Game Theory

Lecture Notes By

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Existence of Nash Equilibrium

Note: *This is a only a draft version, so there could be flaws. If you find any errors, please do send email to* `hari@csa.iisc.ernet.in`. *A more thorough version would be available soon in this space.*

Existence of Nash equilibrium is a key question investigated extensively in game theory. For two person zero sum games, with finite strategy sets, Von Nemann and Morgenstern were able to establish the existence of randomized saddle point through their celebrated minimax theorem [1] in 1928. John Nash, in his brilliant work [2, 3], generalized the notion of an equilibrium to games with three or more players and also established the existence of at least one mixed strategy, Nash equilibrium for every finite strategic form games (that is, with finite number of players and finite strategy sets for players). The existence of equilibria in games is closely coupled with fixed point theorems such as the Brouwer fixed point theorem [4] and Kakutani's fixed point theorem [5]. We shall study these important results in this chapter. We also present the Sperner's lemma, a celebrated result in combinatorics, which can be used to prove fixed point theorems and the Nash theorem. Mathematical preliminaries required for this chapter are included in an appendix at the end of the chapter.

1 Correspondences and Fixed Point Theorems

Given a set $X \subset \mathbb{R}^n$, a correspondence is a mapping that assigns to each $x \in X$, a subset of $\mathbb{R}^k$. We use the notation $f : X \rightrightarrows \mathbb{R}^k$. It is also referred to as *set function*. If $Y \subset \mathbb{R}^k$ and $f(x) \subset Y \ \forall x \in X$, then we say f is a correspondence from X to Y and denote it as $f : X \rightrightarrows Y$.

Note that when $f(x)$ contains exactly one element for each x, f simply becomes an ordinary function.

Graph of a Correspondence

Let $X \subset \mathbb{R}^n$ and let $Y \subset \mathbb{R}^k$ be a closed set. The graph of the correspondence $f : X \rightrightarrows Y$ is the set $\{(x, y) : x \in X, y \in Y, y \in f(x)\}$.

Closed Graph Correspondence

Given $X \subset \mathbb{R}^n$ and a closed set $Y \subset \mathbb{R}^k$, the correspondence $f : X \rightrightarrows Y$ is said to have a closed graph if for any two sequences $x^m \to x \in X$ and $y^m \to y \in Y$, with $x^m \in X$ and $y^m \in f(x^m)$ for every m, we have $y \in f(x)$.

Upper Hemicontinuity

Given $X \subset \mathbb{R}^n$ and a closed set $Y \subset \mathbb{R}^k$, the correspondence $f : X \rightrightarrows Y$ is said to be upper hemicontinuous (uhc) if it has a closed graph and the images of compact sets are bounded.

The above definition also implies that the images of compact sets are in fact compact. Upper hemicontinuity of correspondences is a natural generalization of the notion of continuity of functions.

1.1 Fixed Point Theorems

Fixed point theorems provide very useful results for establishing the existence of solutions of an equilibrium system of equations. The idea is to formulate the problem as the search for a fixed point of a suitably constructed function or correspondence.

Fixed Point of a Function

Let $X \subset \mathbb{R}^n$ and $f : X \to X$ be a mapping. A point $x \in X$ is called a *fixed point* of f if $f(x) = x$.

As an immediate example, suppose $X : [0, 1] \to [0, 1]$ and $f(x) = x^2$. f has two fixed points 0 and 1 since $f(0) = 0$ and $f(1) = 1$.

Tarsky's Fixed Point Theorem

Suppose that $f : [0, 1]^n \to [0, 1]^n$ is a non-decreasing function. Then f has a fixed point.

Brouwer's Fixed Point Theorem

This famous theorem which dates back to 1912 [4] can be used to show that every two player zerosum game has at least one mixed strategy Nash equilibrium. The statement of the theorem is as follows: Let $X \subset \mathbb{R}^n$ be compact and convex. If $f : X \to X$ is continuous, then f has a fixed point.

Fixed Point of a Correspondence

Let $X \subset \mathbb{R}^n$ and $f : X \rightrightarrows X$ be a correspondence. A point $x \in X$ is called a fixed point of f if $x \in f(x)$.

Kakutani's Fixed Point Theorem

This famous theorem due to Kakutani [5] is used extensively in game theory. In fact, John Nash used it in his doctoral work to prove the existence of a mixed Nash equilibrium in finite strategic form games. The theorem is also used in arriving at sufficient conditions for existence of pure strategy Nash equilibrium in finite games.

(1) Suppose that $X \subset \mathbb{R}^n$ is a non-empty, compact, and convex subset of $\mathbb{R}^n$.

(2) Let $f : X \rightrightarrows X$ be a correspondence such that

(a) f is upper hemicontinuous

(b) $f(x) \subset X \; \forall \, x \in X$ is non-empty and convex.

Then f has a fixed point in X.

2 Nash Equilibrium as a Fixed Point

2.1 Pure Strategy Nash Equilibrium

Consider a strategic form game $\Gamma =< N, (S_i), (u_i) >$. We have seen that a strategy profile $(s_1^*, \ldots, s_n^*)$ is a pure strategy Nash equilibrium iff

$$s_i^* \in b_i(s_{-i}^*) \quad \forall \, i \in N$$

where b_i is the best response correspondence:

$$b_i(s_{-i}) = \{s_i \in S_i : u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \; \forall s_i' \in S_i\}.$$

Let us define the composite correspondence $b : S \rightrightarrows S$ as follows:

$$b(s_1, \ldots, s_n) = b_1(s_{-1}) \times \ldots \times b_n(s_{-n})$$

Now, $(s_1^*, \ldots, s_n^*)$ is a Nash equilibrium is equivalent to $s_i^* \in b_i(s_{-i}^*) \quad \forall i \in N$. This means

$$(s_i^*, \ldots, s_n^*) \in b_1(s_{-1}^*) \times \cdots \times b_n(s_{-n}^*)$$

which in turn means

$$s_i^*, \ldots, s_n^*) \in b(s_{-i}^*, \ldots, s_{-n}^*)$$

This is the same as saying that s^* is a fixed point of the best response correspondence b.

An Example: Prisoner's Dilemma

We provide below the payoff matrix of the prisoner's dilemma problem for ready reference.

	2	
1	NC	C
NC	$-2, -2$	$-10, -1$
C	$-1, -10$	$-5, -5$

Note that $S_1 = S_2 = \{C, NC\}$ and $b_1(C) = b_1(NC) = b_2(C) = b_2(NC) = \{C\}$. The best response correspondence for different profiles is given by

$$\begin{aligned} b(NC, NC) &= b_1(C) \times b_2(C) = \{(C, C)\} \\ b(NC, C) &= b_1(C) \times b_2(C) = \{(C, C)\} \\ b(C, NC) &= b_1(NC) \times b_2(C) = \{(C, C)\} \\ b(C, C) &= b_1(NC) \times b_2(NC) = \{(C, C)\}. \end{aligned}$$

The above implies that (C, C) is a Nash equilibrium.

2.2 Mixed Strategy Nash Equilibrium

Now consider mixed strategies. Recall that a mixed strategy profile $(\sigma_1^*, \ldots, \sigma_n^*)$ is a Nash equilibrium iff

$$\sigma_i^* \in b_i(\sigma_{-i}^*) \ \forall \ i \in N$$

where b_i the best response correspondence defined by

$$b_i(\sigma_{-i}) = \{\sigma_i \in \Delta(S_i) : u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i}) \ \forall \ \sigma_i' \in \Delta(S_i)\}.$$

Let us define the composite correspondence

$$b : \Delta(S_1) \times \ldots \Delta(S_n) \quad \rightarrow \quad \Delta(S_1) \times \ldots \Delta(S_n)$$

as we defined earlier in the case of pure strategies:

$$b(\sigma_1, \ldots, \sigma_n) = b_1(\sigma_{-i}) \times \ldots b_n(\sigma_{-n})$$

Clearly, a mixed strategy profile $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ is a Nash equilibrium iff $\sigma^* \in b(\sigma^*)$, that is, iff σ^* is a fixed point of the best response correspondence b.

3 An Important Lemma

We now state and prove an important lemma which is crucial for providing sufficient conditions for existence of pure strategy Nash equilibria in finite games as well as in proving the Nash theorem.

1. Suppose the sets $S_1, S_2, \ldots, S_n$ are non-empty, compact, and convex.
2. $u_i : S_1 \times \cdots \times S_n \to \mathbb{R}$ is continuous in $(s_1, \ldots, s_n)$, and
3. $u_i(s_i, s_{-i})$ quasi-concave in s_i.

Then the best response correspondence of player i defined by

$$b_i(s_{-i}) = \{s_i \in S_i : u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \ \forall \ s_i' \in S_i\}$$

is (a) non-empty, (b) convex, and (c) upper hemicontinuous.

Proof of the Lemma

First we show that $b_i(s_{-i})$ is non-empty. Note that $b_i(s_{-i})$ is the set of maximizers of a continuous function u_i over a compact set S. (S is compact since $S_1, \ldots, S_n$ are compact). Therefore by Weirstrass's theorem, the function u_i attains a maximum and hence $b_i(s_{-i})$ is non-empty.

Next we show that $b_i(s_{-i})$ is convex-valued. It is given that u_i is quasi-concave in s_i. This means $u_i(., s_{-i})$ is quasi concave on S_i. Fix s_{-i} and call $u_i(x, s_{-i}) = u_i(x)$. By definition of quasi-concavity, each of the the sets $U_f(a) = \{x \in S_i : u_i(x) \geq a\}$ is convex $\forall a \in \mathbb{R}$. Let

$$a = \max_{x \in S_i} u_i(x)$$

This is guaranteed to exist, thanks to Weirstrass's theorem. Then the definition implies that the set of all maximizers of a quasi-concave function is a convex set. Therefore $b_i(s_{-i})$ is a convex set.

Now we show that b_i is upper hemicontinuous. To show this, we have to show that

1. b_i has a closed graph. That is, for any sequence $(s_i^n, s_{-i}^n) \to (s_i, s_{-i})$ such that $s_i^n \in b_i(s_{-i}^n)\ \forall n$, we have that $s_i \in b_i(s_{-i})$.

2. b_i maps compact sets into bounded sets. It is easy to see that the set $U_c = b_i(s_{-i})$ is bounded.

To show (1), we are given that $(s_i^n, s_{-i}^n) \to (s_i, s_{-i})$ with $s_i^n \in b_i(s_{-i}^n)\ \forall n$ This implies that $u_i(s_i^n, s_{-i}^n) \geq u_i(s_i', s_{-i}^n)\ \forall s_i' \in S_i\ \forall n$. Since u_i is continuous, we have

$$\begin{aligned} u_i(s_i^n, s_{-i}^n) &\to u_i(s_i, s_{-i}) \\ u_i(s_i', s_{-i}^n) &\to u_i(s_i', s_{-i}) \end{aligned}$$

This implies that

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \quad \forall s_i' \in S_i \Rightarrow s_i \in b_i(s_{-i})$$

This shows that b_i has a closed graph.

4 Sufficient Conditions for Existence of Pure Strategy Nash Equilibria

This result is due to Debreu (1952) [6]. Given a strategic form game game $\Gamma = \langle N, (S_i), (u_i) \rangle$, a pure strategy Nash equilibrium exists if $\forall i = 1, \ldots, n$,

(1) S_i is a non-empty, convex, and compact subset of some Euclidean space

(2) $u_i(s_1, \ldots, s_n)$ is continuous in $(s_1, \ldots, s_n)$

(3) $u_i(s_i, s_{-i})$ is quasi-concave in s_i.

To prove the above result, recall the definition of the best response correspondence $b : S \rightrightarrows S$:

$$b(s_1, \ldots, s_n) = b_1(s_{-1}) \times \cdots \times b_n(s_{-n})$$

Also recall that a pure strategy Nash equilibrium is a fixed point of this correspondence.

Note that $S = S_1 \times \cdots S_n$ is non-empty, compact, and convex since each S_i is non-empty, compact, and convex. Also note by the previous lemma that $b(.)$ is a non-empty, convex-valued, and upper hemicontinuous correspondence. The correspondence b satisfies all the conditions of Kakutani's fixed point theorem and therefore we can now apply the theorem to conclude that the correspondence b has a fixed point which is precisely a pure strategy Nash equilibrium.

We make the following observations.

- The above result does not apply to games which have finite strategy sets because finite sets are not convex.

- The result provides only a sufficient condition. Many finite games having pure strategy Nash equilibria immediately show that the conditions are not necessary.

- The result does not say when a Nash equilibrium is unique.

- Allowing for mixed strategies leads to *convexification* and *compactification* of the strategy sets and ensures that mixed strategy Nash equilibria always exist. We show this in the next section.

- In the above theorem, quasi-concavity cannot be relaxed. For example, consider the game with $N = \{1, 2\}$ where each player picks a point on the unit circle simultaneously. The payoffs are defined as

$$u_1(s_1, s_2) = d(s_1, s_2) \quad \forall\, s_1, s_2 \in [1, 2] \times [1, 2]$$

$$u_2(s_1, s_2) = -d(s_1, s_2) \quad \forall\, s_1, s_2 \in [1, 2] \times [1, 2]$$

  where $d(s_1, s_2)$ is the Euclidean distance between the points s_1 and s_2. In this game, if the two players pick the same location, there is an incentive for player 1 to deviate. On the other hand, if they pick up different locations, there is an incentive for player 2 to deviate. Hence there does not exist any pure strategy Nash equilibrium for this game.

5 The Nash Theorem

We are now finally ready to prove the Nash theorem which states that every finite strategic form game $\Gamma = \langle N, (S_i), (u_i) \rangle$ has at least one mixed strategy Nash equilibrium. To prove this result, we proceed as follows.

Let $S_i = \{s_{i1}, \ldots, s_{im_i}\}$. Note that

$$\Delta(S_i) = \left\{ (x_1, \ldots, x_{m_i}) : \; 0 \leq x_j \leq 1 \; \forall \; j = 1, \ldots, m_i; \; \sum_{j=1}^{m_i} x_j = 1 \right\}$$

Thus $\Delta(S_i) \subset \mathbb{R}^{m_i}$. In fact $\Delta(S_i) \subset [0, 1]^{m_i}$, which means

$$\Delta(S_1) \times \ldots \Delta(S_n) \;\subset\; [0, 1]^{m_1} \times \ldots [0, 1]^{m_n}$$

(1) Clearly, $\Delta(S_i)$, which is the collection of all probability distributions over S_i, is non-empty $\forall i = 1, \ldots, n$.

(2) Now we show that $\Delta(S_i)$ is convex. Let $\lambda \in (0, 1)$ be any number. Consider any two elements of $\Delta(S_i)$, $x = (x_1, \ldots, x_{m_i})$ and $y = (y_1, \ldots, y_{m_i})$ and consider any convex combination:

$$\begin{aligned} \lambda x + (1 - \lambda) y &= \lambda(x_1, \ldots, x_{m_i}) + (1 - \lambda)\; (y_1, \ldots, y_{m_i}) \\ &= (\lambda x_1 + (1 - \lambda) y_1, \ldots, \lambda x_{m_i} + (1 - \lambda) y_{m_i}) \end{aligned}$$

Consider

$$\begin{aligned} \sum_{j=1}^{m_i} \lambda x_j + (1 - \lambda) y_j &= \sum_{j=1}^{m_i} \lambda x_j + \sum_{j=1}^{m_i} (1 - \lambda) y_j \\ &= \lambda + (1 - \lambda) = 1 \end{aligned}$$

Therefore, $\lambda x + (1 - \lambda) y \in \Delta(S_i)$ and hence $\Delta(S_i)$ is convex.

(3) Next we show that $\Delta(S_i)$ is compact for $i = 1, \ldots, n$. Since $\Delta(S_i) \subset \mathbb{R}^n$, compactness is equivalent to closedness and boundedness. Since $\Delta(S_i) \subset [0, 1]^{m_i}$, boundedness is immediate. To prove closedness, assume that $\Delta(S_i)$ is not closed. This means there exists a sequence $x^n \to x$, such that $x^n \in \Delta(S_i)\ \forall n$ but $x \notin \Delta(S_i)$. The proof of this is left as an exercise.

(4) Next we show that $u_i(\sigma_1, \ldots, \sigma_n)$ is continuous in $(\sigma_1, \ldots, \sigma_n)$. Recall that

$$u_i(\sigma_1, \ldots, \sigma_n) = \sum_{(s_1, \ldots, s_n) \in S} \left(\prod_{j=1}^{n} \sigma_j(s_j) \right) u_i(s_1, \ldots, s_n)$$

Since $u_i(s_1, \ldots, s_n)$ is continuous in $(s_1, \ldots, s_n)$, the above implies that $u_i(\sigma_1, \ldots, \sigma_n)$ is also continuous because the RHS is a weighted sum.

(5) Finally we show that $u_i(\sigma_1, \ldots, \sigma_n)$ is quasi-concave in σ_i. This follows from the quasi-concavity of $u_i(s_1, \ldots, s_n)$ over s_i.

From (1), (2), (3), (4), (5), all the conditions of Kakutani's fixed point theorem are satisfied and this immediately proves the Nash theorem.

Needless to say, the above theorem only provides sufficient conditions for the existence of equilibria. The conditions are not necessary, which means that even if these conditions are not satisfied, Nash equilibria may still exist. As an immediate example, we have shown the existence of Nash equilibrium (in fact, a weakly dominant strategy equilibrium) for the strategic form game underlying the Vickrey auction (this example has infinite strategy sets).

6 Sperner's Lemma

Sperner's Lemma is a famous result in combinatorics due to Emanuel Sperner, a celebrated 20^{th} century mathematician from Germany. The lemma is immensely valuable in proving important theorems in fixed point theory and game theory. In particular, the lemma can be used to prove the Brouwer's fixed point theorem, the Kakutani's fixed point theorem, and the famous result by John Nash that every finite strategic form game has at least one mixed strategy Nash equilibrium. To state the Sperner's lemma, we need the following setup. The material in this chapter is based on the discussion in the book by Vohra [7].

6.1 Setup for Sperner's Lemma

Consider a triangle with vertices A, B, and C and triangulate it, (that is, divide the triangle into smaller triangles) in the following way. Identify midpoints of AC, BC, and AB say X, Y and Z, respectively and make a triangle out of X, Y, and Z. This results in four smaller triangles. If we repeat the procedure on each of these four triangles, we obtain 16 triangles, as depicted in Figure 1 (let us call all the smaller triangles baby triangles). The triangulation can be continued further to result in 64, 256,... smaller triangles.

Suppose we follow the rules below for labeling the vertices of all the baby triangles.

1. Vertices along the edge AB can only be labeled with A or B but not C.
2. Vertices along the edge AC can only be labeled with A or C but not B.
3. Vertices along the edge BC can only be labeled with B or C but not A.
4. Vertices lying completely inside the original big triangle can be labeled with A or B or C.

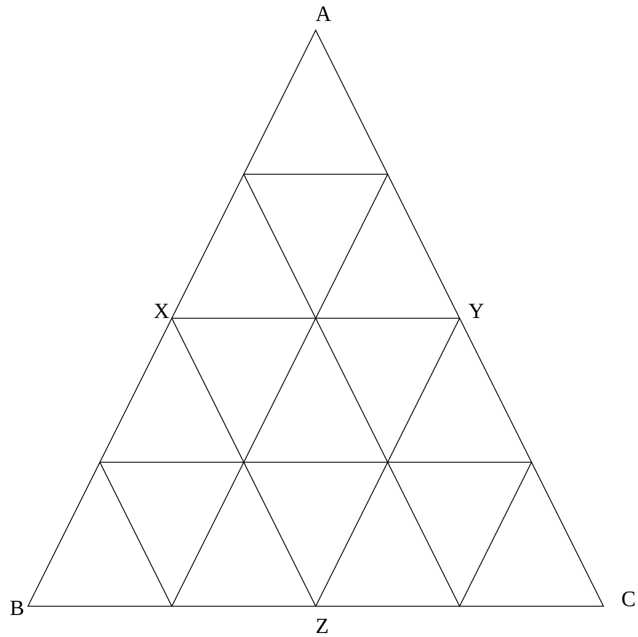

Figure 1: Triangulation into 16 triangles

Identify all the baby triangles such that the three vertices have three distinct colors A, B, C. Let us shade all such baby triangles. Among these shaded baby triangles, there will be two types: (a) those in which the vertices A, B, C appear in clockwise fashion (b) those in which the vertices A, B, C appear in counter-clockwise fashion. We will distinguish these two kinds of shaded baby triangles through different kinds of shades. For example, Figure 2 shows a triangulation with 16 baby triangles where the vertices are labeled following the rules above. Baby triangles where vertices labeled A, B, C appear in clockwise fashion have one kind of shading while baby triangles where vertices labeled A, B, C appear in counter-clockwise fashion have a darker type of shading. Note that there are two shaded baby triangles of the clockwise type and three shaded triangles of the counter-clockwise type. Let N_C (N_{AC}) be the number of shaded triangles of clockwise (counter-clockwise) type.

Figures 3 and 4 show two other instances of triangulations with one shaded triangle and three shaded triangles respectively.

6.2 Sperner's Lemma and Proof

Sperner's lemma states that the number of baby triangles ($N_C + N_{AC}$) having A, B, C as vertex labels following the above labeling rules is always odd. In fact, $N_{AC} = 1 + N_C$. This remarkable result can be proved as follows. The proof relies on labeling the edges and triangles in the following way.

- If the vertices of an edge have the same label, the edge is labeled with 0.
- If the vertices of an edge have different labels, and the labels appear in counter-clockwise sense (same sense as the original big triangle), the edge is labeled with 1.
- If the vertices of an edge have different labels, and the labels appear in clockwise sense, the edge is labeled with -1.

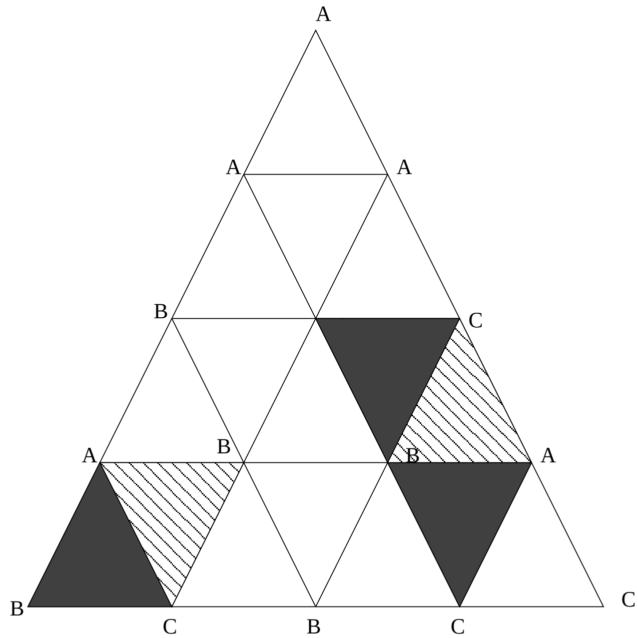

Figure 2: Triangulation with $N_C = 2$; $N_{AC} = 3$

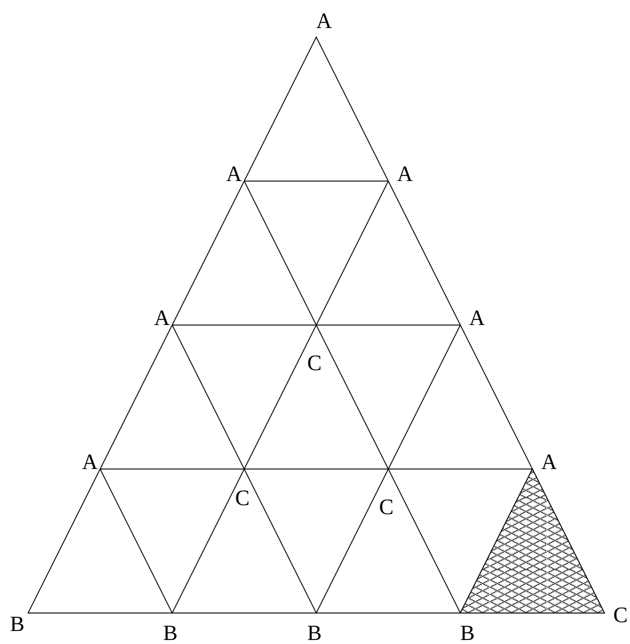

Figure 3: A triangulation with 16 triangles and $N_C = 0$; $N_{AC} = 1$

- For each baby triangle, add the above numbers on the edges and write the sum in a little circle in the middle of the triangle. Call this sum as *Sperner number* of the baby triangle.

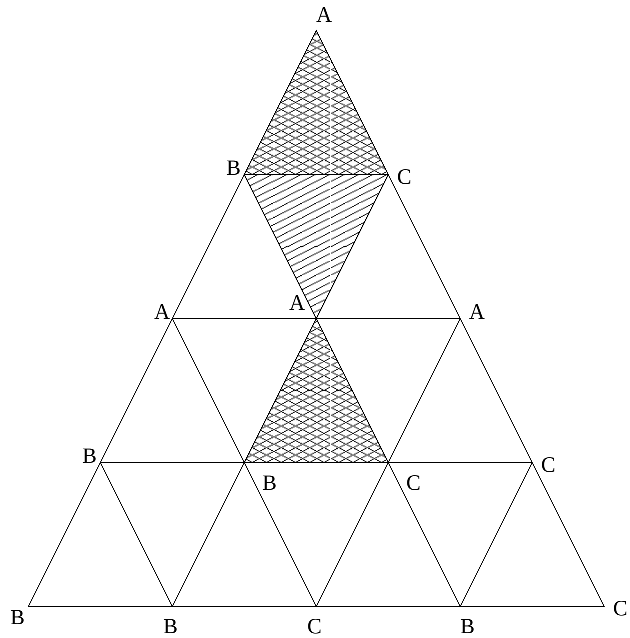

Figure 4: A triangulation with 16 triangles, $N_C = 1$; $N_{AC} = 2$

Figure 5 shows the triangulation of Figure 2 with the Sperner numbers included for each of the 16 baby triangles. In general, with the above numbering scheme, there exist four possibilities.

1. The three vertices are different and are labeled counter-clockwise (Sperner number = 3).
2. The three vertices are different and are labeled clockwise (Sperner number = −3).
3. The three vertices have the same label (Sperner number = 0).
4. Two vertices have the same label and the third vertex has a different label (Sperner number = 0).

Let us analyze a typical edge of the big triangle. Suppose we focus on edge AB. On this edge, there are smaller edges (belonging to the baby triangles). Note that

- number 1 on a small edge indicates a change from A to B.
- number -1 indicates a change from B to A.
- number 0 indicates no change.

On the edge AB of the big triangle, the overall change is from A to B. Therefore the sum of the numbers on all small edges along $AB = 1$. Similarly, the sum of all numbers on all small edges along BC is 1 and this sum is 1 on AC. Thus the sum of all numbers on all small edges lying along the outside edges of the big triangle is equal to 3.

Now let us analyze the numbers along all small edges inside the big triangle. Each small edge inside is a part of two baby triangles and is labeled either 0 on both the triangles or 1 on one triangle

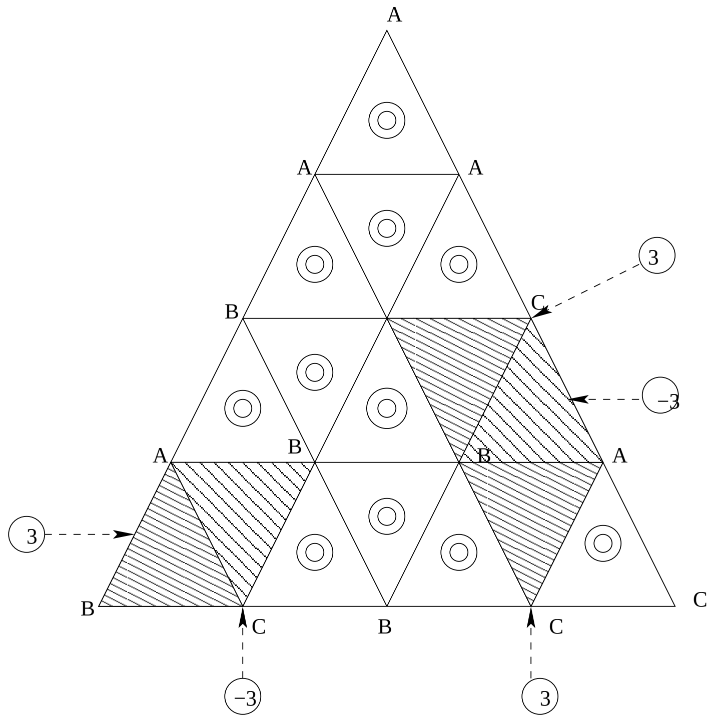

Figure 5: Triangulation of Figure 2 with Sperner numbers included for each triangle

and -1 on the other triangle. Thus the numbers on all small inside edges add up to zero. Thus the sum of all numbers on all edges of all baby triangles is equal to 3.

Notice that the sum of all Sperner numbers inside all baby triangles must be the same as the sum of all numbers on all the edges and hence is equal to 3. Clearly, the Sperner number of a shaded baby triangle is either 3 (for the counter-clockwise case) or -3 (for the clockwise case) and the Sperner number for each of the other baby triangles is 0. This means $N_{AC} = 1 + N_C$ and therefore $N_C + N_{AC}$ must be an odd number.

7 Sperner's Lemma to Brouwer's Fixed Point Theorem

7.1 Brouwer's Fixed Point Theorem

Let us recall Brouwer's fixed point theorem for ready reference [4]. The theorem states that if $X \subset \mathbb{R}^n$ is a compact and convex subset of $\mathbb{R}^n$ and $f : X \to X$ is a continuous function, then f has a fixed point, that is,

$$\exists \text{ an } x \in X \ni f(x) = x$$

To prove the general Brouwer's theorem, we first state and prove a supporting lemma. We prove this supporting lemma using Sperner's lemma. The proof given here closely follows the one that appears in [7]. First we define the n-simplex.

Definition 1 (n-Simplex) *The n-simplex is defined as the set*

$$\Delta^n = \{x \in \mathbb{R}^n : x_i \geq 0 \text{ for } i = 1, 2, \ldots, n; \sum_{i=1}^{n} x_i = 1\}$$

Note immediately that Δ^n is simply the collection of all probability distributions $(x_1, x_2, \ldots, x_n)$. This set is known to be convex and compact.

Lemma 1 *If $f : \Delta^n \rightarrow \Delta^n$ is a continuous function then f has a fixed point. That is, there exists an $x \in \Delta^n$ such that $f(x) = x$.*

Proof : This lemma can be proved easily for $n = 1$ and $n = 2$. For a proof for $n = 2$, see [7]. We present the proof for $n = 3$ using Sperner's lemma. The proof can be generalized in a natural way for $n > 3$. We now set out to show that if a function $f : \Delta^3 \rightarrow \Delta^3$ is continuous, then it has a fixed point. Suppose that f does not have a fixed point. Let

$$f(x) = (f_1(x), f_2(x), f_3(x))$$

The set Δ^3 is a two dimensional triangle which can be triangulated in the standard way yielding smaller baby triangles. The first subdivision leads to 4 triangles, the second subdivision leads to 16 triangles, etc. Let us focus on the m^{th} subdivision. Suppose we color the vertices (1,2, or 3) following the rule below:

$$c(x) = min\{i : f_i(x) < x_i\}$$

This is a well defined rule if f does not have a fixed point. To show this, suppose it is not a well defined rule. That would mean

$$f_i(x) \geq x_i \text{ for } i = 1, 2, 3, \text{ for some } (x_1, x_2, x_3) \in \Delta^3$$

Since $x, f(x) \in \Delta^3$, we have $x_1 + x_2 + x_3 = 1$ and $f_1(x) + f_2(x) + f_3(x) = 1$ which implies that

$$f_i(x) = x_i \text{ for } i = 1, 2, 3$$

This contradicts our assumption that f has no fixed point.

The above coloring scheme satisfies the setup for the Sperner's lemma. This is seen as follows.

- Note that $c(1, 0, 0) = 1; c(0, 1, 0) = 2; c(0, 0, 1) = 3$. That is, the three corners of the big triangle have three different colors.

- Suppose x is a point on an edge of the triangle Δ^3. For example, suppose x is a point on the edge between (1,0,0) and (0,1,0). Note that

  $$x = \lambda(1, 0, 0) + (1 - \lambda)(0, 1, 0) = (\lambda, (1 - \lambda), 0)$$

  for a suitable value of λ, $0 \leq \lambda \leq 1$. Our coloring rule can be applied to show that

  $$c\ (\lambda,\ 1 - \lambda,\ 0) = 1 \text{ or } 2$$

  This means points on the boundaries of Δ^3 have colors in accordance with the requirements of Sperner's Lemma.

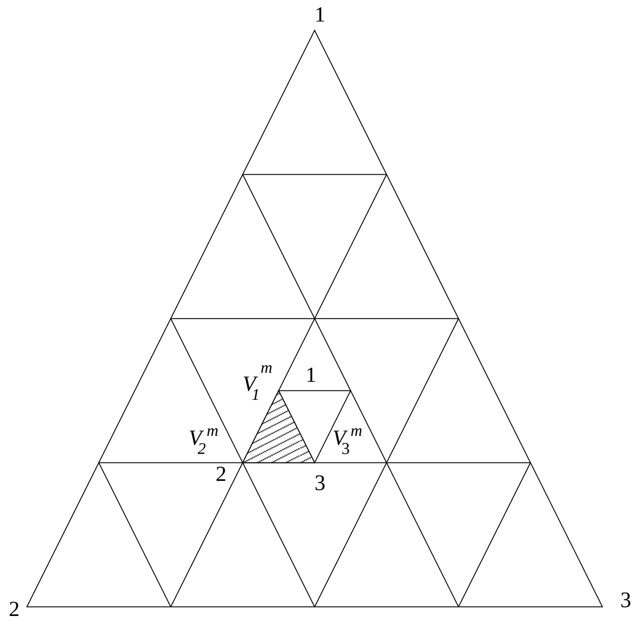

Figure 6: A triangulation showing a baby triangle with three distinct colors on m^{th} subdivision

Applying Sperner's Lemma to the current situation, we conclude that in the m^{th} subdivision of Δ^3, there must exist a baby triangle with corners (v_1^m, v_2^m, v_3^m) that are differently colored. See Figure 6. Without loss of generality, we have

$$c\left(v_1^m\right) = 1;\ c\left(v_2^m\right) = 2;\ c\left(v_3^m\right) = 3$$

Let $m \to \infty$ and consider the infinite sequence

$$\{v_1^m\}_{m \geq 1}$$

This sequence may itself not have a limit but since Δ^3 is a compact set, the above sequence has a convergent subsequence. For an appropriate subsequence, we therefore get that

$$v_1^m \to x \in \Delta^3 \text{ for some } x$$

Similarly, the sequences v_2^m and v_3^m also have convergent subsequences. As $m \to \infty$, the baby triangles shrink to infinitesimal sizes and therefore we have

$$v_2^m \to x \quad \text{as} \quad m \to \infty$$

$$v_3^m \to x \quad \text{as} \quad m \to \infty$$

Since the function $f : \Delta^3 \to \Delta^3$ is continuous, we have

$$f(v_1^m) \to f(x)$$

$$f(v_2^m) \to f(x)$$

$$f(v_3^m) \to f(x)$$

Since $c(v_1^m) = 1$, $f_1(v_1^m)$ is strictly less than the first component of v_1^m. Since $c(v_2^m) = 2$, $f_2(v_2^m)$ is strictly less than the second component of v_2^m. Since $c(v_3^m) = 3$, $f_3(v_3^m)$ is strictly less than the third component of v_2^m. This would imply that

$$f_1(x) \leq x_1; \;\; f_2(x) \leq x_2; \;\; f_3(x) \leq x_3$$

The above implies that

$$\sum_{i=1}^{3} f_i(x) \leq \sum_{i=1}^{3} x_i$$

However, since $f : \Delta^3 \to \Delta^3$, we have

$$\sum_{i=1}^{3} f_i(x) = \sum_{i=1}^{3} x_i = 1$$

The above is possible if and only if

$$f_i(x) = x_i \text{ for } i = 1, 2, 3$$

which contradicts our original assumption that f does not have a fixed point. This then proves the lemma. Based on this lemma, we now prove Brouwer's fixed point theorem.

7.2 Proof of Brouwer's Fixed Point Theorem

The proof crucially uses a result on the topological equivalence of compact and convex sets. The result says that if X is compact and convex of dimension $(n-1)$, then X and Δ^n are topologically equivalent in the following sense: there exist $g : X \to \Delta^n$ and $g^{-1} : \Delta^n \to X$ such that both g and g^{-1} are continuous. Now define $h : \Delta^n \to \Delta^n$ as follows.

$$h(x) = g(f(g^{-1}(x)))$$

Since f and g are continuous, so is h and by invoking our lemma, h will have a fixed point, say x^*. That is

$$h(x^*) = g(f(g^{-1}(x^*))) = x^*$$

This immediately implies that

$$f(g^{-1}(x^*)) = g^{-1}(x^*)$$

which implies that $\bar{g}^1(x^*)$ is a fixed point of $f : X \to X$. This proves Brouwer's fixed point theorem.

8 Proving Nash Theorem Via Brouwer's Theorem

Consider a finite strategic form game $\Gamma = \langle N, (S_i), (U_i) \rangle$ with $N = \{1, 2, \ldots, n\}$. Let us use the notation:

$$S_i = \{s_{ij} : j = 1, 2, \ldots, |S_i|\} \;\; i = 1, \ldots, n$$

We now prove, using Brouwer's fixed point theorem, that a finite strategic form game such as above will always have at least one mixed strategic equilibrium (which is the Nash Theorem). As usual, let $\Delta(S_i)$ denote the set of all probability distributions on S_i and define

$$M = \Delta(S_1) \times \ldots \times \Delta(S_n)$$

Each $\sigma \in M$ is a vector with $\sum_{i \in N} |S_i|$ components. Suppose s_{ik} is a particular pure strategy of player i and let σ_{ik} be the component in σ corresponding to s_{ik}. Define a mapping $f : M \to M$ that maps vectors from M into itself. Given $\sigma \in M, f(\sigma)$ has as many components as σ and let $f_{ik}(\sigma)$ be the component in $f(\sigma)$ corresponding to pure strategy s_{ik}. Recall that we can represent $\sigma = (\sigma_i, \sigma_{-i})$. Define

$$f_{ik}(\sigma) = \frac{[\sigma_{ik} + \max(0, u_i(s_{ik}, \sigma_{-i}) - u_i(\sigma_i, \sigma_{-i}))]}{\sum_{s_{ij} \in S_i} [\sigma_{ij} + \max(0, u_i(s_{ij}, \sigma_{-i}) - u_i(\sigma_i, \sigma_{-i}))]}$$

Note that the denominator above is certainly greater than or equal to

$$\sum_{s_{ij} \in S_i} \sigma_{ij} = 1$$

Also, we have

$$\sum_{j=1}^{|S_i|} f_{ij}(\sigma) = 1 \quad \text{for } i = 1, 2, \ldots, n$$

Clearly, $f : M \to M$ is a well defined mapping. It can be verified that f is continuous and therefore by Brouwer's theorem, f has a fixed point. That is, there exists a $\sigma^* \in M$ such that

$$f(\sigma^*) = \sigma^*$$

This means that $\forall\, i \in N,\ \forall\, s_{ik} \in S_i$, we have

$$\sigma^*_{ik} = \frac{\sigma^*_{ik} + \max(0, u_i(s_{ik}, \sigma^*_{-i}) - u_i(\sigma^*_i, \sigma^*_{-i}))}{\sum_{s_{ij} \in S_i} [\sigma^*_{ij} + \max(0, u_i(s_{ij}, \sigma^*_{-i}) - u_i(\sigma^*_i, \sigma^*_{-i}))]}$$

We now consider two cases. In case 1,

$$\sum_{s_{ij} \in S_i} [\max(0, u_i(s_{ik}, \sigma^*_{-i}) - u_i(\sigma^*_i, \sigma^*_{-i}))] = 0 \ \ \forall \ \ i \in N$$

and in case 2, the above sum is > 0 for at least some $i \in N$. In case 1, we will get

$$u_i(s_{ij}, \sigma^*_{-i}) - u_i(\sigma^*_i, \sigma^*_{-i}) \leq 0 \ \ \forall \ \ s_{ij} \in S_i \ \ \forall \ \ i \in N$$

which immediately implies that $(\sigma^*_i, \sigma^*_{-i})$ is a Nash equilibrium.

In case 2, there will exist at least one agent i and strategy $s_{ik} \in S_i$ such that

$$\max(0, u_i(s_{ik}, \sigma^*_{-i}) - u_i(\sigma^*_i, \sigma^*_{-i})) > 0$$

This implies that

$$\sigma^*_{ik} = \frac{\max(0, u_i(s_{ik}, \sigma^*_{-i}) - u_i(\sigma^*_i, \sigma^*_{-i}))}{\sum_{S_{ij} \in S_i} [\max(0, u_i(s_{ij}, \sigma^*_{-i}) - u_i(\sigma^*_i, \sigma^*_{-i})] - 1}$$

This in turn implies $\sigma_{ik}^* \neq 0$ which implies $\sigma_{ik}^* > 0$.

Based on the above, we can show that

$$u_i(s_{ik}, \sigma_{-i}^*) > u_i(\sigma_i^*, \sigma_{-i}^*) \; \forall \; s_{ik} \in S_i$$

such that $\sigma_{ik}^* > 0$.

This leads to a contradiction since $u_i(\sigma_i^*, \sigma_{-i}^*)$ is a convex combination of $u_i(s_{ik}, \sigma_{-i}^*)$ such that $S_{ik} \in S_i$ and $\sigma_{ik}^* > 0$ and we assert that $(\sigma_i^*, \sigma_{-i}^*)$ is a Nash equilibrium.

9 Existence of Nash Equilibrium Infinite Strategy Sets

Consider a strategic form game but with infinite strategy sets. Now the definition of mixed strategies and equilibrium analysis require a more technical treatment. In particular the strategy sets are required to be compact metric spaces to make the games amenable for mathematical analysis. Myerson [8] provides an excellent description of the technical issues involved here. A prominent result on existence of mixed strategy Nash equilibrium in this setting is due to Glicksberg [9]. Before stating this result, we define a few relevant notions.

Continuous Games

A strategic form game with finite number of players, $< N, (S_i), (u_i) >$ is called a continuous game if $\forall \; i \in N$, the strategic set S_i is a non-empty, compact metric space and u_i is a continuous function.

ε-Nash Equilibrium

Given a real number $\varepsilon > 0$, a strategy profile $(\sigma_1^*, \ldots, \sigma_n^*,)$ of a strategic form game $< N, (S_i), (u_i) >$ is said to be an ε-Nash equilibrium if $\forall \; i \in N$,

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*) - \varepsilon \; \forall \; \sigma_i \in \Delta(S_i)$$

Note immediately that an ε-Nash equilibrium becomes a Nash equilibrium when $\varepsilon = 0$.

Glicksberg's Results

We now state two results, due to Glicksberg [9].

1. Every continuous game has an ε-Nash equilibrium for all $\varepsilon > 0$.

2. Every continuous game has a Nash equilibrium.

We now provide two examples to show that continuity and compactness are necessary for the above result.

Example 1: Necessity of Continuity

Consider a game with $N = \{1\}$ and $S_1 = [0, 1]$ (compact). Define the utility function as a discontinuous map:

$$\begin{aligned} u_1(s_1) \; &= s_1 \quad \text{if } 0 \leq s_1 < 1 \\ &= 0 \quad \text{if } s_1 = 1 \end{aligned}$$

It can be seen that the above game does not have a mixed strategy equilibrium.

Example 1: Necessity of Compactness

Consider again a game with $N = \{1\}$ but with $S_1 = [0, 1]$ (not compact). Define the utility function as a continuous map:

$$u_1(s_1) = s_1 \; \forall s_1 \in [0, 1]$$

It can be shown that this game also does not have a mixed strategy equilibrium.

It may be noted that any finite set is a compact metric space under the discrete metric. Also, any mapping whose domain is endowed with a discrete metric is automatically continuous. For this reason, all finite games are continuous.

9.1 Some Key Results

Dasgupta and Maskin [10, 11] have come up with sufficient conditions for existence of Nash equilibrium in discontinuous games. Simon [12], Renny [13], and Carmona [14] provide updates on existence results in discontinuous games. Myerson [8] brings out the technical issues involved in a detailed way. Moon [15] provides an up-to-date survey.

10 Appendix: Relevant Mathematical Background

We provide several relevant definitions and key results which are essential for understanding the contents in this chapter.

Metric Space

A metric space (V, d) consists of a set V and a mapping $d : V \times V \to \mathbb{R}$ such that $\forall x, y, z \in V$, the following holds.

1. $d(x, y) \geq 0$
2. $d(x, y) = 0$ iff $x = y$
3. $d(x, y) = d(y, x)$
4. $d(x, z) \leq d(x, y) + d(y, z)$

The mapping d is called a *metric* or *distance* function. Note that the first condition above follows from the other three.

Open Ball

Given a metric space (V, d), an open ball of radius $r > 0$, and centre $x \in V$, is the set

$$B(x, r) = \{y \in V : d(x, y) < r\}$$

.

Open Set

An open set X in a metric space (V, d) is a subset of V such that we can find, at each $x \in X$, an open ball that is contained in X.

Bounded Set

A subset X of a metric space is said to be bounded if X is completely contained in some open ball around 0.

Closed Set

A subset X of a metric space V is said to be a closed set iff every convergent sequence in X converges to a point which lies in X. That is, for all sequences $\{x_k\}$ in X such that $x_k \to x$ for some $x \in V$, it will happen that $x \in X$.

It may be noted that a set X is closed set iff the complement set $X^c = V \setminus X$ is an open set.

Compact Set

Given a subset X of a metric space (V, d), X is said to be compact if every sequence of points in X has a convergent subsequence.

A key result is that if the metric space $V = \mathbb{R}^n$, then a subset X is compact iff it is closed and bounded.

Examples of Compact Sets

The closed interval $[0, 1]$ is compact. None of the sets $[0, \infty), (0, 1), (0, 1], [0, 1), (-\infty, \infty)$ is compact (because none of these sets is closed). Any finite subset of $\mathbb{R}$ is compact.

A Useful Result

Let $X \subset \mathbb{R}n$ and let $f : X \to \mathbb{R}^k$ be a continuous function. Then the image of a compact set under f is also compact.

Weirstrass Theorem

Let $X \subset \mathbb{R}^n$ and let $f : X \to \mathbb{R}$ be a continuous function. If X is compact, then f has and attains a maximum and a minimum in X.

10.1 Convexity

Convex Combination

Given $x_1, \ldots, x_m \in \mathbb{R}^n$, a point $y \in \mathbb{R}^n$ is called a convex combination of $x_1, \ldots, x_m$ if there exist numbers $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ such that

(1) $\lambda_i \geq 0, \quad i = 1, \ldots, m$

(2) $\sum_{i=1}^{m} \lambda_i = 1$

(3) $y = \sum_{i=1}^{m} \lambda_i x_i$

Convex Set

A set $X \subset \mathbb{R}^n$ is said to be convex if the convex combination of any two points in X is also in X. The above definition immediately implies that a finite set cannot be convex. Intuitively, the set X is convex if the straight line joining any two points in X is completely contained in X.

Examples of Convex Sets

The intervals (0,1), [0,1], [0,1], [0,1] are all convex. The set $X = \{x \in \mathbb{R}^2 : ||x|| < 1\}$ is convex The set $X = \{x \in \mathbb{R}^2 : ||x|| = 1\}$ is not convex.

Concave and Convex Functions

Let $X \subset \mathbb{R}^n$ be a convex set. A function $f : X \to \mathbb{R}$ is said to be concave iff $\forall x, y, \in X$ and $\forall \lambda \in (0, 1)$,

$$f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y)$$

f is said to be convex iff $\forall x, y \in X$ and $\forall \lambda \in (0, 1)$

$$f[\lambda x + (1 - \lambda)y] \leq \lambda f(x) + (1 - \lambda)f(y)$$

An alternative definition for convex and concave functions is as follows.
If $X \subset \mathbb{R}^n$ is a convex set and $f : X \to \mathbb{R}$ is a function, define

$$\begin{aligned} \text{sub } f &= \{(x, y) : x \in X, y \in \mathbb{R}, f(x) \geq y\} \\ \text{epi } f &= \{(x, y) : x \in X, y \in \mathbb{R}, f(x) \leq y\} \end{aligned}$$

f is *concave* if sub f is *convex* ; f is *convex* if epi f is *convex*.

Examples of Concave and Convex Functions

(1) $f(x) = x^3, x \in \mathbb{R}$ is neither convex nor concave

(2) $f(x) = ax + b, x \in \mathbb{R}^n$ and $a, b \in \mathbb{R}$ is both convex and concave.

(3) $f : \mathbb{R}^+ \to \mathbb{R}$ defined by $f(x) = x^\alpha$ where $\mathbb{R}^+$ is the set of all positive real numbers,

- strictly concave for $0 < \alpha < 1$
- strictly convex for $\alpha > 1$
- Both concave and convex for $\alpha = 1$

Some Results on Convexity

(1) A function $f : X \to \mathbb{R}$ is concave iff the function $-f$ is convex.

(2) Let $f : X \to \mathbb{R}$ be concave. Then If X is an open set, then f is continuous on X. If X is not an open set, then f is continuous on the interior of X

Quasi-Concavity and Quasi-Convexity

Let $X \subset \mathbb{R}^n$ be a convex set and let $f : X \to \mathbb{R}$ be a function. The *upper contour set* of f at $a \in \mathbb{R}$ is defined as

$$U_f(a) = \{x \in X : f(x) \geq a\}$$

The *lower contour set* of f at $a \in \mathbb{R}$ is defined as

$$L_f(a) = \{x \in X : f(x) \leq a\}$$

A function $f : X \to \mathbb{R}$ is said to be *quasi-concave* if $U_f(a)$ is convex for all $x \in \mathbb{R}$ and is said to be *quasi-convex* if $L_f(a)$ is convex for all $a \in \mathbb{R}$.

Alternatively, $f : X \to \mathbb{R}$ is *quasi-concave* on X iff $\forall\, x, y \in X$ and $\forall\, \lambda \in (0, 1)$,

$$f[\lambda x + (1 - \lambda)y] \geq \min(f(x), f(y))$$

and *quasi-convex* on X iff $\forall x, y \in X$ and $\forall \lambda \in (0, 1)$,

$$f[\lambda x + (1 - \lambda)y] \leq \max(f(x), f(y))$$

Examples: Quasi-Concavity and Quasi-Convexity

(1) $f(x) = x^3$ on $\mathbb{R}$ is quasi-convex and also quasi-concave on $\mathbb{R}$. But it is neither convex nor concave on $\mathbb{R}$.

(2) Any non-decreasing function $f : \mathbb{R} \to \mathbb{R}$ is quasi-convex and quasi-concave. But it need not be convex and need not be concave. In the above, the upper contour set and also the lower contour set are both convex and hence the function is both quasi-convex and quasi-concave. Also, for every pair of points x_1 and x_2, the values of the function for points between x_1 and x_2 lie between $\min(f(x_1), f(x_2))$ and $\max(f(x_1), f(x_2))$ and therefore the function is both quasi-convex and quasi-concave.

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