rule, and in this sense, the proposed model becomes a generic model. Using this model, we derive several interesting topological predictions in the network formation context.

The game theoretic model that we work with in this paper is a strategic form game where individuals announce independently the links they wish to form to other individuals and the links are formed under mutual consent. For each graph that emerges due to the strategies of the individuals, we define a network value of the graph using a value function. This value function satisfies certain desirable properties. The network value is divided among the nodes as utilities of the nodes, using any allocation rule that satisfies a set of appropriate axioms. The class of allocation rules considered includes well known allocation rules such as the Myerson value[3] (Myerson (1977), Hart and Kurz (1983), Owen (1986), Moulin (1988), Jackson and Wolinsky (1996), Dutta et. al. (1998), Chwe (2000), Slikker and Nouweland (2001), Kar (2002), Jackson and Watts (2002), Faigle and Kern (1992), Goyal and Joshi (2003)) and makes the network formation model a generic one. The combination of the value function and the allocation rule ensures that the utilities of the nodes are decided by key determinants of network formation such as:

1. *Link Benefits:* Benefits that individual nodes derive from immediate neighbors through direct links (Jackson and Wolinsky (1996), Slikker and Nouweland (2001), Demange and Wooders (2005), Goyal (2007), Jackson (2008), Hummon (2000), Doreian (2006), Buskens and Rijt (2008), Goyal and Vega-Redondo (2007), Kleinberg et. al. (2008)).
2. *Link Costs:* Costs to individual nodes to maintain the above links (Jackson and Wolinsky (1996), Slikker and Nouweland (2001), Demange and Wooders (2005), Goyal (2007), Jackson (2008), Hummon (2000), Doreian (2006), Goyal and Vega-Redondo (2007), Kleinberg et. al. (2008)).
3. *Benefits from Non-neighbor Nodes:* It is well understood in social networks that individuals gain more advantages from their immediate neighbors compared to that from the neighbors of these neighbors and so on (Jackson and Wolinsky (1996), Slikker and Nouweland (2001), Demange and Wooders (2005), Goyal (2007), Jackson (2008), Hummon (2000), Doreian (2006), Buskens and Rijt (2008), Goyal and Vega-Redondo (2007), Kleinberg et. al. (2008), Johnson and Gilles (2000), Bloch (2007), Calvo-Armengol (2004), Dutta et. al. (1998), Dutta and Jackson (2000), Galeotti et. al. (2006)). We model the benefits from non-neighbor nodes through a benefit function that captures the decay of benefits with the (shortest) distance between the source node and the target node.
4. *Intermediary Benefits:* When information flows from one individual node to another in a network using multi-step paths, the individual nodes on the multi-step paths can/do gain appropriate intermediary benefits. These intermediary benefits are known as bridging benefits and

[3]Please refer to Appendix A for more details on Myerson value.

the theory of structural holes describes this phenomenon in detail (Burt (2001), Burt (2004), Burt (2002), Burt (2005), Burt (2007), Burt (1992), Goyal (2007), Jackson (2008)). Such situations are prevalent in many practical situations such as (i) decentralized peer-to-peer file-sharing systems, for example Gnutella, (Kleinberg and Raghavan (2005), Lua et. al. (2005)), (ii) job finding through social networks that provide indirect access to a large set of people connected through multi-step paths of acquaintances (Jackson (2008)), and (iii) query incentive networks (Kleinberg and Raghavan (2005)). In particular, the notion of structural holes is effectively captured in a few models of social network formation in the literature (Goyal and Vega-Redondo (2007), Buskens and Rijt (2008), Kleinberg et. al. (2008)).

In this network formation setting, we investigate the properties as well as the topological structure of networks that are efficient and that are pairwise stable (Jackson and Wolinsky (1996)). The notion of efficiency that we consider is maximization of the sum of utilities of the nodes. Efficient networks are important as they are the most productive from the overall network viewpoint. A network is said to be pairwise stable if (i) no node has an incentive to sever a link, and (ii) no pair of nodes has an incentive to form a new link. Pairwise stable networks are important as such networks are robust to the strategic behavior of the nodes. We also investigate the compatibility of pairwise stability and efficiency.

We emphasize that our model is more general than several existing models in the literature in terms of capturing key determinants of network formation and investigating the properties as well as the topological structure of networks that satisfy pairwise stability and efficiency. In the following section, we present the relevant work and bring out the contributions of our work.

1.1. Relevant Work

We must note that the field of network formation is very rich (Jackson (2008), Goyal (2007), Demange and Wooders (2005), Slikker and Nouweland (2001), Hummon (2000), Doreian (2006), Buskens and Rijt (2008), Goyal and Vega-Redondo (2007), Kleinberg et. al. (2008), Johnson and Gilles (2000), Bloch (2007), Calvo-Armengol (2004), Dutta et. al. (1998), Dutta and Jackson (2000), Galeotti et. al. (2006), Jackson and Wolinsky (1996), Jackson and Watts (2002), Jackson (2005), Jackson and Nouweland (2005), Jackson (2003), Doreian (2008), Doreian (2008a)). We have only included a discussion of the models that are most relevant to our work. At the end of this section, we highlight the research gap in the literature. When there is no confusion, we use the words *graph* and *network* synonymously.

The modeling of strategic formation in a general network setting was first studied in the seminal work of Jackson and Wolinsky (1996). They basically consider a value function and an allocation rule model where the value function defines a value to each network and the allocation rule distributes this value to the nodes in the network. They investigate whether efficient networks will form when self-interested individuals can choose to form links and/or break links. The authors define two stylized models[4]. For these models, the authors observe that for high and low costs the efficient networks are pairwise stable, but not always for medium level costs. They also examine the tension between efficiency and stability and derive various conditions and allocation rules for

[4]They are the connections model and the co-authorship model (Jackson and Wolinsky (1996)).

which efficiency and pairwise stability are compatible. An important feature their model does not capture is that of the intermediary benefits that nodes gain by being intermediaries lying on the paths between non-neighbor nodes. In particular, they do not capture the benefits due to structural holes.

Hummon (2000) carries out several interesting investigations to unravel more specific topologies using a specific model[5] proposed by Jackson and Wolinsky (1996). Two different agent-based simulation approaches, the multi-thread model and the discrete event simulation model, are used in the analysis of Hummon (2000) to explore the dynamics of network evolution based on a model proposed in Jackson and Wolinsky (1996). Hummon identifies certain pairwise stable structures that are more specific than those anticipated by the formal analysis of Jackson and Wolinsky (1996). Doreian (2006) explores the same issue in a systematic manner and establishes the conditions under which different pairwise structures are generated. Some gaps in the analysis of Doreian (2006) are addressed by Xie and Cui (2008a) and Xie and Cui (2008b).

Jackson (2003) reviews several models of network formation in the literature with an emphasis on the tradeoffs between efficiency with stability. The author initially highlights that pairwise stable networks may not exist in some settings and provides a few example settings where this happens. Then, the author shows the existence of pairwise stable networks for the egalitarian and component-wise egalitarian allocation rules. Further, the author proves, under the Myerson value allocation rule, that there always exists a pairwise stable network. This work also studies the relationship between pairwise stable and efficient networks in a variety of contexts and under three different definitions of efficiency.

A later paper by Jackson (2005) presents a family of allocation rules (for example, networkolus) that incorporate information about alternative network structures when allocating the network value to the individual nodes[6]. The author provides a general method of defining allocation rules in network formation games. However, this paper does not however investigate the structure of networks under the allocation rules discussed.

Goyal and Vega-Redondo (2007) propose a non-cooperative game model in which a node i can benefit from serving as an intermediary between a pair of nodes x and y. In their model, a node i could lie on an arbitrarily long path between x and y. The authors assume, however, that the benefits from farther nodes are not subject to decay. They also assume that the benefit of communication between any pair of nodes is always 1 unit. This 1 unit is distributed to the two communicating nodes and only to certain so called *essential nodes* (Goyal and Vega-Redondo (2007)) on the paths between the two communicating nodes. In this setting, the authors show that a star graph is the only non-empty robust equilibrium graph. The authors also study the implications of capacity constraints in the ability of individual nodes to form links to other nodes and show that a cycle network emerges.

The network formation model proposed by Buskens and Rijt (2008) captures the cost to nodes of wasting resources on redundant length 2 paths. This model does not capture the bridging bene-

[5]In the context of the symmetric connections model (Jackson and Wolinsky (1996)).

[6]In particular, evaluating the contribution to value of a given node depends on the contribution of that node to various networks. Jackson (2005) highlights that the *Myerson value* based allocation rule implicitly works with a fixed network structure while distributing the network value across the nodes.

fits and the authors restrict their attention to the setting where all edges have the same cost.

Kleinberg et. al. (2008) propose a non-cooperative game model of network formation that incorporates the intermediary benefit received by a node by bridging any pair of non-neighbor nodes separated by a path of length 2. In this setting, the authors characterize the structure of stable networks with *Nash equilibrium* as the notion of stability. They propose a polynomial time algorithm for a node to determine its best response in a given graph as nodes can choose to link to any subset of other nodes. The authors also show that stable networks have a rich combinatorial structure. Their model does not capture the bridging benefits that nodes can gain by being intermediaries on (shortest) paths of length greater than 2 between non-neighbor nodes.

Model	**Benefits from Direct Links**	**Costs of Direct Links**	**Decaying Benefits from Non-neighbors**	**Bridging Benefits**	**Analysis of Stable Networks**	**Analysis of Efficient Networks**
Jackson and Wolinsky (1996)	✓	✓	✓	×	✓	✓
Jackson (2005)	✓	✓	✓	✓	×	×
Goyal and VegaRedondo (2007)	✓	✓	×	✓	✓	✓
Buskens and Rijt (2008)	×	✓	✓ [Length 2 Paths]	×	✓	✓
Kleinberg et al. (2008)	✓	✓	×	✓ [Length 2 Paths]	✓	×
Our Model	✓	✓	✓	✓	✓	✓

Table 1: Comparison of our model with the most relevant models in the literature

To summarize, there is no model in the literature that simultaneously takes into account all four key determinants of network formation in analyzing the topologies of stable and efficient networks. We propose to fill this research gap in this paper. Table 1 provides a quick summary of the relevant work in comparison with our model.

1.2. Our Contributions

The proposed *Value Function - Allocation Rule* model, takes several ideas from the models discussed by Jackson and Wolinsky (1996), Jackson (2003), and Jackson (2005). Specifically, the model that we work with in this paper is a strategic form game where individuals announce independently the links they wish to form to other individuals and the links are formed under mutual consent. For the graph that emerges due to the strategies of the individuals, we define an overall value of the graph using a value function. This value function satisfies certain desirable properties. The value of the graph is divided among the nodes as utilities using an allocation rule that satisfies a set of representative axioms. Section 2 presents the details of the proposed model. With this model in place, we undertake a detailed study of topologies of networks that could result if efficiency and pairwise stability properties are satisfied.

In Section 3, we characterize the topologies of efficient networks. We point out that our characterization of efficient networks is only dependent on the value function and it is independent of the allocation rule. We show that the complete graph and the star graph are the only possible non-empty efficient networks under certain conditions. These findings are consistent with prior work in the literature.

In Section 4, we analyze the topologies of pairwise stable networks. We note that the study of pairwise stable networks depend both on the value function and the allocation rule. We also note that the class of allocation rules that we work with is based on a standard set of axioms and includes allocation rules such as the Myerson value (please refer to Appendix A for more details). Each member of this class of allocation rules results in a specific utility function. Thus our model can capture a range of utility functions. Moreover, our analysis shows that the set of pairwise stable networks derived using the proposed model enriches the set of pairwise stable networks that are derived using various models in the literature[7] (Jackson and Wolinsky (1996), Goyal and Vega-Redondo (2007), Buskens and Rijt (2008), Kleinberg et. al. (2008)).

In Section 5, we highlight the tradeoffs between efficiency and pairwise stability. Then we present a necessary and sufficient condition for any efficient network to be pairwise stable. We refer to this condition as *regularity condition (RC)* and this turns out to be a simple consistency condition.

Finally, in Section 6, we study the topologies of pairwise stable networks in more detail by considering some specific settings. In particular, we focus our attention on situations where the topology happens to be a minimal edge graph with diameter 2 and derive more specific topologies by looking at these situations in more detail. We do this investigation using the framework proposed by Doreian (2006). In particular, we consider all networks with 4 nodes and then study the resulting topologies. Our investigation leads to unravelling of more specific topologies.

Before formally presenting the model and the results, we would like to emphasize the need for the proposed model. In our view, social contacts (i.e. links) form or get deleted in social networks based on the previously mentioned four key determinants of network formation. Existing models in the literature capture only proper subsets of these four key determinants. Our proposed model

[7]In particular, we show that the complete graph and any minimal edge graphs with diameter 2 (for example star graph, any completely connected bipartite graph) are pairwise stable under appropriate conditions. We now highlight that the following network structures are stable using various models in the literature: (a) the complete graph and the star graph are stable using Jackson and Wolinsky (1996) model, (b) the star graph and the cycle graph are stable networks under appropriate conditions using Goyal and Vega-Redondo (2007) model, (c) completely connected bipartite graphs are pairwise stable using Buskens and Rijt (2008) model, (d) (appropriately defined) multipartite graph is stable using Kleinberg et. al. (2008) model.

captures all of these simultaneously and hence provides a more encompassing model[8].

Further, based on the analysis of the proposed model in the rest of this paper, we also note that our analytical findings are consistent with many findings of empirical research in the area. For example, the work of Burt (2007, 2005) on the theory of structural holes suggests that in practice most of the bridging benefits arise from bridging pairs of non-neighbor nodes (i.e. being intermediary on paths of length two), rather than serving as an intermediary on paths of length three or more. In other words, paths of length two in the network play a crucial role for the bridging benefits, compared to that of paths of length three or more. Our results in this paper suggest topologies which are broader than those found in the related literature and also prominently include some specific networks[9] with diameter two. With analytical models, such as ours, reproducing satisfactorily the findings of empirical research, we believe social analysts will find these models useful.

2. A Novel Value Function - Allocation Rule Model

Here we present the model of the network formation that we work with in the sequel of this paper. Let $N = \{1, 2, \ldots, n\}$ be the set of n (≥ 3) nodes (or players) in the network formation game. A strategy s_i of a node i is any subset of nodes with which the node would like to establish links and these links are formed under mutual consent. Assume that S_i is the set of strategies of node i. Let $s = (s_1, s_2, \ldots, s_n)$ be a profile of strategies of the nodes. Also let S be the set of all such strategy profiles. Each strategy profile s leads to an undirected graph and we represent it by $g(s)$. Let $\Psi(S)$ be the set of all such undirected graphs. When the context is clear, we use g and Ψ instead of $g(s)$ and $\Psi(S)$ respectively. If nodes i and j are connected by a link in g, then we say that $(i, j) \in g$, otherwise we say that $(i, j) \notin g$. If nodes x and y form a link (x, y) in a graph g, then we represent the new graph by $g \cup \{(x, y)\}$.

Distances: Let $d_g(.,.)$ be a distance function that specifies the length of a shortest path between any pair of nodes in g. Since g is undirected, $d_g(i, j) = d_g(j, i), \forall i, j \in N$. If two nodes i and j are not connected by a path in g, then we assume that $d_g(i, j)$ is infinity. It is implicitly assumed that nodes communicate through shortest paths. This assumption is adequate as often individuals seek for such shortest paths in social networks (Jackson and Wolinsky (1996), Johnson and Gilles (2000), Dutta and Jackson (2000), Hummon (2000)).

Costs: If nodes i and j are connected by a link in g, then we assume that the link incurs a cost $c > 0$.

Benefits: Assume that $b(1) \in (0, 1)$. Given a pair of nodes i and j, it is reasonable to assume that the communication between i and j leads to a benefit of $b(d_g(i, j))$. In particular, if a link exists between i and j in g (i.e. $d_g(i, j) = 1$), then this link generates a benefit of $b(1)$. We assume that $b(.)$ is a non-increasing function, implying that the benefit of communication decays as the length of shortest path increases.

We now define a value function $v : \Psi \to \mathbb{R}$ for a given graph $g \in \Psi$ as follows:

[8]We find that many results derived using our model are consistent with the findings in the literature and moreover, our proposed model expands the range of model-predicted topologies for social networks.

[9]Minimal edge graphs with diameter 2 (see Section 3)

$$v(g) = \sum_{\substack{x,y \in N, \\ (x,y) \in g}} [b(1) - c] + \sum_{i \in N} \sum_{\substack{j \in N, \\ j > i, \\ (i,j) \notin g}} b(d_g(i, j)) \tag{1}$$

This value function consists of two terms. The first term represents the benefits and the costs derived due to the direct links (or immediate neighbors). The second term indicates the sum of benefits derived due to communication between non-neighbor nodes.

We now present two important properties of the value function $v(.)$ namely anonymity and component additivity. Anonymity of the value function requires that the value of the graph be not dependant on the identities of the nodes. Since the proposed value function (1) does not depend on the identities of the nodes, it is anonymous. Component additivity requires that the value of a group of nodes be the sum of values of the components induced by the links between the nodes in that group. More formally; let $\Omega(N)$ be the set of all components of g. Note that if C_1 and C_2 are two different components in $\Omega(N)$, then there does not exist an edge between any node in C_1 and any node in C_2. From the definition of the value function (1), we get that $v(g) = \sum_{C \in \Omega(N)} v(C)$. This shows that the value function $v(.)$ is component additive. Component additivity is a condition that rules out externalities across components.

The network value $v(g)$ is divided among the nodes in g as utilities using an allocation rule. An allocation rule is a function $Y : \Psi \rightarrow \mathbb{R}^n$ that associates an n-dimensional vector $Y(g)$ for every g such that $\sum_{i \in N} Y_i(g) = v(g)$ for all g. Here $Y_i(g)$ represents the i-th component in the n-dimensional vector $Y(g)$ and we define $Y_i(g)$ to be the utility ($u_i(g)$) of node i. That is,

$$u_i(g) = Y_i(g), \ \forall i \in N.$$

We choose an allocation rule $Y(.)$ that satisfies the following four properties: (i) anonymity, (ii) component balance, (iii) weak link symmetry, and (iv) improvement property. We note that the last two properties namely weak link symmetry and improvement property are proposed in Dutta et. al. (1998). We now describe these four properties.

(i) *Anonymity:* Given a permutation of players π (a bijection from N to N) and any $g \in \Psi$, let $g^\pi = \{(\pi(i), \pi(j)) | (i, j) \in g\}$. Thus g^π and g share the same structure but with a relabeling of players according to π. Given a permutation π, let v^π be defined as $v^\pi(g^\pi) = v(g)$ for each $g \in \Psi$. This is just the value function obtained when the names of agents are relabeled according to π. The allocation rule Y is anonymous if for any v, $g \in \Psi$, and any permutation of the players π, $Y_{\pi(i)}(g^\pi) = Y_i(g)$.

Anonymity of the allocation rule requires that, if all that has changed are the labels of the players and the value generated by the network has changed correspondingly, then the allocation rule only changes according to the relabeling.

(ii) *Component Balance:* The allocation rule Y is *component balanced* if $\sum_{i \in C} Y_i(g) = v(C)$ for each component additive v, $g \in \Psi$, and for each component C in $\Omega(g)$ (the set of all components of the graph g).

If v is component additive, then a component balanced allocation rule must distribute the value of the component among the nodes in that component only.

(iii) *Weak Link Symmetry:* The allocation rule Y satisfies *weak link symmetry* if for each link $e = (i, j) \notin g$, it holds that if $Y_i(g \cup \{e\}) > Y_i(g)$, then $Y_j(g \cup \{e\}) > Y_j(g)$. Weak link symmetry implies that if a new link is added to a graph and the utility of an end point of the link strictly increases, then the utility of the other end point point should also strictly increase.

(iv) *Improvement Property:* The allocation rule Y satisfies *improvement property* if for each link $e = (i, j) \notin g$, whenever there exists a node $z \in N \setminus \{i, j\}$ such that $Y_z(g \cup \{e\}) > Y_z(g)$, then $Y_i(g \cup \{e\}) > Y_i(g)$ or $Y_j(g \cup \{e\}) > Y_j(g)$. Improvement property implies that if a new link, say (i, j), is added to a graph and the utility of a node other than i and j strictly increases, then the utility of either node i or node j should strictly increase.

The above four properties have rich practical implications for an allocation rule (Dutta, Nouweland, and Tijs (1998). We note that the Myerson value (Myerson (1977)) belongs to the above class of allocation rules (Dutta et. al. (1998)).

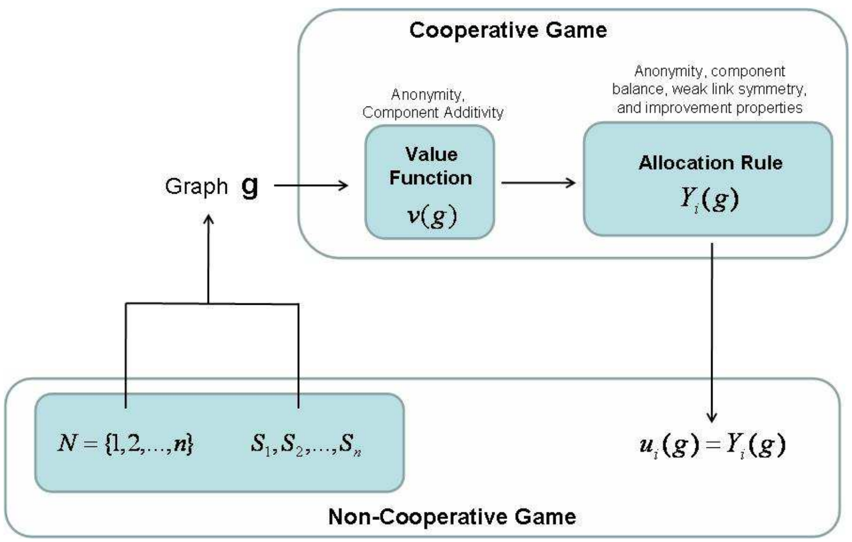

Figure 1: A high level view of the model

2.1. Strategic Form Game of Network Formation

Based on the above discussion, we can define a strategic form game $\Gamma = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$, to describe network formation. Figure 1 shows a high level idea of the model. We note that this model uses ideas from both non-cooperative game theory and cooperative game theory. We now present an example which illustrates how the proposed model captures several key determinants of network formation (mentioned previously in Section 1).

Example 1. *Let* $N = \{1, 2, 3, 4, 5\}$. *Let* $s_1 = \{2, 5\}$, $s_2 = \{1, 3\}$, $s_3 = \{2, 4\}$, $s_4 = \{3, 5\}$, *and* $s_5 = \{1, 4\}$. *This results in the network g as shown in Figure 2(i). Note that there are* 5 *links in g. These links incur a cost* 5*c and lead to a benefit* $5b(1)$. *We now focus on the benefits due to communication between all pairs of non-neighbor nodes in g. Start with the pair of nodes* 1 *and* 3. *The shortest*

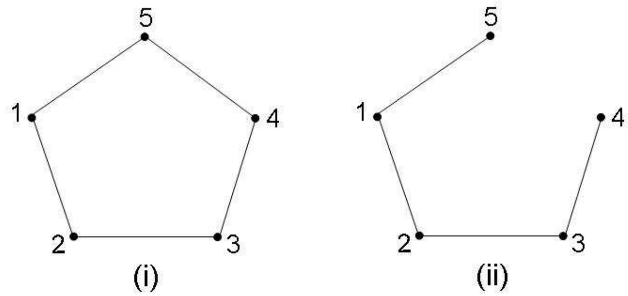

Figure 2: Two simple networks (i) g and (ii) g'

distance between these nodes is 2*; i.e.* $d_g(1,3) = 2$*. Hence the communication between* 1 *and* 3 *leads to a benefit of* $b(2)$*. Proceeding on similar lines, we get that* $v(g) = 5(b(1) - c) + 5b(2)$*.*

Now consider the network that results when $s_1 = \{2,5\}$*,* $s_2 = \{1,3\}$*,* $s_3 = \{2,4\}$*,* $s_4 = \{3\}$*, and* $s_5 = \{1\}$*. The resulting network is* g' *as shown in Figure 2(ii). If we compute the value of* g' *on similar lines as we did on g above, we get that* $v(g') = 4(b(1) - c) + 3b(2) + 2b(3) + b(4)$*.*

If we use the Myerson value as the allocation rule, the utilities of the nodes in g would be as follows:

$$u_i(g) = (b(1) - c) + b(2) \quad \forall i \in \{1,2,3,4,5\}.$$

Similarly, if we use the Myerson value as the allocation rule, the utilities of the nodes in g' *shown in Figure 2(ii) are determined as follows:*

$$\begin{aligned}
u_1(g') &= (b(1) - c) + \tfrac{2}{3}b(2) + \tfrac{1}{2}b(3) + \tfrac{1}{5}b(4) \\
u_2(g') &= (b(1) - c) + b(2) + \tfrac{1}{2}b(3) + \tfrac{1}{5}b(4) \\
u_3(g') &= (b(1) - c) + \tfrac{2}{3}b(2) + \tfrac{1}{2}b(3) + \tfrac{1}{5}b(4) \\
u_4(g') &= \tfrac{1}{2}(b(1) - c) + \tfrac{1}{3}b(2) + \tfrac{1}{4}b(3) + \tfrac{1}{5}b(4) \\
u_5(g') &= \tfrac{1}{2}(b(1) - c) + \tfrac{1}{3}b(2) + \tfrac{1}{4}b(3) + \tfrac{1}{5}b(4)
\end{aligned}$$

We now consider the utility $u_1(g')$ *of node* 1*: the term* $(b(1) - c)$ *represents the benefits and costs due to direct links; the term* $\frac{2}{3}b(2) + \frac{1}{2}b(3)$ *represents the benefits from non-neighbors and also captures the decaying of these benefits with distance; and the term* $\frac{1}{5}b(4)$ *represents the bridging benefits. Similarly, we can observe the same with the other nodes as well. Hence the proposed value function (*$v(.)$*) - allocation rule (*$Y(.)$*) model captures several key determinants of network formation.*

This example also demonstrates the fact that the network value of a graph depends on not only which nodes are connected but also on how those nodes are connected. The next example compares and contrasts our model with important existing models in the literature.

Example 2. *In this example, let* $N = \{1,2,3,4,5,6\}$*. Let* $s_1 = \{5\}$*,* $s_2 = \{5\}$*,* $s_3 = \{6\}$*,* $s_4 = \{6\}$*,* $s_5 = \{1,2,6\}$*, and* $s_6 = \{3,4,5\}$*. This results in the network* g_1 *as shown in Figure 3. We now derive the individual utilities of the nodes and the value of the graph using various models as follows.*

Using the model proposed in this paper, the value of the graph g_1 *is:*

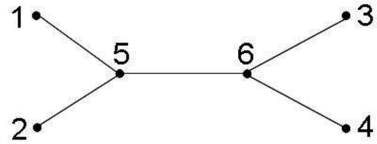

Figure 3: A stylized example network

$$v(g_1) = 5(b(1) - c) + 6b(2) + 4b(3).$$

The utilities of the nodes using the Myerson value are:

$$u_1(g_1) = u_2(g_1) = u_3(g_1) = u_4(g_1) = \tfrac{17}{30}(b(1) - c) + \tfrac{2}{3}b(2) + \tfrac{1}{2}b(3),$$
$$u_5(g_1) = u_6(g_1) = \tfrac{41}{30}(b(1) - c) + \tfrac{5}{3}b(2) + b(3).$$

Consider the utilities of nodes 1, 2, 3, *and* 4*: the first term represents the benefits and costs due to direct links; and the last two terms represent the decaying benefits from non-neighbors. Similarly, consider the utilities of the nodes* 5 *and* 6*: the first term represents the benefits and costs due to direct links; the second term represents the decaying benefits from non-neighbors; and the third term represents the bridging benefits.*

If we use the Connections Model (Jackson and Wolinsky (1996)), the individual utilities of the nodes are as follows:

$$u_1^{JW}(g_1) = u_2^{JW}(g_1) = u_3^{JW}(g_1) = u_4^{JW}(g_1) = (b(1) - c) + 2b(2) + 2b(3),$$
$$u_5^{JW}(g_1) = u_6^{JW}(g_1) = 3(b(1) - c) + 2b(2).$$

Note that the utilities of the nodes in this model do not capture the bridging benefits that nodes can gain by serving as intermediaries on various paths. Here, the value of the graph is:

$$v^{JW}(g_1) = 10(b(1) - c) + 12b(2) + 8b(3).$$

Using the model of Goyal and Vega-Redondo (2007), the utilities of the nodes in g_1 *are obtained as:*

$$u_1^G(g_1) = u_2^G(g_1) = u_3^G(g_1) = u_4^G(g_1) = \tfrac{5}{3} - c,$$
$$u_5^G(g_1) = u_6^G(g_1) = \tfrac{13}{6} - 3c.$$

Note that this model does not capture that the fact that the benefits to nodes from non-neighbor nodes could decay with distance. The value of the graph is given by:

$$v^G(g_1) = 11 - 10c.$$

Using the model of Kleinberg et. al. (2008), the individual utilities of the nodes are obtained as:

$$u_1^K(g_1) = u_2^K(g_1) = u_3^K(g_1) = u_4^K(g_1) = \alpha_0 - c,$$
$$u_5^K(g_1) = u_6^K(g_1) = 9\alpha_0 - 3c.$$

Here this model does not capture the bridging benefits arising from arbitrary length paths and it does not also consider the benefits to nodes from non-neighbor nodes. If we define the value of the graph to be sum of the utilities of nodes, then its value is:

$$v^K(g_1) = 22\alpha_0 - 10c.$$

Finally, using the model of Buskens and Rijt (2008), the constraint measures of the nodes are computed as follows:

$$c_1(g_1) = c_2(g_1) = c_3(g_1) = c_4(g_1) = 1,$$
$$c_4(g_1) = c_5(g_1) = \tfrac{1}{3}.$$

This model does not capture the benefits to nodes from direct links (or immediate neighbors) and also it does not model the bridging benefits to nodes.

2.2. Pairwise Stability and Efficiency

Our goal is to predict the network structures that emerge due to the dynamics of network formation using the above proposed model. In particular, we wish to study the topologies of the networks that satisfy two important properties namely efficiency and pairwise stability. Now we describe these two notions. We call a network efficient if the sum of the utilities of the nodes in the network is maximal. More formally,

Definition 1. *A network $g \in \Psi$ is said to be efficient if $v(g) \geq v(g') \;\; \forall g' \in \Psi$.*

The notion of pairwise stability was proposed by Jackson and Wolinsky (1996). We call a network pairwise stable if (i) there is no incentive for any node to sever a link, and (ii) there is no incentive for any pair of nodes to form a new link. Formally,

Definition 2. A network g is said to be *pairwise stable* with respect to the value function v and the allocation rule Y if (i) for each edge $e = (i, j) \in g$, $Y_i(g) \geq Y_i(g \setminus \{e\})$ and $Y_j(g) \geq Y_j(g \setminus \{e\})$, and (ii) for each edge $e' = (i, j) \notin g$, if $Y_i(g) < Y_i(g \cup \{e'\})$ then $Y_j(g) > Y_j(g \cup \{e'\})$.

In what follows, we first characterize the structures of efficient networks in Section 3 and we emphasize that this characterization is independent of the allocation rule. In Section 4, we proceed to the analysis of pairwise stable networks.

3. Analysis of Efficient Networks

In this section, we characterize the structure of the efficient networks. Before we proceed, we present an important class of graphs that are useful for the analysis. First we recall the definition of diameter of a graph.

Definition 3. The diameter of a graph is the length of a longest shortest path between any two vertices of the graph.

Definition 4. A graph with diameter p is said to be a *minimal edge graph with diameter p* if the deletion of any edge in the graph results in a graph with diameter greater than p.

We now present a few observations:

(i) In a star graph, a single node is connected to all the remaining nodes and there are no other edges. The star graph is an example for a minimal edge graph with diameter 2.
(ii) Given a set of n nodes, there may be multiple minimal edge graphs with diameter p for $1 < p < n$. For example, when $n = 6$, Figure 4 shows three different minimal edge graphs with diameter 2.
(iii) We note that any minimal edge graph with diameter p is a k-partite graph for some appropriate value of k.

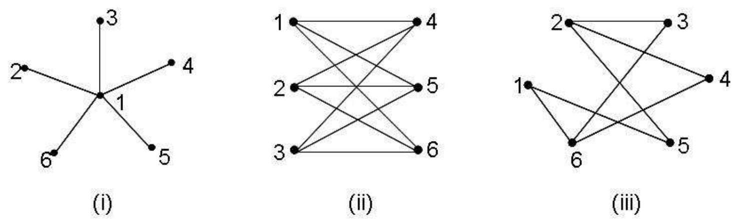

Figure 4: Three different minimal edge graphs with diameter 2 having 6 nodes

We now present two useful results in the form of Lemma 1 and Lemma 2 that are useful in the characterization of the efficient networks.

Lemma 1. *Given a graph g, if* $(b(1) - b(2)) < c < (b(1) - b(3))$ *and there exists a pair of nodes x and y such that* $d_g(x, y) > 2$*, then forming a link between x and y strictly increases the value of g.*

Proof. We are given that $(b(1)-b(2)) < c < (b(1)-b(3))$. Consider any network g and assume that there exists a pair of nodes x and y such that $d_g(x, y) > 2$. Note that the communication between nodes x and y in g leads to a benefit of $b(d_g(x, y))$.

Now assume that the nodes x and y form a link and call the link $e = (x, y)$. Also call the new graph g'; i.e. $g' = g \cup \{e\}$. The link (x, y) leads to a net benefit of $(b(1) - c)$. Note that the length of a shortest path between any pair of nodes in g' either remains same or decreases when compared to that in g. From these observations, we get that $v(g') - v(g) \geq \left[(b(1) - c) - b(d_g(x, y))\right] > 0$, since $d_g(x, y) > 2$ and $(b(1) - c) > b(3) \geq b(d_g(x, y))$. That is, $v(g') > v(g)$. □

Lemma 2. *If* $(b(1) - b(2)) < c < (b(1) - b(3))$*, then every efficient network is a minimal edge graph with diameter* 2.

Proof. Consider that g is an efficient graph. Due to Lemma 1, the shortest distance between any pair of nodes is at most 2 in g. That is, g is a graph with diameter 2.

Suppose that g is not a minimal edge graph with diameter 2. Then g contains a link (x, y) such that severing the link (x, y) does not lead the diameter to exceed 2. Thus, if we remove the link (x, y), only the shortest distance between the nodes x and y increases to 2. Since $(b(1) - b(2)) < c$, the value of g strictly increases if the link (x, y) is severed. This is a contradiction to the fact that g is efficient. Hence g is a minimal edge graph with diameter p. □

We now present a theorem that characterizes efficient networks. It is important to emphasize the distinction between this following theorem and Proposition 1 (the connections model) in Jackson and Wolinsky (1996). First, Jackson and Wolinsky work with detailed information, namely, the individual utilities defined as per the connections model whereas we do not assume any individual utilities and work only with the overall network value. Our theorem therefore holds good immaterial of how this network value is allocated to the individual players and as a consequence will hold good for a wide variety of utility functions. Also, because of the generality of the situation handled by us, our line of arguments in the proof is different from that of Jackson and Wolinsky (1996).

Theorem 1. *Following our proposed model,*

(i) if $c < (b(1) - b(2))$*, then the complete graph is the unique topology possible for an efficient network*

(ii) if $(b(1) - b(2)) < c \leq b(1) + (\frac{n-2}{2})b(2)$*, then the star network is the unique topology possible for an efficient network*

(iii) if $c > b(1) + (\frac{n-2}{2})b(2)$*, then the only efficient network is the empty graph.*

Proof. (i) Given that $c < (b(1) - b(2))$. Consider a graph g such that a pair of nodes, say, i and j, is not adjacent. Now consider a graph g' such that $g' = g \cup \{(i, j)\}$. We note that the length of the shortest path between any pair of nodes in g' is less than or equal to that in g. From this observation and due to $c < (b(1) - b(2))$, we get that $v(g') > v(g)$. That is, when $c < (b(1) - b(2))$, the value of a graph strictly increases when two non-neighbor nodes form a link. Hence the complete graph emerges as the efficient network. Moreover, the complete graph is the unique efficient graph as there does not exist any other graph with as many edges as that of the complete graph.

(ii) We prove this in two steps as follows. In the first step, we show that the unique efficient graph is the star graph when $(b(1) - b(2)) < c < (b(1) - b(3))$. In the second step, we show that the star graph is indeed the unique efficient graph when $(b(1) - b(2)) < c < b(1) + (\frac{n-2}{2})b(2)$.

Let us start with the first step, that is, $(b(1) - b(2)) < c < (b(1) - b(3))$. Now from Lemma 2, any efficient graph is a minimal edge graph with diameter 2. Hence we focus on the class of minimal edge graphs with diameter 2. Now we consider two different minimal edge graphs with diameter 2. First, consider that g_1 is the star graph. In g_1, note that there are $(n-1)$ links (or edges) and $\binom{n-1}{2}$ pairs of nodes are separated by shortest paths of length 2. Hence the value, $v(g_1)$, of g_1 is

$$v(g_1) = (n-1)(b(1) - c) + \binom{n-1}{2} b(2). \tag{2}$$

Secondly, consider that g_2 is an arbitrary minimal edge graph with diameter 2 that does not have the same topology of g_1. Observe that g_2 is a k-partite graph for some appropriate value of k and assume that $a_1, a_2, \ldots, a_k$ are the cardinalities of the respective k partitions. Consider all partitions such that $a_i \geq 2$, $\forall i \in \{1, 2, \ldots, k\}$ and assume that there are r such partitions. Let $b_1, b_2, \ldots, b_r$ be those r partitions respectively.

Construct a new graph, call it g_3, that is a completely connected k-partite graph for the same value of k as in g_2 and also for the same sizes of the k partitions as in g_2. Now we can assume that g_2 is obtained from g_3 by deleting say q (≥ 0) links. We note that the number of links in g_3 is $\left[\sum_{i=1}^{i=k} \sum_{j=i+1}^{j=k} a_i a_j\right]$ and the number of links in g_2 is $\left[\sum_{i=1}^{i=k} \sum_{j=i+1}^{j=k} a_i a_j - q\right]$. Also note that the number of pairs of non-neighbor nodes that are separated by shortest paths of length 2 in g_3 is $\left[\binom{b_1}{2} + \binom{b_2}{2} + \ldots + \binom{b_r}{2}\right]$. Since g_2 is a minimal edge graph with diameter 2 and it is obtained by deleting q links from g_3, the number of pairs of non-neighbor nodes that are separated by shortest paths of length 2 in g_2 is $\left[\binom{b_1}{2} + \binom{b_2}{2} + \ldots + \binom{b_r}{2} + q\right]$. Hence the value ($v(g_2)$) of g_2 is:

$$v(g_2) = (b(1) - c)\left[\sum_{i=1}^{i=k} \sum_{j=i+1}^{j=k} a_i a_j - q\right] + b(2)\left[\sum_{i=1}^{i=r} \binom{b_i}{2} + q\right] \tag{3}$$

Now subtracting equation (2) from equation (3), we get the following, after simple algebra:

$$v(g_1) - v(g_2) = [b(2) - (b(1) - c)]\left[\sum_{i=1}^{i=k} \sum_{j=i+1}^{j=k} a_i a_j - q - (n-1)\right] \tag{4}$$

Since g_2 is a minimal edge graph with diameter 2, it is a connected graph and therefore g_2 must have at least $(n-1)$ links. If g_2 contains $(n-1)$ links, then it must be a star graph because g_2 is a minimal edge graph with diameter 2. But we started with the fact that g_2 is not a star graph. Hence g_2 must contain more than $(n-1)$ links. In other words, we get that

$$\left[\sum_{i=1}^{i=k} \sum_{j=i+1}^{j=k} a_i a_j - q\right] > (n-1). \tag{5}$$

Now from expression (5) and the fact that $(b(1) - c) < b(2)$, we conclude from expression (4) that $v(g_1) > v(g_2)$. Since g_2 is an arbitrary minimal edge graph with diameter 2 that is not the same as g_1, we can conclude that g_1 is the unique efficient network. That is, the star graph is uniquely efficient.

We note that a star graph consisting of $n_1 + n_2$ nodes has higher value than that of separate star graphs of n_1 and n_2 nodes respectively. Hence the star graph consisting of all nodes is the unique efficient graph. For the star graph with n nodes to have positive value, we should have $v(g_1) \geq 0$ and this implies that $c \leq b(1) + (\frac{n-2}{2})b(2)$.

We now show that the star graph is in fact the unique efficient graph when $(b(1) - b(2)) < c \leq b(1) + (\frac{n-2}{2})b(2)$ as follows: (a) Observe that any network (call it g') with $(n-1)$ links that is not a star network must have at least one pair of nodes that is separated by a path of length greater than 2. Hence such a network g' is not efficient. (b) Consider any network (call it g'') with more than $(n-1)$ edges such that any pair of nodes is separated by a shortest path of length at most 2. Since the graph g'' has more edges than that of the star graph and $(b(1) - b(2)) < c$, it is immediate that g'' is not efficient.

(iii) From the above argument, the value of the graph is negative when $c > b(1) + (\frac{n-2}{2})b(2)$. This implies that the empty graph is the only efficient graph when $c > b(1) + (\frac{n-2}{2})b(2)$. □

From this theorem, we get the following observation. When $c = (b(1) - b(2))$, then both the complete graph and the star graph are efficient as the values of these graphs are one and the same. In other words, when $c = (b(1) - b(2))$, the efficient graph is not unique.

4. Analysis of Pairwise Stable Networks

In this section, we study the topologies of social network structures that are pairwise stable. We first present two useful results in the form of Lemma 3 and Lemma 4.

Lemma 3. *For any graph g, if a pair of non-neighbor nodes i and j form a link (i, j) such that $v(g \cup \{(i, j)\}) > v(g)$, then it holds that both $Y_i(g \cup \{(i, j)\}) > Y_i(g)$ and $Y_j(g \cup \{(i, j)\}) > Y_j(g)$.*

Proof. Consider any graph g such that two non-neighbor nodes i and j form a link (i, j) such that $v(g \cup \{(i, j)\}) > v(g)$. Let the new graph be $g' = g \cup \{(i, j)\}$. Assume that M is the set of all nodes except the nodes i and j. That is, $M = N \setminus \{i, j\}$. We prove this lemma in two parts.

Part 1: Consider that the value of g' is distributed such that the allocation to each node in M is less than or equal to that in g. In other words, $\forall k \in M$, $Y_k(g') \leq Y_k(g)$. Since $v(g') > v(g)$, it must be that either $Y_i(g') > Y_i(g)$ or $Y_j(g') > Y_j(g)$. Now since the allocation rule satisfies weak link symmetry, it holds that $Y_i(g') > Y_i(g)$ and $Y_j(g') > Y_j(g)$.

Part 2: Consider that the value of g' is allocated such that the allocation to some of the nodes in M is greater than that in g. That is, $\exists z \in M$ such that $Y_z(g') > Y_z(g)$. Since there is one such node z, and the allocation rule satisfies improvement property, it must be the case that either $Y_i(g') > Y_i(g)$ or $Y_j(g') > Y_j(g)$. Since the allocation rule satisfies weak link symmetry, it holds that $Y_i(g') > Y_i(g)$ and $Y_j(g') > Y_j(g)$. □

Lemma 4. *For any graph g, if a node i severs a link $e = (i, j) \in g$ with a node j such that $v(g \setminus \{(i, j)\}) \leq v(g)$, then it holds that $Y_i(g \setminus \{(i, j)\}) \leq Y_i(g)$.*

Proof. Suppose node i severs a link $e = (i, j) \in g$ with a node j such that $v(g \setminus \{(i, j)\}) \leq v(g)$. Let g_1 be the new graph; i.e. $g_1 = g \setminus \{(i, j)\}$. Now assume that

$$Y_i(g_1) > Y_i(g). \tag{6}$$

Since $v(g_1) \leq v(g)$, there exists a $z \in N \setminus \{i\}$ such that $Y_z(g_1) < Y_z(g)$. We now consider two cases.

Case 1: If $z = j$, then due to weak link symmetry (observe that g can be thought of as $g_1 \cup \{(i, j)\}$), then it must hold that $Y_i(g_1) < Y_i(g)$. This is a contradiction to our assumption (6). Hence our assumption is wrong and it must hold that $Y_i(g_1) \leq Y_i(g)$.

Case 2: If $z \neq j$, then due to improvement property, it holds that either (a) $Y_i(g_1) < Y_i(g)$ or (b) $Y_j(g_1) < Y_j(g)$. If (*a*) holds, then it is a contradiction to our expression (6). If (*b*) holds, then due to weak link symmetry, it must hold that $Y_i(g_1) < Y_i(g)$. Again it is a contradiction to the expression (6). Thus our assumption (6) is wrong and it must hold that $Y_i(g_1) \leq Y_i(g)$. □

Following the line of argument in Lemma 4, we can prove the following corollary.

Corollary 1. *For any graph g, under our model, if a node i severs a link $e = (i, j) \in g$ with a node j such that $v(g \setminus \{(i, j)\}) < v(g)$, then it must hold that $Y_i(g \setminus \{(i, j)\}) < Y_i(g)$.*

The above results are helpful in the analysis of the structure of pairwise stable graphs. We start with the case where $(b(1) - c) > b(2)$ and show that the complete graph is the only topology possible for pairwise stability.

Lemma 5. *If $c < (b(1) - b(2))$, then the complete graph is the unique topology possible for a pairwise stable graph.*

Proof. Given that $c < (b(1) - b(2))$. Consider an arbitrary graph g such that there exists a pair of non-neighbor nodes, say i and j. Assume that i and j form a link, call it (i, j). Let the new graph be g'; that is $g' = g \cup \{(i, j)\}$. We note that the shortest distance between any pair of nodes in g' is less than or equal to that in g. From this observation and the fact that $(b(1) - c) > b(2)$, we get that $v(g') > v(g)$. Now from Lemma 3, it turns out that $Y_i(g') > Y_i(g)$ and $Y_j(g') > Y_j(g)$. That is, nodes i and j strictly gain by forming a link. Hence a complete graph emerges as any two non-neighbor nodes will form a link to improve their respective utilities. We conclude that the complete graph is pairwise stable as no node severs a link to improve utility due to the fact that $(b(1) - c) > b(2)$ and following Corollary 1. For same reasons mentioned above, complete graph is the unique pairwise stable network. □

We now consider the case of $0 \leq (b(1)-c) < b(2)$. This condition is equivalent to the following condition: $c \in (b(1) - b(2), b(1)]$. In this case, our intention is to highlight the fact that pairwise stable networks may not exist for some specific values of cost c.

Example 3. *Consider for example a star graph g with 5 nodes as shown in Figure 5(i). We*

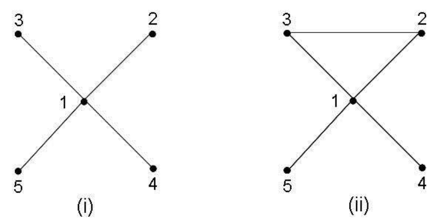

Figure 5: (i) A star graph g and (ii) A star graph with an edge g'

determine the utilities of the nodes using the Myerson value as follows: $Y_1(g) = 2(\delta - c) + 2\delta^2$, $Y_2(g) = Y_3(g) = Y_4(g) = Y_5(g) = \frac{1}{2}(\delta - c) + \delta^2$.

Now consider that nodes 2 and 3 form a link in the star graph g and call the new graph g' (as shown in Figure 5(ii)). As was done previously, we determine the utilities of the nodes in g' as follows: $Y_1(g') = 2(\delta - c) + \frac{5}{3}\delta^2$, $Y_2(g') = Y_3(g) = (\delta - c) + \frac{2}{3}\delta^2$, *and* $Y_4(g') = Y_5(g') = \frac{1}{2}(\delta - c) + \delta^2$.

First observe that $v(g') < v(g)$ as $c \in (b(1) - b(2), b(1)]$. In particular, when $c \in (b(1) - b(2),\ b(1) - \frac{2}{3}b(2))$, it happens that $Y_2(g') > Y_2(g)$ and $Y_3(g') > Y_3(g)$. That is, nodes 2 and 3 are strictly better off by forming a link in the star graph g. Hence the star graph is not pairwise stable at least when $c \in (b(1) - b(2),\ b(1) - \frac{2}{3}b(2))$. Q.E.D.

By virtue of Lemma 3, we note that the kind of situations exemplified in Example 3 may possibly occur only in situations when a pair of non-neighbor nodes i and j in g form a link (i, j) such that $v(g \cup \{(i, j)\}) \leq v(g)$ happens. Having said this, we take into account such scenarios with the help of a condition (which we call *regularity condition (RC)*) as follows.

Regularity Condition (RC): This involves a couple of conditions:

(a) If a pair of nodes i and j in a graph g are not neighbors and form a link (i, j) such that $v(g \cup \{(i, j)\}) \leq v(g)$, then it implies that either $Y_i(g \cup \{(i, j)\}) \leq Y_i(g)$ or $Y_j(g \cup \{(i, j)\}) \leq Y_j(g)$.
(b) $Y_i(g) \geq 0, \forall i \in N$.

We now determine the topologies of the pairwise stable networks when $c \in (b(1) - b(2), b(1)]$ and the regularity condition (RC) is satisfied. The following lemma states the result.

Lemma 6. *If $c \in (b(1)-b(2), b(1)]$ and* RC *is satisfied, then any minimal edge graph with diameter* 2 *is pairwise stable.*

Proof. Given that $c \in (b(1) - b(2), b(1)]$. Consider that g is any minimal edge graph with diameter 2. We prove this by first stating and proving two claims.

Claim 1: No node in g severs any link.
We prove this claim as follows. Consider that i is any node in g. Assume that i severs a link (call it (i, j)) with an arbitrary node j and also assume that the new graph is $g1$; i.e. $g1 = g \setminus \{(i, j)\}$. Since g is a minimal edge graph, the shortest path distance between at least one pair of nodes in $g1$ is greater than 2. Since $c \in (b(1) - b(2), b(1)]$, we get that $v(g1) < v(g)$. Due to Corollary 1, we get that $Y_i(g1) < Y_i(g)$. Hence node i does not sever the link (i, j).

Claim 2: No two nodes form a link.
To prove this claim, assume that a pair of nodes i and j form a link (i, j) in g and call the new graph $g2$. That is, $g2 = g \cup \{(i, j)\}$. Since g is a minimal edge graph with diameter 2, the new edge (i, j) only decreases the shortest distance between nodes i and j from two to one. From this observation and the fact that $c \in (b(1) - b(2), b(1)]$ we get that $v(g2) < v(g)$. Now since RC is satisfied, we get that $Y_i(g2) \leq Y_i(g)$ or $Y_j(g2) \leq Y_j(g)$. That is, at least one of node i and node j is not strictly benefited. At this point, weak link symmetry ensures that $Y_i(g2) \leq Y_i(g)$ and $Y_j(g2) \leq Y_j(g)$. Thus neither of the nodes i and j is better off by forming a new link in g.

Hence g is pairwise stable. □

We note that the star graph and any completely connected bipartite graph are minimal edge graphs with diameter 2 and hence they are pairwise stable due to the above lemma. We emphasize this fact as these graphs happen to be stable graphs using various models (Buskens and Rijt (2008); Goyal and Vega-Redondo (2007)) and various equilibrium notions. Thus Lemma 6 allows a wide variety of graphs to be pairwise stable compared to that of the several previous models. The following corollary conveys the same.

Corollary 2. *When $c \in (b(1) - b(2), b(1)]$ and* RC *is satisfied, then the star graph and the completely connected bi-partite graph are pairwise stable.*

However, there can be other graph structures that are pairwise stable under appropriate conditions. This further refinement of stable network structures could be carried out if we know the value of n. In what follows, we present a lemma provides more structure to the possible topologies for pairwise stable graphs.

Lemma 7. *If $(b(1) - b(p)) < c < (b(1) - b(p + 1))$ for any integer $p > 1$ and if g is a pairwise stable graph, then g is a graph with diameter p.*

Proof. Given that $(b(1) - b(p)) < c < (b(1) - b(p+1))$ for any integer $p > 1$. Suppose that g is a pairwise stable graph. Now we show that g is a graph with diameter p. Assume that i and j in g are such that $d_g(i, j) > p$ and they form a link (i, j). Call the new graph $g'' = g \cup \{(i, j)\}$. Since $b(p+1) < (b(1) - c) < b(p)$ and the shortest distance between any pair of nodes either remains same or decreases in g'' compared to that in g, we get that $v(g'') - v(g) > [(b(1) - c) - b(d_g(i, j))] \geq [(b(1) - c) - b(p+1)] > 0$. That is, $v(g'') > v(g)$. Now due to Lemma 3, nodes i and j are strictly better off. This is a contradiction since g is pairwise stable. Thus the shortest distance between any pair of nodes in g is at most p. Hence g is a graph with diameter p. □

Finally, when $c > b(1) + b(2)$, the empty graph is pairwise stable as forming an edge makes the value of the resultant network negative.

Lemma 8. *If* $c > b(1) + b(2)$, *then the empty graph is pairwise stable.*

5. Efficiency Versus Pairwise Stability

In this section, we investigate the tension between pairwise stability and efficiency and then we propose a necessary and sufficient condition for every efficient network to be pairwise stable.

Recall from Theorem 1 that the star graph is the unique efficient network when $c \in (b(1) - b(2), b(1)]$. Now consider for example a star graph g with 5 nodes as shown in Figure 5(i). We determine the utilities of the nodes using the Myerson value as shown in Example 3. Now consider that nodes 2 and 3 form a link in g and call the new graph g' (as shown in Figure 5(ii)). Again using the Myerson value, we determine the utilities of the nodes in g' as shown in Example 3. We first observe that $v(g') < v(g)$ as $c \in (b(1)-b(2), b(1)]$. In particular, when $c \in (b(1)-b(2),\ b(1)-\frac{2}{3}b(2))$, it holds that $Y_2(g') > Y_2(g)$ and $Y_3(g') > Y_3(g)$. That is, nodes 2 and 3 are strictly better off by forming a link in the star graph g. Hence the star graph is efficient but not pairwise stable at least when $c \in (b(1) - b(2),\ b(1) - \frac{2}{3}b(2))$.

These kinds tradeoffs between efficiency and stability are not specific to our model. In fact, such tradeoffs have been highlighted by Jackson and Wolinsky (1996) and are pointed out by several models of network formation in the literature. In this regard, we show that the *Regularity Condition (RC)*) turns out to be a necessary and sufficient condition for any efficient network to be pairwise stable following our model. In the following, we show the same result.

Theorem 2. *Consider an anonymous and component additive value function v; and an anonymous, component balanced allocation rule $Y(.)$ satisfying weak link symmetry and improvement properties. Suppose g is an efficient graph relative to v. Then g is pairwise stable if and only if v, Y, and g satisfy the* regularity condition (RC).

Proof. We prove this in two parts.

Part 1: Given that g is an efficient graph. Assume that g is pairwise stable. We now show that the regularity condition is satisfied. Let i and j be a pair of non-neighbor nodes in g and form a link, (i, j). Assume that the new graph is $g' = g \cup \{(i, j)\}$. Since g is efficient and v is component additive, we get that $v(g') \leq v(g)$. If $Y_i(g') > Y_i(g)$ (or $Y_j(g') > Y_j(g)$), then due to weak link symmetry, we get that $Y_j(g') > Y_j(g)$ (or $Y_i(g') > Y_i(g)$). This contradicts the fact that g is pairwise stable. This implies that neither i nor j is strictly better off in g'. At the same time as g is pairwise stable, we have that $Y_i(g) \geq 0, \forall i \in N$. Hence RC is satisfied.

Part 2: Now we prove the other way. Given that g is an efficient graph relative to a component additive v and that the regularity condition is satisfied. We show that g is pairwise stable for Y relative to v. We now deal with severing or adding a link separately as follows.

Suppose a node x_1 severs a link (x_1, y_1) with node y_1 in g. Assume that the new graph is g_1; i.e. $g_1 = g \setminus \{(x_1, y_1)\}$. Since g is efficient and v is component additive, we get that $v(g_1) \leq v(g)$. Now from Lemma 4, node x_1 is not strictly better off by severing the link.

Suppose two non-neighbor nodes i and j form a link (i, j) in g. Call the new graph g'; i.e. $g' = g \cup \{(i, j)\}$. Since g is efficient and v is component additive, it holds that $v(g') \leq v(g)$. If $Y_i(g') > Y_i(g)$ (or $Y_j(g') > Y_j(g)$), then due to weak link symmetry, we get that $Y_j(g') > Y_j(g)$ (or $Y_i(g') > Y_i(g)$). This contradicts RC. This implies that neither i nor j is strictly better off. Again since RC is satisfied, we get that $Y_i(g) \geq 0, \forall i \in N$.

Thus g is pairwise stable. □

6. Pairwise Stable Topologies under More Specified Settings

We have shown in Section 4 that the complete graph and any minimal edge graph with diameter 2 (for example: star graph, any completely connected bipartite graph) are pairwise stable under appropriate conditions, following the generic model proposed here in this paper. In this section, we focus our attention on situations where the topology happens to be a minimal edge graph with diameter 2 and derive more specific topologies by looking at these situations in more detail. We do this investigation following the framework proposed by Doreian (2006). In particular, we consider all networks with 4 nodes and then study the resulting topologies. The set of all possible (in this case 11) network structures are shown in Figure 6.

Following the framework of Doreian (2006), if we restrict our attention only to changes in the graphs that occur through the addition of a single edge, the only possible changes among the 11 graphs (in Figure 6) would be as shown in Figure 7. Now following our model, we determine the value of each graph in the set of 11 possible graphs with 4 nodes and these are shown in Table 2. The value of each graph is divided among the nodes in the graph using the Myerson value (Myerson (1977)) and the respective utilities of the nodes for each graph is shown in Table 3.

We now concentrate on deriving appropriate conditions for the transitions between various graphs as shown in Figure 7. Let us first consider the transition from $g1$ to $g2$ and note that $g1$ is obtained from $g0$ by the addition of an edge, call it (a, b). From Table 3, the utilities of both node a and node b will be positive if $c < b(1)$. On the other hand, if $c > b(1)$, then the link (a, b) would not form as the utility of each node would decrease.

Now consider the transition from $g2$ to $g3$ where a link between any two nodes can form, call it (b, d). Note that the utility of node b in $g3$ is $(b(1) - c) + \frac{b(2)}{3}$ and in $g2$ is $\frac{b(1)-c}{2}$. For the utility of node b to increase in $g3$, we should have $\frac{b(1)-c}{2} + \frac{b(2)}{3} > 0 \Rightarrow c < b(1) + \frac{2b(2)}{3}$. Similarly, the condition for the utility of node d to increase in $g3$ over $g2$ is $c < b(1) + \frac{2b(2)}{3}$. Hence if $c < b(1) + \frac{2b(2)}{3}$, the utilities of both node b and node d would increase and hence $g3$ emerges.

Next consider the transition from $g2$ to $g4$ through the addition of a link, call it (c, d). From Table 3, we can see that the utility of node c would increase in $g4$ compared to that of $g2$ when

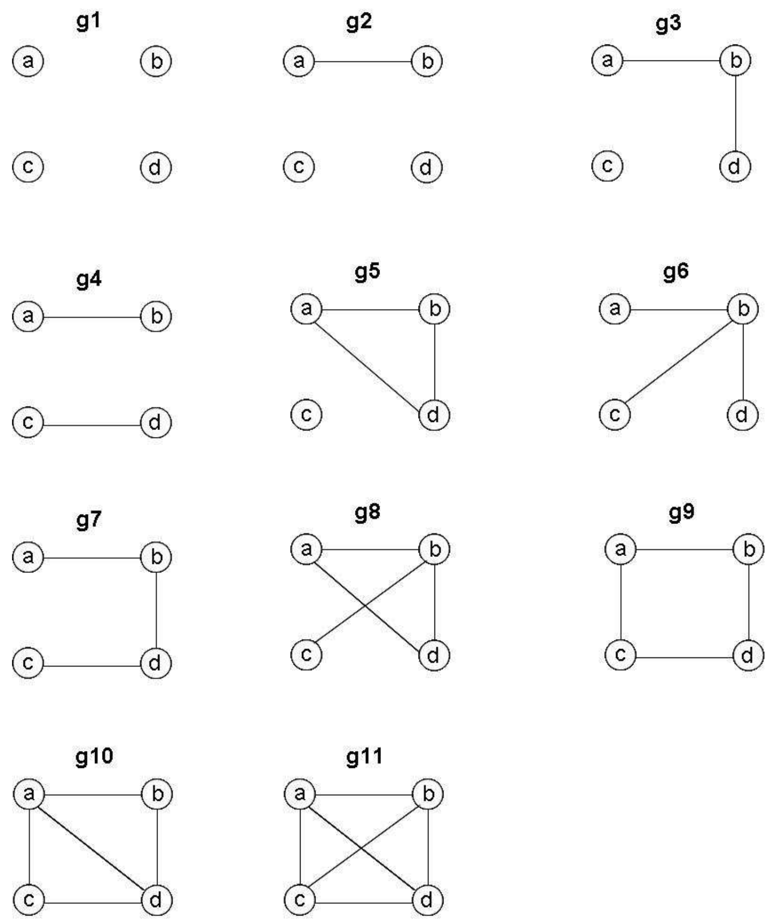

Figure 6: All possible graph structures with 4 nodes

Graph	**Value of Graph**
$g1$	0
$g2$	$b(1) - c$
$g3$	$2(b(1) - c) + b(2)$
$g4$	$2(b(1) - c)$
$g5$	$3(b(1) - c)$
$g6$	$3(b(1) - c) + 3b(2)$
$g7$	$3(b(1) - c) + 2b(2) + b(3)$
$g8$	$4(b(1) - c) + 2b(2)$
$g9$	$4(b(1) - c) + 2b(2)$
$g10$	$5(b(1) - c) + b(2)$
$g11$	$6(b(1) - c)$

Table 2: Value of each graph in the set of all possible graphs with 4 nodes

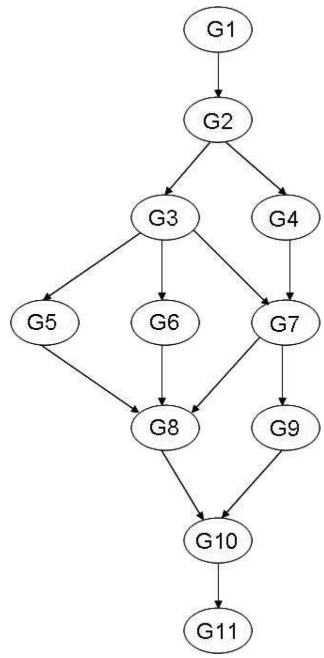

Figure 7: Lattice of (edge) graphs with 4 vertices

Graph	**Node** a	**Node** b	**Node** c	**Node** d
$g1$	0	0	0	0
$g2$	$\frac{b(1)-c}{2}$	$\frac{b(1)-c}{2}$	0	0
$g3$	$\frac{b(1)-c}{2} + \frac{b(2)}{3}$	$(b(1)-c) + \frac{b(2)}{3}$	0	$\frac{b(1)-c}{2} + \frac{b(2)}{3}$
$g4$	$\frac{b(1)-c}{2}$	$\frac{b(1)-c}{2}$	$\frac{b(1)-c}{2}$	$\frac{b(1)-c}{2}$
$g5$	$b(1)-c$	$b(1)-c$	0	$b(1)-c$
$g6$	$\frac{b(1)-c}{2} + \frac{2b(2)}{3}$	$\frac{3(b(1)-c)}{2} + b(2)$	$\frac{b(1)-c}{2} + \frac{2b(2)}{3}$	$\frac{b(1)-c}{2} + \frac{2b(2)}{3}$
$g7$	$\frac{b(1)-c}{2} + \frac{b(2)}{3} + \frac{b(3)}{4}$	$(b(1)-c) + \frac{2b(2)}{3} + \frac{b(3)}{4}$	$\frac{b(1)-c}{2} + \frac{b(2)}{3} + \frac{b(3)}{4}$	$(b(1)-c) + \frac{2b(2)}{3} + \frac{b(3)}{4}$
$g8$	$(b(1)-c) + \frac{b(2)}{3}$	$\frac{3(b(1)-c)}{2} + \frac{2b(2)}{3}$	$\frac{b(1)-c}{2} + \frac{2b(2)}{3}$	$(b(1)-c) + \frac{b(2)}{3}$
$g9$	$(b(1)-c) + \frac{b(2)}{2}$	$(b(1)-c) + \frac{b(2)}{2}$	$(b(1)-c) + \frac{b(2)}{2}$	$(b(1)-c) + \frac{b(2)}{2}$
$g10$	$\frac{3(b(1)-c)}{2} + \frac{b(2)}{12}$	$(b(1)-c) + \frac{5b(2)}{12}$	$(b(1)-c) + \frac{5b(2)}{12}$	$\frac{3(b(1)-c)}{2} + \frac{b(2)}{12}$
$g11$	$\frac{3(b(1)-c)}{2}$	$\frac{3(b(1)-c)}{2}$	$\frac{3(b(1)-c)}{2}$	$\frac{3(b(1)-c)}{2}$

Table 3: Utilities of nodes for all graphs with 4 nodes, computed using the Myerson value

$c < b(1)$. Again, the same condition holds good for node d as well for its utility to increase in $g4$. Hence under the condition $c < b(1)$, the utility of both node c and node d would increase in $g4$ compared to that of $g2$.

On similar lines, one can derive appropriate conditions for the transitions between graphs in the sequence shown in Figure 7. Table 4 provides the conditions required for each transition from one graph to other graph.

From Graph	**To Graph**	**Link added**	**Condition for First Node**	**Condition for Second Node**
$g1$	$g2$	(a, b)	$c < b(1)$	$c < b(1)$
$g2$	$g3$	(b, d)	$c < b(1) + \frac{2b(2)}{3}$	$c < b(1) + \frac{2b(2)}{3}$
$g2$	$g4$	(c, d)	$c < b(1)$	$c < b(1)$
$g3$	$g5$	(a, d)	$c < b(1) - \frac{2b(2)}{3}$	$c < b(1) - \frac{2b(2)}{3}$
$g3$	$g6$	(b, c)	$c < b(1) + \frac{4b(2)}{3}$	$c < b(1) + \frac{4b(2)}{3}$
$g3$	$g7$	(c, d)	$c < b(1) + \frac{2b(2)}{3} + \frac{b(3)}{2}$	$c < b(1) + \frac{2b(2)}{3} + \frac{b(3)}{2}$
$g4$	$g7$	(b, d)	$c < b(1) + \frac{4b(2)}{3} + \frac{b(3)}{2}$	$c < b(1) + \frac{4b(2)}{3} + \frac{b(3)}{2}$
$g5$	$g8$	(b, c)	$c < b(1) + \frac{4b(2)}{3}$	$c < b(1) + \frac{4b(2)}{3}$
$g6$	$g8$	(a, d)	$c < b(1) - \frac{2b(2)}{3}$	$c < b(1) - \frac{2b(2)}{3}$
$g7$	$g8$	(b, d)	$c < b(1) - \frac{b(3)}{2}$	$c < b(1) - \frac{b(3)}{2}$
$g7$	$g9$	(a, c)	$c < b(1) + \frac{b(2)}{3} - \frac{b(3)}{2}$	$c < b(1) + \frac{b(2)}{3} - \frac{b(3)}{2}$
$g8$	$g10$	(c, d)	$c < b(1) - \frac{b(2)}{2}$	$c < b(1) - \frac{b(2)}{2}$
$g9$	$g10$	(a, d)	$c < b(1) - \frac{5b(2)}{6}$	$c < b(1) - \frac{5b(2)}{6}$
$g10$	$g11$	(b, c)	$c < b(1) - \frac{5b(2)}{6}$	$c < b(1) - \frac{5b(2)}{6}$

Table 4: Conditions for the transitions between graphs with 4 nodes

Note 1: Consider (for example) the transition from graph $g2$ to graph $g3$ through the formation of an edge (b, d). From Table 4, we get that this transition occurs if $c < b(1) + \frac{2b(2)}{3}$ and in this case the utilities of both node b and node d are strictly higher in $g3$ compared to that of in $g2$. On the other hand, if $c = b(1) + \frac{2b(2)}{3}$, then the utilities of both b and d are the same in $g2$ and $g3$ respectively. That is, nodes b and d are indifferent in $g2$ and $g3$ in terms of their utilities. The same holds good for all the transitions between the graphs shown in Figure 7.

Note 2: We assume that a pair of nodes would form an edge if it strictly increases the utilities of those two nodes. That is, nodes deviate from their current state (by forming a link or deleting a link with other nodes) if such a deviation is beneficial. For this reason, all the conditions for transitions between graphs shown in Table 4 do not involve any boundary (or equality) conditions. Also we follow this convention throughout our analysis.

From the previous analysis and the information in Table 4, we arrive at the following conclusions:

- If $c < b(1) - b(2)$, then all possible transitions are possible in Figure 7 and hence $g11$ (i.e. complete graph) emerges as the unique pairwise stable graph. This observation is consistent with Lemma 5.

- If $(b(1) - b(2)) < c < (b(1) - \frac{5b(2)}{6})$, then all possible transitions are possible between graphs in Figure 7 and hence $g11$ (i.e. complete graph) emerges as pairwise stable graph. We note that this observation is in line with Corollary 2. This is because the regularity condition (RC) is not satisfied by the star graph (i.e. $g6$) and the completely connected bi-partite graph (i.e. $g9$) in the range $(b(1) - b(2)) < c < (b(1) - \frac{5b(2)}{6})$.

- If $(b(1) - \frac{5b(2)}{6}) < c < (b(1) - \frac{2b(2)}{3})$, then we show the possible transitions in Figure 8 with solid lines. From this figure, we obtain that graphs $g9$ and $g10$ are stable graphs. We note that $g9$ is a cycle graph (or a completely connected bi-partite graph) and $g10$ is a *near complete graph* (using the terminology of Hummon (2000) and Doreian (2006)).

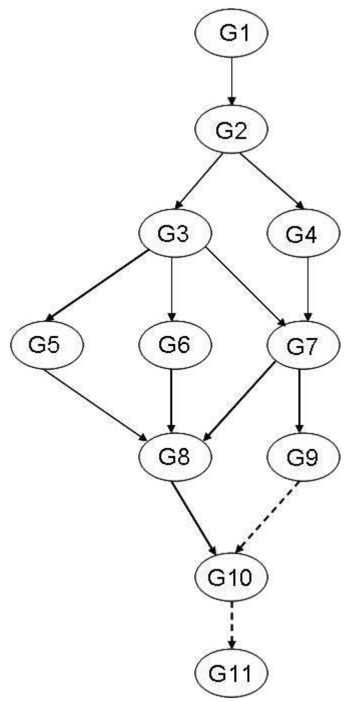

Figure 8: Transitions for $(b(1) - \frac{5b(2)}{6}) < c < (b(1) - \frac{2b(2)}{3})$ in lattice of graphs with 4 vertices. Solid lines indicate possible transitions and Dotted lines indicate transitions that are not possible between graphs.

- Consider that $(b(1) - \frac{2b(2)}{3}) < c < (b(1) - \frac{b(2)}{2})$. If we perform similar analysis as we did above, we get that $g6$ (i.e. star graph) and $g10$ (i.e. near complete graph) as pairwise stable graphs. Again this result is consistent with Corollary 2 as the regularity condition (RC) is not satisfied by the completely connected bi-partite graph (i.e. $g9$) in the range $(b(1) - \frac{2b(2)}{3}) < c < (b(1) - \frac{b(2)}{2})$.

- Now consider that $(b(1) - \frac{b(2)}{2}) < c < (b(1) - b(2))$. In this range, we get that $g6$ (i.e. star graph), $g8$ (i.e. near star graph), and $g9$ (i.e. completely connected bi-partite graph) are pairwise stable. Note that this observation is in favor of Corollary 2 as the regularity condition (RC) is satisfied by both $g6$ and $g9$ in the cost rage $(b(1) - \frac{b(2)}{2}) < c < (b(1) - b(2))$.

- If $(b(1) - b(2)) < c < (b(1) - \frac{b(3)}{2})$, the usual analysis reveals that $g6$ (i.e. star graph), $g8$ (i.e. near star graph), and $g9$ (i.e. completely connected bi-partite graph) are pairwise stable.
- Consider that $(b(1) - \frac{b(3)}{2}) < c < (b(1) - b(3))$. In this case, our analysis finds that $g6$ (i.e. star graph) and $g9$ (i.e. completely connected bi-partite graph) are pairwise stable.
- Now consider that $(b(1) - b(3)) < c < b(1)$. If we perform analysis on similar lines as above, we identify that $g6$ (i.e. star graph) and $g9$ (i.e. completely connected bi-partite graph) are pairwise stable when $b(1) < \frac{2}{3}$. Further, when $b(1) > \frac{2}{3}$, we get that $g6$ (i.e. star graph) and $g7$ (i.e. near completely connected bi-partite graph) as pairwise stable.
- Finally, when $c > b(1)$, then we find that only the null graph is pairwise stable.

We can perform similar analysis on networks with 5 nodes, 6 nodes, etc. as well and unravel the possible topologies using the proposed network formation model. However, there are limitations of this analysis as exhaustive searches start becoming prohibitive (Doreian (2006)). In other words, for networks with a large number of nodes, it is difficult to conduct the previous analysis and for such scenarios, we may have to rely more on a formal analysis of the model.

7. Conclusions and Future Work

In this paper, we worked with a generic model of social network formation that takes into account four key determinants of network formation. We characterized the topologies of the networks that are efficient and that are pairwise stable. We also provided a necessary and sufficient condition for any efficient network to be pairwise stable.

We note that there are several ways to extend this model. We provide a few such possibilities:

- One can study the topologies of networks under other notions of stability such as Nash stability, strong pairwise stability, etc.
- It would be interesting to further extend the model presented in this paper as follows: (a) one can try defining the value function differently and verify the topologies of the efficient and pairwise stable networks, and (b) exploring and verifying the applicability of other specific allocation rules such as the core, networkolus (Jackson (2005)), etc.
- We considered that all nodes in the network are homogeneous. In real world social networks, we find individuals with varying degrees of influence. We can study such contexts by assigning weights to individual nodes.

8. Acknowledgements

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Appendix A. Myerson Value

Let N be a nonempty finite set and it represents the set of players. A graph on N is a set of unordered pairs of distinct members of N. We will refer to these unordered pairs as links, and we will denote the link between n and m by (n, m). Note that the links are unordered, both (n, m) and (m, n) mean the same. Let g^N be the complete graph of all links: $g^N = (n, m) \mid n \in N, m \in N,\ n \neq m$. Suppose GR is the set of all graphs on N, so that $GR = \{g \mid g \subseteq g^N\}$. In other words, each graph g in GN represents a possible cooperation structure among the players. Also, let z be a game in characteristic function form (Myerson (1977)). We try to stick to the notation that is used in Myerson (1977) and we introduce a few definitions before presenting the main result on Myerson value.

Coalition: A coalition is a nonempty subset of N.

Connectedness: We will need a few basic concepts of connectedness, to relate coalitions and cooperation graphs. Suppose $S \subseteq N$, $g \in GR$, $n \in S$, and $m \in S$ are given. Then we say that n and m are connected in S by g if and only if there is a path in g which goes from n to m and stays within S. That is, n and m are connected in S by g if $n = m$ or if there is some $k > 1$ and a sequence $(n^0, n^1, \ldots, n^k)$ such that $n^0 = n$, $n^k = m$, and $(n^{i-1}, n^i) \in g$ and $n^i \in S$ for all i from 1 to k.

Partition of a Coalition: Given $g \in GR$ and $S \subseteq N$, there is a unique partition of S which groups players together if and only if they are connected in S by g. We will denote this partition by S/g (reads as S divided by g), so: $S/g = \{\{i \mid i \textit{ and } j \textit{ are connected in } S \textit{ by } g\} \mid j \in S\}$. We can interpret S/g as the collection of smaller coalitions into which S would break up, if players could only coordinate along the links in g.

Allocation Rule: We define an allocation rule for z to be any function $Y : GR \rightarrow R^{|\mathbb{N}|}$ such that $\forall g \in GR$, $\forall S \in N/g$, $\sum_{n \in S} Y_n(g) = z(S)$.

We use the symbol \ to denote removal of a member from a set. Thus $g \setminus (n, m) = \{(i, j) \mid (i, j) \in g,\ (i, j) \neq (n, m)\}$.

Stable Allocation Rule: An allocation rule $Y : GR \rightarrow R^{|\mathbb{N}|}$ is stable if and only if $\forall g \in GR$, $\forall (n, m) \in g$, $Y_n(g) > Y_n(g \setminus (n, m))$ and $Y_m(g) > Y_m(g \setminus (n, m))$.

Fair Allocation Rule: We define an allocation rule $Y : GR \rightarrow R^{|\mathbb{N}|}$ to be fair if and only if $\forall g \in GR$, $\forall (n, m) \in g$, $Y_n(g) - Y_n(g \setminus (n, m)) = Y_m(g) - Y_m(g \setminus (n, m))$.

Given a characteristic function game z and a graph g, define z/g to be a characteristic function game so that $\forall S \subseteq N$, $(z/g)(S) = \sum_{T \in S/g} z(T)$. One may interpret z/g as the characteristic function game which would result if we altered the situation represented by z, requiring that players can only communicate along links in g.

With the above definitions in place, we can define the Myerson value as follows.

Theorem 3. *(Myerson (1977)) Given a characteristic function game z, there is a unique fair allocation rule $Y : GR \rightarrow R^{|\mathbb{N}|}$. This fair allocation rule also satisfies $Y(g) = \phi(z/g)$, $\forall g \in GR$, where $\phi(.)$ is the Shapley value operator. Furthermore, if z is superadditive then the fair allocation rule is stable.*

The following example from Myerson (1977) illustrates the above concepts.

Example 4. *Let $N = \{1, 2, 3\}$, and consider the characteristic function game z where:*

$$z(\{1\}) = z(\{2\}) = z(\{3\}) = 0,$$
$$z(\{1,3\}) = z(\{2,3\}) = 6, \textit{ and}$$
$$z(\{1,2\}) = z(\{1,2,3\}) = 12.$$

The Myerson value for this game is as follows:

$$\begin{aligned}
Y(\Phi) &= (0,0,0),\\
Y(\{(1,2),(1,3)\}) &= (7,4,1),\\
Y(\{(1,2)\}) &= (6,6,0),\\
Y(\{(1,2),(2,3)\}) &= (4,7,1),\\
Y(\{(1,3)\}) &= (3,0,3),\\
Y(\{(1,3),(2,3)\}) &= (3,3,6),\\
Y(\{(2,3)\}) &= (0,3,3),\\
Y(\{(1,2),(1,3),(2,3)\}) &= (5,5,2).
\end{aligned}$$

Note: Chapter 12 in Jackson (2008) also provides a description on Myerson value.

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