Game Theory

Lecture Notes By

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Rationalizable Strategies

Note: *This is a only a draft version, so there could be flaws. If you find any errors, please do send email to* `hari@csa.iisc.ernet.in`. *A more thorough version would be available soon in this space.*

In this chapter, we define the notion of dominance in the context of mixed strategies and describe how elimination of dominated strategies simplifies equilibrium analysis. Our discussion leads to the important notion of rationalizability.

1 Dominating Mixed Strategies and Dominated Mixed Strategies

Suppose $< N, (S_i), (u_i) >$ is strategic form game. In an earlier chapter (Chapter 4), we have discussed the notion of dominance in the context of pure strategies. We now extend this notion to mixed strategies.

Definition 1 (Dominance in Mixed Strategies) . *Given two mixed strategies $\sigma_i, \sigma_i' \in \Delta(S_i)$, we say*

- σ_i *strictly dominates* σ_i' *if*

$$u_i(\sigma_i, \sigma_{-i}) > u_i(\sigma_i', \sigma_{-i}) \ \ \forall \sigma_{-i} \in \Delta(S_{-i})$$

- σ_i *weakly dominates* σ_i' *if*

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i}) \ \ \forall \sigma_{-i} \in \Delta(S_{-i})$$

and

$$u_i(\sigma_i, \sigma_{-i}) > u_i(\sigma_i', \sigma_{-i}) \textit{ for some } \ \ \sigma_{-i} \in \Delta(S_i)$$

- σ_i *very weakly dominates* σ_i' *if*

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i}) \ \ \forall \sigma_{-i} \in \Delta(S_{-i}$$

In all cases above, we say the strategy σ_i' is strongly (weakly) (very weakly) dominated by σ_i. If σ_i strongly (weakly) (very weakly) dominates all other strategies $\sigma_i' \in \Delta(S_i)$, we say σ_i is a strongly (weakly) (very weakly) dominant strategy of player i. A strategy profile $(\sigma_1^*, \cdots, \sigma_n^*)$ such that σ_i^* is a strictly (weakly) (very weakly) dominant strategy for player i, $\forall i \in N$, is called a strictly (weakly) (very weakly) dominant mixed strategy equilibrium.

Note 1 *It can be shown that any dominant mixed strategy equilibrium is also a mixed strategy Nash equilibrium.*

Note 2 *A strictly dominant mixed strategy for any player, if one exists, is unique. Therefore a strictly dominant mixed strategy equilibrium, if one exists, is unique.*

Example 1 *Consider the Prisoner's Dilemma game whose payoff matrix is reproduced in Figure 1(a) for ready reference.*

(a)

	2	
1	*NC*	*C*
NC	$-2, -2$	$-1, -10$
C	$-10, -1$	$-5, -5$

(b)

	2
1	*C*
NC	$-1, -10$
C	$-5, -5$

(c)

	2
1	*C*
C	$-5, -5$

Figure 1: Prisoner's dilemma problem and elimination of strictly dominated strategies.

Since the strategy NC is strictly dominated by strategy C for player 2, the player will never play NC. So, strategy NC of player 2 can be eliminated leading to the reduced payoff matrix as in Figure 1(b). Now the strategy NC of player 1 which is dominated by strategy C can also be eliminated, leading to the degenerate payoff matrix with a single entry corresponding to the profile (C,C) which in this case happens to be a strongly dominant strategy equilibrium.

Example 2 *Consider a two player game shown in Figure 2 (this game is a modified version of an example that appears in Shoham and Leyton-Brown [1]).*

	2		
1	*X*	*Y*	*Z*
A	*3,1*	*0,1*	*0,0*
B	*0,1*	*4,1*	*0,0*
C	*1,1*	*1,1*	*5,0*

Figure 2: A two player game to illustrate elimination of strictly dominated strategies.

Note that the strategy Z of player 2 is strictly dominated by the strategy X and also the strategy Y. Therefore player 2 will never like to play strategy Z (whatever the strategy chosen by player 1). Thus strategy Z can be safely eliminated, leading to the reduced game as shown in Figure 3.

1	2	
	X	*Y*
A	*3,1*	*0,1*
B	*0,1*	*4,1*
C	*1,1*	*1,1*

Figure 3: Game obtained after eliminating strategy Z of player 2.

Now notice that none of the pure strategies of player 1 is dominated by any of the other pure strategies of player 1. However the strategy C is strictly dominated by the mixed strategy of player 1 that assigns equal probability to A and B. It has to be noted with care that the strategy C was not dominated by any mixed strategy in the original game. A strategy that was not dominated may become dominated. Also note that a pure strategy may not be dominated by any of the other pure strategies but could be dominated by a mixture of those pure strategies. Figure 4 shows the game obtained after eliminating the strategy C of player 1. No more strategies can be eliminated from this game.

1	2	
	X	*Y*
A	*3,1*	*0,1*
B	*0,1*	*4,1*

Figure 4: Game obtained after eliminating strategy C.

2 Iterated Elimination of Dominated Strategies

We have observed that elimination of strictly dominated strategies simplifies analysis of games. We shall formalize this as follows. Consider a finite strategic form game $< N, (S_i), (u_i) >$. Let $k = 1, 2, \ldots, K$ denote the successive rounds in which strictly dominated strategies are eliminated. For each player $i \in N$, define the sets of strategies S_i^k as follows.

- $S_i^1 = S_i$
- $S_i^{k+1} \subseteq S_i^k \quad \text{for} \quad k = 1, 2, \ldots, K-1.$
- For $k = 1, 2, \ldots K-1$, all strategies $s_i \in S_i^k \setminus S_i^{k+1}$ are strictly dominated strategies which are eliminated in the kth round from the game in which the set of strategies of $j \in N$ is S_j^k.
- No strategy in S_i^K is strictly dominated in the game in which the set of strategies of each player $j \in N$ is S_j^K.

The above steps define the process of iterated elimination of strongly dominated strategies. The set of strategy profiles

$$\{(s_i, s_2, \ldots s_n) : s_i \in S_i^k \quad \text{for} \quad i = 1, \ldots, n\}$$

is said to survive the iterated elimination of strictly dominated strategies.

Example 3 *Consider the two player game of Figure 2 where $S_1 = \{A, B, C\}$; $S_2 = \{X, Y, Z\}$. For this game,*

- $S_1^1 = S_1 = \{A, B, C\}$; $S_2^1 = S_2 = \{X, Y, Z\}$
- $S_1^2 = \{A, B, C\}$; $S_2^2 = \{X, Y\}$
- $S_1^3 = \{A, B\}$; $S_2^3 = \{X, Y\}$.

Therefore the strategy profiles

$$\{(A, X), (A, Y), (B, X), (B, Y)\}$$

survive the iterated removal of strongly dominated strategies.

Consider the set of all strategy profiles $(\sigma_1, \sigma_2, \ldots \sigma_n)$ such that $\forall \quad i \in N, \quad \sigma_i(s_i) = 0$ for all strategies $s_i \in S_i$ which are eliminated during iterated removal of strongly dominated strategies. It can be proved that this set of strategy profiles will contain all Nash equilibria. If we are lucky (as it happened in the case of Prisoner's dilemma), the above set may be exactly the set of all Nash equilibria. In the other extreme case, it may as well be the entire set of all strategy profiles! Thus iterated removal of strongly dominated strategies may sometimes simplify Nash equilibrium computation.

If the game is finite, iterated elimination of dominated strategies must end in a finite number of rounds. The Church–Rosser property states that the order of removal of strongly dominated strategies does not affect the final outcome of the iterated elimination process. However, the order in which weakly or very weakly dominated strategies are eliminated does influence the final outcome of the process. However, elimination of weakly or very weakly dominated strategies yields smaller reduced games compared to elimination of strongly dominated strategies.

3 Rationalizable Strategies

We start this section by defining the notion of a *Never a Best Response strategy*.

Definition 2 (Never a Best Response Strategy) . *Given a finite strategic form game $< N_1(S_i), (u_i) >$, a pure strategy $s_i' \in S_i$ is said to be never a best response if $\forall \mu_i \in \Delta(S_{-i})$, there exists some $\sigma_i \in \Delta(S_i)$ such that*

$$\sum_{s_{-i} \in S_{-i}} \mu_i(s_{-i}) u_i(\sigma_i, s_{-i}) > \sum_{s_{-i} \in S_{-i}} \mu_i(s_{-i}) u_i(s_i', s_{-i})$$

This means that whatever the belief of player i about the rest of the players, there always exists a mixed strategy of player i that produces strictly higher utility than the strategy s_i'. One can define iterated removal of never a best response strategies in the same way as iterated removal of strongly dominated strategies.

It can be shown in finite games $< N, (S_i), (u_i) >$ that a strategy $s_i \in S_i$ of player i is never a best response if and only if it is strictly dominated. Thus in finite games, iterated removal of never a best response strategies yields the same outcomes as iterated removal of strongly dominated strategies.

3.1 Notion of Rationalizability

Rationalizable strategies represent strategies of players which are deduced fully taking into account the preferences of the other players and a complete analysis of the other players reasoning about their own rational strategies. To understand this concept, we first define two notions: belief and rational strategy.

Definition 3 (Belief) . *Given a strategic form game* $\langle N, (S_i), (u_i) \rangle$*, a belief of player* i *about the strategies of the rest of the players is a probability distribution* $\mu_i \in \Delta(S_{-i})$.

Definition 4 (Rational Strategy) *Given a strategic form game, a mixed strategy* $\sigma_i \in \Delta(S_i)$ *is called a rational strategy of player* i *if there exists a belief* $\mu_i \in \Delta(S_{-i})$ *under which* σ_i *maximizes player* i*'s expected payoff:*

$$\sum_{s_{-i} \in S_{-i}} \mu_i(s_{-i}) u_i(\sigma_i, s_{-i})$$

Given a strategic form game, it can be shown that a mixed strategy profile $(\sigma_1^*, \ldots, \sigma_n^*)$ is a Nash equilibrium iff $\forall i \in N$, σ_i^* is rational for player i under belief μ_i given by

$$\mu_i(s_{-i}) = \prod_{j \neq i} \sigma_i^*(s_j) \quad \forall s_{-i} \in S_{-i}$$

Example 4 *In the prisoner's dilemma problem, the only rational action for each player in the strategy "C".*

Example 5 *Consider the two player game shown in Figure 4.*

	2	
1	X	Y
A	*3,2*	*0,3*
B	*2,0*	*1,1*

If the belief of player 1 is that player 2 will choose strategy X*, then player 1's best response is* A*. On the other hand, if player 1 believes that player 2 will choose strategy* Y*, player 1's best response is* B*. Thus, based on the belief of player 1,* A *or* B *are both rational for player 1. If we now turn to player 2, whether player 1 chooses* A *or* B*, it is rational for player 2 to choose* Y *since* Y *is a strictly dominant strategy and* X *is a strictly dominated strategy. It is reasonable to restrict the belief of player 1 to be consistent with player 2's rationality. In such a case, player 1 believes that player 2 chooses* Y *and therefore it is rational for player 1 to choose* B*. We make the statement that* B *is the only strategy of player 1 that is rational as well as it is supported by a belief that is consistent with the rational behavior of player 2.*

Example 6 *Let us consider the game shown in Figure 4.*

	2	
1	*X*	*Y*
A	*4,2*	*0,3*
B	*1,1*	*1,0*
C	*3,0*	*2,2*

If player 1's belief about player 2 is $X(Y)$, then the rational strategy for player 1 is $A(C)$. If player 2's belief about player 1 is $A(B)(C)$, then player 2's rational strategy is $Y(X)(Y)$.

If we now consider that player 1 assumes that player 2 assumes that player 1 is rational, then player 2 will assume that player 1 will not choose because B is not a best response to any belief of player 1 about player 2. This means that player 2 will assign zero probability to strategy B in any belief player 2 has about player 1. This in turn means that player 2 will choose Y since Y is a best response to A or C or any randomization of A and C. Since player 1 assumes that player 2 assumes that player 1 is rational, player 1 knows that player 2 will choose Y and hence player 1 will choose C. Thus the profile (C, Y) is the only profile which is consistent with the following:

- *each player assumes that the other player assumes that she is rational*
- *each player assumes that the other player is rational*
- *each player is rational*

It can be shown that the profile (C, Y) will emerge as being consistent with any recursive reasoning of the type each player assumes that the other player assumes that ... she is rational. *We also say that the strategy C is rationalizable for player 1 and strategy Y is rationalizable for player 2. Formally, a rationalizable strategy is defined as follows.*

Definition 5 (Rationalizable Strategy) . *Given a strategic form game $< N, (S_i), (u_i) >$, a strategy $s_i^* \in S_i$ is rationalizable for player i if $\forall j \in N$, there exists a set $R_j \subseteq S_i$ such that*

- $s_i^* \in R_i$
- *For each $j \in N$, every action $s_j \in R_j$ is a best response to any belief of player j that belongs to $\Delta(R_{-j})$, that is, assigns positive probabilities only to profiles in R_{-j}.*

Example 7 *In the game discussed in Example 5, we have $S_1 = \{A, B\}$; $S_2 = \{X, Y\}$ and $R_1 = \{B\}$; $R_2 = \{Y\}$. In the game discussed in Example 6, we have $S_1 = \{A, B, C\}$; $S_2 = \{X, Y\}$ and $R_1 = \{C\}$ and $R_2 = \{Y\}$.*

Example 8 *In the game shown in Figure 5, all strategies are rationalizable for all players.*

	2	
1	*A*	*B*
A	*2,2*	*1,0*
B	*0,1*	*1,1*

Figure 5: A two player game

Intuitively, a strategy $s_i^* \in S_i$ is rationalizable for player i if player i can perfectly justify playing it against perfectly rational players. The action s_i^* must be rational against "justifiable" beliefs that take into account player i's knowledge of rationality of the other players, which incorporates the other players' knowledge of player i's rationality, their knowledge of player i's knowledge of their rationality, etc. [1].

Suppose $< N, (S_i), (u_i) >$ is a finite strategic form game. Then the following results hold in respect of rationalizable strategies.

- If $(\sigma_1^*, \ldots, \sigma_n^*)$ is a mixed strategy Nash equilibrium, then $\forall i \in N$, all strategies belonging to the support of σ_i^* (that is all strategies having non-zero probabilities in σ_i^*) are rationalizable for player i.

- The set of all rationalizable strategies of all players is precisely the set of all strategies that survive the iterative elimination of never a best response strategies which is also the set of all strategies that survive the iterative elimination of strongly dominated strategies.

4 Problems

1. Apply iterative elimination of strongly dominated strategies to the following problem ([2]).

	2		
1	X	Y	Z
A	2,3	3,0	0,1
B	0,0	1,6	4,2

2. For the game shown below, apply the following elimination steps and observe the outcomes ([2]). What can you infer from these.

	2	
1	X	Y
A	3,2	2,2
B	1,1	0,0
C	0,0	1,1

   - Eliminate C followed by Y
   - Eliminate B followed by X
   - Eliminate B and C at the same time

3. Show using the game below that the final outcomes of iterated elimination of weakly dominated strategies depends on the order of elimination ([3])

	2		
1	X	Y	Z
A	1,1	1,1	0,0
B	0,0	1,2	1,2

4. Find the set of rationalizable strategies for each player in the following games ([3]).

1	2 X	2 Y	2 Z
A	3,1	0,0	−1,2
B	0,0	1,3	0,2

1	2 X	2 Y	2 Z
A	4,3	0,0	2,1
B	0,0	3,4	1,1

1	2 X	2 Y	2 Z
A	2,1	1,4	0,3
B	1,8	0.2	1,3

1	2 X	2 Y	2 Z
A	0,7	2,5	7,0
B	5,2	3,3	5,2
C	7,0	2,5	0,7

5. Given a strategic form game $< N, (S_i), (u_i) >$, consider the set of all strategy profiles $(\sigma_1, \ldots, \sigma_n)$ such that $\forall i \in N$, $\sigma(s_i) = 0$ for all strategies $s_i \in S_i$ that are eliminated during the iterated removal of strongly dominated strategies show that the above set includes all mixed strategy Nash equilibria.

6. Prove the property that in a finite strategic form game, a strategy $s_i \in S_i$ of player i is never a best response if and only if it is strictly dominated.

7. Given a strategic form game, show that a mixed strategy profile $(\sigma_1^*, \ldots, \sigma_n^*)$ is a Nash equilibrium if and only if $\forall i \in N$, σ_i^* is rational for player i under belief $\mu_i \in \Delta(S_{-i})$ given by

$$\mu_i(s_{-i}) = \prod_{j \neq i} \sigma_j^*(s_j) \quad \forall s_{-i} \in S_{-i}$$

5 To Probe Further

The material in this chapter is mostly called out from the books by Osborne [3], Shoham and Leyton-Brown [1], and Myerson [2]. The proofs for some of the problems mentioned above can also be found in these references.

References

[1] Yoam Shoham and Kevin Leyton-Brown. *Multiagent systems: Algorithmic, Game-Theoretic, and Logical Foundations.* Cambridge University Press, New York, USA, 2009, 2009.

[2] Roger B. Myerson. *Game Theory: Analysis of Conflict.* Harvard University Press, Cambridge, Massachusetts, USA, 1997.

[3] Martin J. Osborne. *An Introduction to Game Theory.* The MIT Press, 2003.