Game Theory

Lecture Notes By

Y. Narahari

Department of Computer Science and Automation

Indian Institute of Science

Bangalore, India

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Chapter 6: Mixed Strategies and Mixed Strategy Nash Equilibrium

Note: *This is a only a draft version, so there could be flaws. If you find any errors, please do send email to* `hari@csa.iisc.ernet.in`. *A more thorough version would be available soon in this space.*

In this chapter, we introduce randomized strategies or mixed strategies and define a mixed strategy Nash equilibrium. We state and prove a crucial theorem that provides an extremely useful necessary and sufficient condition for a mixed strategy profile to be a Nash equilibrium. We provide several examples to get an intuitive understanding of this important notion.

1 Randomized Strategies or Mixed Strategies

Consider a strategic form game: $\Gamma = \langle N, (S_i), (u_i) \rangle$. The elements of S_i are called *pure strategies* of player i ($i = 1, \ldots, n$). If player i chooses a strategy in S_i according to a probability distribution, we have a mixed strategy or a randomized strategy. In the discussion that follows, we assume that S_i is a finite for each $i = 1, 2, \ldots, n$.

Definition 1 (Mixed Strategy) . *Given a player i with S_i as the set of pure strategies, a mixed strategy σ_i of player i is a probability distribution over S_i. That is, $\sigma_i : S_i \to [0, 1]$ assigns to each pure strategy $s_i \in S_i$, a probability $\sigma_i(s_i)$ such that*

$$\sum_{s_i \in S_i} \sigma_i(s_i) = 1.$$

A pure strategy of a player, say $s_i \in S_i$, can be considered as a mixed strategy that assigns probability 1 to s_i and probability 0 to all other strategies of player i. Such a mixed strategy is called a *degenerate mixed strategy* and is denoted by $e(s_i)$ or simply by s_i.

If $S_i = \{s_{i1}, s_{i2}, \ldots, s_{im}\}$, then clearly, the set of all mixed strategies of player i is the set of all probability distributions on the set S_i. In other words, it is the simplex:

$$\Delta(S_i) = \left\{ (\sigma_{i1}, \ldots, \sigma_{im}) \in \mathbb{R}^m : \sigma_{ij} \geq 0 \text{ for } j = 1, \ldots, m \text{ and } \sum_{j=1}^{m} \sigma_{ij} = 1 \right\}.$$

The above simplex is called the *mixed extension* of S_i. Using the mixed extensions of strategy sets, we would like to define a mixed extension of the pure strategy game $\Gamma = \langle N, (S_i), (u_i) \rangle$. Let us denote the mixed extension of Γ by

$$\Gamma_{ME} = \langle N, (\Delta(S_i)), (U_i) \rangle.$$

Note that, for $i = 1, 2, \ldots, n$,

$$U_i : \times_{i \in N} \Delta(S_i) \to \mathbb{R}.$$

Given $\sigma_i \in \Delta(S_i)$ for $i = 1, \ldots, n$, a natural way of defining and computing $U_i(\sigma_1, \ldots, \sigma_n)$ as follows. First, we make the standard assumption that the randomizations of individual players are mutually independent. This implies that given a profile $(\sigma_1, \ldots, \sigma_n)$, the random variables $\sigma_1, \ldots, \sigma_n$ are mutually independent. Therefore the probability of a pure strategy profile $(s_1, \ldots, s_n)$ is given by

$$\sigma(s_1, \ldots, s_n) = \prod_{i \in N} \sigma_i(s_i).$$

The payoff functions U_i are defined as

$$U_i(\sigma_1, \ldots, \sigma_n) = \sum_{(s_1, \ldots, s_n) \in S} \sigma(s_1, \ldots, s_n)\ u_i(s_1, \ldots, s_n).$$

In the sequel, when there is no confusion, we will write u_i instead of U_i. For example, instead of writing $U_i(\sigma_1, \ldots, \sigma_n)$, we will simply write $u_i(\sigma_1, \ldots, \sigma_n)$.

1.1 Example: Mixed Strategies in the BOS Problem

Recall the BOS game discussed in Chapters 3 and 5, having the following payoff matrix:

	2	
1	A	B
A	2,1	0,0
B	0,0	1,2

Suppose (σ_1, σ_2) is a mixed strategy profile. This means that σ_1 is a probability distribution on $S_1 = \{A, B\}$, and σ_2 is a probability distribution on $S_2 = \{A, B\}$. Let us represent

$$\sigma_1 = (\sigma_1(A), \sigma_1(B))$$
$$\sigma_2 = (\sigma_2(A), \sigma_2(B)).$$

We have

$$S = S_1 \times S_2 = \{(A, A), (A, B), (B, A), (B, B)\}.$$

We will now compute the payoff functions u_1 and u_2. Note that, for $i = 1, 2$,

$$u_i(\sigma_1, \sigma_2) = \sum_{(s_1, s_2) \in S} \sigma(s_1, s_2)\ u_i(s_1, s_2).$$

The function u_1 can be computed as

$$\begin{aligned} u_1(\sigma_1, \sigma_2) &= \sigma_1(A)\sigma_2(A)u_1(A, A) + \sigma_1(A)\sigma_2(B)u_1(A, B) \\ &\quad + \sigma_1(B)\sigma_2(A)u_1(B, A) + \sigma_1(B)\sigma_2(B)u_1(B, B) \\ &= 2\sigma_1(A)\sigma_2(A) + \sigma_1(B)\sigma_2(B) \\ &= 2\sigma_1(A)\sigma_2(A) + (1 - \sigma_1(A))(1 - \sigma_2(A)) \\ &= 1 + 3\sigma_1(A)\sigma_2(A) - \sigma_1(A) - \sigma_2(A). \end{aligned}$$

Similarly, we can show that

$$u_2(\sigma_1, \sigma_2) = 2 + 3\sigma_1(A)\sigma_2(A) - 2\sigma_1(A) - 2\sigma_2(A).$$

Suppose $\sigma_1 = \left(\frac{2}{3}, \frac{1}{3}\right)$ and $\sigma_2 = \left(\frac{1}{3}, \frac{2}{3}\right)$. Then it is easy to see that

$$u_1(\sigma_1, \sigma_2) = \frac{2}{3}; \quad u_2(\sigma_1, \sigma_2) = \frac{2}{3}.$$

2 Mixed Strategy Nash Equilibrium

We now define the notion of a mixed strategy Nash equilibrium, which is a natural extension of the notion of pure strategy Nash equilibrium.

Definition 2 (Mixed Strategy Nash Equilibrium) . *Given a strategic form game* $\Gamma = \langle N, (S_i), (u_i) \rangle$, *a mixed strategy profile* $(\sigma_1^*, \ldots, \sigma_n^*)$ *is called a Nash equilibrium if* $\forall i \in N$,

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*) \quad \forall \sigma_i \in \Delta(S_i).$$

Define the best response functions $B_i(.)$ as follows.

$$B_i(\sigma_{-i}) = \{\sigma_i \in \Delta(S_i) : u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i}) \;\; \forall \sigma_i' \in \Delta(S_i)\}.$$

Then, clearly, a mixed strategy profile $(\sigma_1^*, \ldots, \sigma_n^*)$ is a Nash equilibrium iff

$$\sigma_i^* \in B_i(\sigma_{-i}^*) \qquad \forall i = 1, 2, \ldots, n.$$

2.1 Mixed Strategy Nash Equilibria for the BOS Game

Given the BOS game, suppose (σ_1, σ_2) is a mixed strategy profile. We have already shown that

$$\begin{aligned} u_1(\sigma_1, \sigma_2) &= 1 + 3\sigma_1(A)\sigma_2(A) - \sigma_1(A) - \sigma_2(A) \\ u_2(\sigma_1, \sigma_2) &= 2 + 3\sigma_1(A)\sigma_2(A) - 2\sigma_1(A) - 2\sigma_2(A). \end{aligned}$$

Let (σ_1^*, σ_2^*) be a mixed strategy equilibrium. Then

$$u_1(\sigma_1^*, \sigma_2^*) \geq u_1(\sigma_1, \sigma_2^*) \quad \forall \sigma_1 \in \Delta(S_1)$$

$$u_2(\sigma_1^*, \sigma_2^*) \geq u_2(\sigma_1^*, \sigma_2) \quad \forall \sigma_2 \in \Delta(S_2).$$

The above two equations are equivalent to:

$$3\sigma_1^*(A)\sigma_2^*(A) - \sigma_1^*(A) \geq 3\sigma_1(A)\sigma_2^*(A) - \sigma_1(A)$$

$$3\sigma_1^*(A)\sigma_2^*(A) - 2\sigma_2^*(A) \geq 3\sigma_1^*(A)\sigma_2(A) - 2\sigma_2(A).$$

The last two equations are equivalent to:

$$\sigma_1^*(A)\{3\sigma_2^*(A) - 1\} \geq \sigma_1(A)\{3\sigma_2^*(A) - 1\} \quad \forall \sigma_1 \in \Delta(S_1) \tag{1}$$

$$\sigma_2^*(A)\{3\sigma_1^*(A) - 2\} \geq \sigma_2(A)\{3\sigma_1^*(A) - 2\} \quad \forall \sigma_2 \in \Delta(S_2). \tag{2}$$

There are three possible cases.

- Case 1: $3\sigma_2^*(A) > 1$. This leads to the pure strategy Nash equilibrium (A, A).
- Case 2: $3\sigma_2^*(A) < 1$. This leads to the pure strategy Nash equilibrium (B, B).
- Case 3: $3\sigma_2^*(A) = 1$. This leads to the mixed strategy profile:

$$\sigma_1^*(A) = \frac{2}{3};\ \ \sigma_1^*(B) = \frac{1}{3};\ \ \sigma_2^*(A) = \frac{1}{3};\ \ \sigma_2^*(B) = \frac{2}{3}.$$

  We will show later on that this is indeed a mixed strategy Nash equilibrium using a necessary and sufficient condition for a mixed strategy profile to be a Nash equilibrium.

3 Some Results on Mixed Strategies

3.1 Convex Combination

Given numbers $y_1, y_2, \ldots, y_n$, a convex combination of these numbers is a weighted sum of the form $\lambda_1 y_1 + \lambda_2 y_2 + \cdots + \lambda_n y_n$, where

$$0 \leq \lambda_i \leq 1 \quad \text{for } i = 1, 2, \ldots, n; \quad \sum_{i=1}^{n} \lambda_i = 1$$

We shall now prove some interesting properties and results about mixed strategies.

3.2 Result 1

Given a mixed strategy game $\Gamma = \langle N, (\Delta(S_i)), (u_i) \rangle$, then, for any $i \in N$,

$$u_i(\sigma_i, \sigma_{-i}) = \sum_{s_i \in S_i} \sigma_i(s_i) u_i(s_i, \sigma_{-i})$$

The implication of this result is that the payoff for any player under a mixed strategy can be computed as a convex combination of the payoffs obtained with the player playing pure strategies with the rest of the players playing σ_{-i}.

Proof of Result 1

Note that

$$\begin{aligned} u_i(\sigma_i, \sigma_{-i}) &= \sum_{(s_1,\ldots,s_n)\in S} \left(\prod_{j\in N} \sigma_j(s_j)\right) u_i(s_i, s_{-i}) \\ &= \sum_{s_1\in S_1} \sum_{s_2\in S_2} \cdots \sum_{s_n\in S_n} \left(\prod_{j\in N} \sigma_j(s_j)\right) u_i(s_i, s_{-i}) \\ &= \sum_{s_i\in S_i} \sum_{s_{-i}\in S_{-i}} \left(\prod_{j\neq i} \sigma_j(s_j)\right) \sigma_i(s_i) u_i(s_i, s_{-i}) \\ &= \sum_{s_i\in S_i} \sigma_i(s_i) \left\{ \sum_{s_{-i}\in S_{-i}} \left(\prod_{j\neq i} \sigma_j(s_j)\right) u_i(s_i, s_{-i}) \right\} \\ &= \sum_{s_i\in S_i} u_i(s_i, \sigma_{-i}) \end{aligned}$$

3.3 Result 2

Given a mixed strategy game $\Gamma = \langle N, (\Delta(S_i)), (u_i)\rangle$, then, for any $i \in N$,

$$u_i(\sigma_i, \sigma_{-i}) = \sum_{s_{-i}\in S_{-i}} \left(\prod_{j\neq i} \sigma_j(s_j)\right) u_i(\sigma_i, \sigma_{-i})$$

Proof of Result 2

The result is proved using a series of steps as shown.

$$\begin{aligned} u_i(\sigma_i, \sigma_{-i}) &= \sum_{\substack{s_i\in S_i \\ \forall i\in N}} \left(\prod_{j\in N} \sigma_j(s_j)\right) u_i(s_i, s_{-i}) \\ &= \sum_{s_{-i}\in S_{-i}} \left(\prod_{j\neq i} \sigma_j(s_j)\right) \sum_{s_i\in S_i} \sigma_i(s_i) u_i(s_i, s_{-i}) \\ &= \sum_{s_{-i}\in S_{-i}} \left(\prod_{j\neq i} \sigma_j(s_j)\right) u_i(\sigma_i, s_{-i}) \end{aligned}$$

3.4 An Example and Some Observations on Convex Combinations

We motivate a few important observations about convex combinations in the context of mixed strategies with a simple example. Consider a game with

$$\begin{aligned} N &= \{1, 2\} \\ S_1 &= \{x_1, x_2, x_3, x_4, x_5\} \\ u_1(\sigma_1, \sigma_2) &= \sum_{s_1 \in S_1} \sigma_1(s_1) u_1(s_1, \sigma_2) \\ &= \sigma_1(x_1) u_1(x_1, \sigma_2) \\ &+ \sigma_1(x_2) u_1(x_2, \sigma_2) + \sigma_1(x_3) u_1(x_3, \sigma_2) \\ &+ \sigma_1(x_4) u_1(x_4, \sigma_2) + \sigma_1(x_5) u_1(x_5, \sigma_2) \end{aligned}$$

Let $u_1(x_1, \sigma_2) = 5$; $u_1(x_2, \sigma_2) = u_1(x_3, \sigma_2) = 10$; $u_1(x_4, \sigma_2) = u_1(x_5, \sigma_2) = 20$. First note that maximum value of the convex combination $= 20$ and this maximum value is attained when $\sigma_1(x_4) = 1$ or $\sigma_1(x_5) = 1$ or in general when $\sigma_1(x_4) + \sigma_1(x_5) = 1$. That is, when $\sigma_1(x_1) + \sigma_1(x_2) + \sigma_1(x_3) = 0$, or equivalently, when $\sigma_1(x_1) = \sigma_1(x_2) = \sigma_1(x_3) = 0$. Also, note that

$$\max_{\sigma_1 \in \Delta(s_1)} u_1(\sigma_1, \sigma_2) = 20$$

$$\max_{\sigma_1 \in \Delta(s_1)} u_1(\sigma_1, \sigma_2) = \max_{s_1 \in S} u_1(s_1, \sigma_2)$$

Let $\rho \in \{\sigma_1 \in \Delta(s_1) : u_1(\sigma_1, \sigma_2) \geq u_1(\sigma_1^{'}, \sigma_2) \ \ \forall \sigma_1^{'} \in \Delta(s_1)\}$.

$$\begin{aligned} &\iff \rho(x_4) + \rho(x_5) = 1 \\ &\iff \rho(x_1) + \rho(x_2) + \rho(x_3) = 0 \\ &\iff \rho(x_1) = \rho(x_2) = \rho(x_3) = 0 \\ &\iff \rho(y) = 0 \ \ \forall y \notin \overset{argmax}{s_1 \in S_1} \ u_1(s_1, \sigma_2) \end{aligned}$$

The above example motivates the following important result.

3.5 Result 3

Given $\langle N_1(\Delta(S_i), (u_i) \rangle$ for any $\sigma \in i \overset{\times}{\in} N \ \Delta(S_i)$ and for any player $i \in N$,

$$\max_{\sigma_i \in \Delta(S_i)} u_i(\sigma_i, \sigma_{-i}) = \max_{s_i \in S_i} u_i(s_i, \sigma_{-i})$$

Furthermore

$$\rho_i \in \ \overset{argmax}{\sigma_i \in \Delta(S_i)} \ u_i(\sigma_i, \sigma_{-i})$$

iff

$$\rho_i(x) = 0 \quad \forall x \notin \ \overset{argmax}{s_i \in S_i} \ u_i(s_i, \sigma_{-i})$$

Proof of Result 3

The first step is to express $u_i(\sigma_i, \sigma_{-i})$ as a convex combination:

$$u_i(\sigma_i, \sigma_{-i}) = \sum_{s_i \in S_i} \sigma_i(s_i) u_i(s_i, \sigma_{-i})$$

The maximum value of a convex combination of values is simply the maximum of values. Hence

$$\max_{\sigma_i \in \Delta(S_i)} u_i(\sigma_i, \sigma_{-i}) = \max_{s_i \in S_i} u_i(s_i, \sigma_{-i})$$

A mixed strategy $\rho_i \in \Delta(S_i)$ will attain this maximum value iff

$$\sum_{xinX} \rho_i(x) = 1 \text{ where } X = \overset{argmax}{s_i \in S_i} u_i(s_i, \sigma_{-i})$$

$$\iff \rho_i(x) = 0 \quad \forall x \notin \overset{argmax}{s_i \in S_i} u_i(s_i, \sigma_{-i})$$

4 A Necessary and Sufficient Condition for Nash Equilibrium

We now prove an extremely useful characterization for a mixed strategy Nash equilibrium profile. First we define the notion of support of a mixed strategy.

Definition 3 (Support of a Mixed Strategy) . *Let σ_i be any mixed strategy of a player i. The support of σ_i, denoted by $\delta(\sigma_i)$, is the set of all pure strategies which have non-zero probabilities under σ_i, that is:*

$$\delta(\sigma_i) = \{s_i \in S_i : \sigma_i(s_i) > 0\}$$

Definition 4 (Support of a Mixed Strategy Profile) . *Let $\sigma = (\sigma_1, \ldots, \sigma_n)$ be a mixed strategy profile with $\delta(\sigma_i)$ as the support of σ_i for $i = 1, \ldots, n$. Then the support $\delta(\sigma)$ of the profile σ is the Cartesian product of the individual supports, that is $\delta(\sigma_1) \times \ldots \delta(\sigma_n)$.*

4.1 NASC for a Mixed Strategy Nash Equilibrium

The mixed strategy profile $(\sigma_1^*, \ldots, \sigma_n^*)$ is a mixed strategy Nash equilibrium iff

1. $u_i(s_i, \sigma_{-i}^*)$ is the same $\quad \forall s_i \in \delta(\sigma_i^*)$
2. $u_i(s_i, \sigma_{-i}^*) \geq u_i(s_i^{'}, \sigma_{-i}^*) \ \ \forall s_i \in \delta(\sigma_i^*) \ \ \forall s_i^{'} \notin \delta(\sigma_i^*)$ (that is, the payoff of the player i for each pure strategy having positive probability is the same and is at least the payoff for each pure strategy having zero probability).

This theorem has a great deal of significance in many contexts, including computation of Nash equilibria. We now prove this theorem.

Proof of Necessity

Given that $(\sigma_1^*, \ldots, \sigma_n^*)$ is a Nash equilibrium. We have to show that the profile will satisfy the two conditions above. It is clear from the definition of Nash equilibrium that

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*) \;\; \forall \sigma_i \in \Delta(S_i)$$

This implies that

$$u_i(\sigma_i^*, \sigma_{-i}^*) = \max_{\sigma_i \in \Delta(S_i}) u_i(\sigma_i, \sigma_{-i}^*)$$

Using Result 3, we can now write that

$$u_i(\sigma_i^*, \sigma_{-i}^*) = \max_{s_i \in S_i} u_i(s_i, \sigma_{-i}^*)$$

This immediately implies that

$$\sum_{s_i \in S_i} \sigma_i^*(s_i) u_i(s_i, \sigma_{-i}^*) = \max_{s_i \in S_i} u_i(s_i, \sigma_{-i}^*)$$

This in turn leads to

$$\sum_{s_i \in \delta(\sigma_i^*} \sigma_i^*(s_i) u_i(s_i, \sigma_{-i}^*) = \max_{s_i \in S_i} u_i(s_i, \sigma_{-i}^*)$$

Since $\sigma_i^*(s_i) = 0 \;\; \forall s_i \notin \delta(\sigma_i)$, we have

$$u_i(s_i, \sigma_{-i}^*) = \max_{s_i \in S_i} u_i(s_i, \sigma_{-i}^*) \forall s_i \in \delta(\sigma_i^*)$$

While deriving the above, we have used the standard property of a convex combination that, if $\pi_1 + \ldots + \pi_n = 1$ and $\pi_1 x_1 + \ldots + \pi_n x_n = \max(x_1 + \ldots + x_n)$, then,

$$x_1 = x_2 = \ldots = x_n = \max(x_1, \ldots, x_n) \Rightarrow u_i(s_i, \sigma_{-i}^*) = u_i(\sigma_i^*, \sigma_{-i}^*) \;\; \forall s_i \in \delta(\sigma_i^*)$$

Since $u_i(s_i, \sigma_{-i}^*) = u_i(\sigma_i^*, \sigma_{-i}^*) \;\; \forall s_i \in \delta(\sigma_i^*)$, it is clear that

$$u_i(s_i, \sigma_{-i}^*) \geq u_i(s_i^{'}, \sigma_{-i}^*) \;\; \forall s_i^{'} \notin \delta(\sigma_i^*) \text{ and } \; \forall s_i \in \delta(\sigma_i^*)$$

This proves the necessity.

Proof of Sufficiency

We are given that

1. $u_i(s_i, \sigma_{-i}^*)$ has the same value, say, w_i, for all $s_i \in \delta(\sigma_i^*) \;\;\; \forall i \in N$
2. $u_i(s_i, \sigma_{-i}^*) \geq u_i(s_i^{'}, \sigma_{-i}^*), \forall s_i \in \delta(\sigma_i^*) \;\; \forall s_i^{'} \notin \delta(\sigma_i^*) \;\;\; \forall i \in N.$

Then, we have to show that $u(\sigma_i^*, \ldots, (\sigma_n^*)$ is a Nash equilibrium. Consider

$$\begin{aligned}
u_i(s_i, \sigma_{-i}^*) &= \sum_{s_i \in S_i} \sigma_i^*(s_i) u_i(s_i, \sigma_{-i}^*) \\
&= \sum_{s_i \in \delta(\sigma_i^*)} \sigma_i^*(s_i) u_i(s_i, \sigma_{-i}^*) \;\text{ since } \sigma_i^*(s_i) = 0 \;\; \forall s_i \notin \delta(\sigma_i^*) \\
&= \sum_{s_i \in \delta(\sigma_i^*)} \sigma_i^*(s_i).w_i \\
&= w_i \\
&= \sum_{s_i \in S_i} \sigma_i(s_i) w_i \;\; \forall \sigma_i \in \Delta(S_i) \\
&\geq \sum_{s_i \in S_i} \sigma_i(s_i) u_i(s_i, \sigma_i^*)
\end{aligned}$$

since $w_i = u_i(s_i, \sigma_{-i}^*) \;\; \forall s_i \in \delta(\sigma_i^*)$ and $u_i(s_i, \sigma_{-i}^*) \geq u_i(s_i^{-1}, \sigma_{-i}^*) \;\; \forall s_i^{'} \notin \delta(\sigma_i^*) = u_i(\sigma_i, \sigma_{-i}^*)$ where $\sigma_i \in \Delta(\sigma_i^*)$. Therefore, $(\sigma_1^*, \ldots, \sigma_n^*)$ is a Nash equilibrium.

4.2 Implications of the Necessary and Sufficient Conditions

The NASC above has the following implications.

1. In a mixed strategy Nash equilibrium, each player gets the same payoff (as in Nash equilibrium) by playing *any pure strategy* having positive probability in his Nash equilibrium strategy.
2. The above implies that the player can be indifferent about which of the pure strategies (with positive probability) he/she will play.
3. To verify whether or not a mixed strategy profile is a Nash equilibrium, it is enough to consider the effects of only pure strategy deviations.
4. Another important implication is described in the following result.

A Result on Degenerate Mixed Strategies

Given $s_i \in S_i$, let $e(s_i)$ denote the degenerate mixed strategy that assigns probability 1 to s_i and probability 0 to elements of S_i other than s_i. The pure strategy profile $(s_i^*, \ldots, s_n^*)$ is a Nash equilibrium of the game $(N, (S_i), (u_i))$ iff the mixed strategy profile $(e(s_1^*), \ldots, e(s_n^*))$ is a mixed strategy Nash equilibrium of the game $(N, (\Delta(S_i)), (u_i))$.

The proof of this proceeds as follows. First we prove the sufficiency. Let $(e(s_i^*), \ldots, e(s_n^*))$ be a mixed strategy Nash equilibrium.

$$\begin{aligned}
&\Rightarrow \;\; u_i(e(s_i^*), e(s_{-i}^*)) \geq u_i(\sigma_i, e(s_{-i}^*)) \;\; \forall \sigma_i \in \Delta(S_i) \\
&\Rightarrow \;\; u_i(s_i^*, s_{-i}^*) \geq u_i(\sigma_i, s_{-i}^*) \;\; \forall \sigma_i \in \Delta(S_i) \\
&\Rightarrow \;\; u_i(s_i^*, s_{-i}^*) \geq u_i(e(s_i), s_{-i}^*) \;\; \forall \sigma_i \in S_i \\
&\Rightarrow \;\; u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*) \;\; \forall s_i \in S_i \\
&\Rightarrow \;\; (s_1^*, \ldots, s_n^*) \text{ is a pure strategy Nash equilibrium}
\end{aligned}$$

The above proves sufficiency. The necessity is proved as follows. Given that $(s_1^*, \ldots, s_n^*)$ is a pure strategy Nash equilibrium

$$\begin{aligned}
&\Rightarrow\ \ u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*) \ \ \forall s_i \in S_i \quad \forall i \in N \\
&\Rightarrow\ \ u_i(e(s_i^*), e(s_{-i}^*)) \geq u_i(s_i, e(s_{-i}^*)) \ \ \forall s_i \in S_i \quad \forall i \in N \\
&\Rightarrow\ \ u_i(e(s_i^*), e(s_{-i}^*)) = \max_{s_i \in S_i} u_i(s_i, e(s_{-i}^*)) \ \ \forall i \in N \\
&\Rightarrow\ \ u_i(e(s_i^*), e(s_{-i}^*)) = \max_{\sigma_i \in \Delta(S_i)} u_i(\sigma_i, e(s_{-i}^*)) \qquad \forall i \in N \text{using a previous result} \\
&\Rightarrow\ \ u_i(e(s_i^*), e(s_{-i}^*)) \geq u_i(\sigma_i, e(s_{-i}^*)) \qquad \forall \sigma_i \in N\Delta(S_i) \qquad \forall i \in N \\
&\Rightarrow\ \ (e(s_1^*), \ldots, e(s_n^*)) \text{ is a mixed strategy Nash equilibrium}
\end{aligned}$$

The implication of this result is that to identify the pure strategy equilibria of the game $(N, (\Delta(S_i)), (u_i))$, it is enough to look at the pure strategy game $(N, (S_i), (u_i))$.

5 Examples to Illustrate Necessary and Sufficient Conditions

5.1 The BOS Game

We recall again the BOS game with the payoff matrix:

	2	
1	A	B
A	2,1	0,0
B	0,0	1,2

We have seen that

$$u_1(\sigma_1, \sigma_2) = 1 + 3\sigma_1(A)\sigma_2(A) - \sigma_1(A) - \sigma_2(A)$$
$$u_2(\sigma_1, \sigma_2) = 2 + 3\sigma_1(A)\sigma_2(A) - 2\sigma_1(A) - 2\sigma_2(A)$$

First we verify that (A, A) is a Nash equilibrium. Surely it satisfies the NASC of the theorem.

$$\sigma_1^*(A) = 1; \quad \sigma_1^*(B) = 0; \quad \sigma_2^*(A) = 1; \quad \sigma_2^*(B) = 0$$

$$u_1(A, \sigma_2^*) = 2; \quad u_1(B, \sigma_2^*) = 0$$

Condition (1) is trivially true and condition (2) is true because

$$u_1(A, \sigma_2^*) > u_1(B, \sigma_2^*)$$

These conditions are similarly satisfied for player 2 also. Hence (A, A) is a Nash equilibrium. Similarly, (B, B) is also a NE.

Now, let us look at the candidate Nash equilibrium: $((\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3}))$. We have:

$$\sigma_1^*(A) = \frac{2}{3} \quad \sigma_1^*(B) = \frac{1}{3}$$
$$\sigma_2^*(A) = \frac{1}{3} \quad \sigma_2^*(B) = \frac{2}{3}$$

Player 1: Let us check condition (1).

$$u_1(A, \sigma_2^*(A) = \frac{1}{3}(2) + \frac{2}{3}(0) = \frac{2}{3}$$
$$u_1(B, \sigma_2^*(A) = \frac{1}{3}(0) + \frac{2}{3}(1) = \frac{2}{3}$$

Condition (2) is trivially satisfied since $\delta(\sigma_1^*) = \{A, B\}$, the entire set.

Player 2: Let us check condition (1).

$$\left.\begin{array}{ll} u_2(\sigma_1^*, A) & = \frac{2}{3}(1) = \frac{2}{3} \\ u_2(\sigma_1^*, B) & = \frac{2}{3} \end{array}\right\}$$

Condition (2) is trivially satisfied as before.

Let us investigate if there are any other Nash equilibria. The equilibrium (A, A) corresponds to the support $\{A\} \times \{A\}$. The equilibrium (B, B) corresponds to the support $\{B\} \times \{B\}$. The equilibrium $((\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3}))$ corresponds to the support $\{A, B\} \times \{A, B\}$.

- There is no Nash equilibrium with support $\{A\} \times \{A, B\}$. If player 1 plays A, then player 2 has to play only A, which leads to the NE (A, A). There is no way player will play B.

- Similarly, there is no Nash equilibrium with supports

$$\begin{array}{rcl} \{B\} & \times & \{A, B\} \\ \{A, B\} & \times & \{A\} \\ \{A, B\} & \times & \{B\} \\ \{B\} & \times & \{A\} \\ \{A\} & \times & \{B\} \end{array}$$

- Let us see if there is any other Nash equilibrium with support $\{A, B\} \times \{A, B\}$. To see this, let (σ_1^*, σ_2^*) defined by

$$\begin{array}{ll} \sigma_1^*(A) = x & \sigma_1^*(B) = 1 - x \\ \sigma_2^*(A) = y & \sigma_2^*(B) = 1 - y \end{array}$$

  be a Nash equilibrium such that neither $x \neq o, x \neq 1, y \neq o$ and $y \neq 1 (0 < x < 1; 0 < y < 1)$. Then by condition (1) of the theorem, we have:

$$\begin{array}{ll} u_1(A, \sigma_2^*) & = u_1(B, \sigma_2^*) \\ u_2(\sigma_1^*, A) & = u_2(\sigma_1^*, B) \end{array}$$

  This implies $2y = 1 - y$ and $x = 2(1 - x)$. This in turn implies $y = \frac{1}{3}$; $x = \frac{2}{3}$. This leads to the NE

$$\sigma_1^* = \left(\frac{2}{3}, \frac{1}{3}\right); \quad \sigma_2^* = \left(\frac{1}{3}, \frac{2}{3}\right)$$

5.2 Example 2: Coordination Game

Let us consider a variant of the coordination game with the payoff matrix:

	2	
1	A	B
A	10,10	0,0
B	0,0	1,1

In one interpretation of this game, the two players are students studying in a college and option A corresponds to staying in college and option B corresponds to going to a movie. We have already seen that (A, A) and (B, B) are pure strategy Nash equilibria. These correspond to the supports $\{A\} \times \{A\}$ and $\{B\} \times \{B\}$, respectively. It can be shown that the supports $\{A\} \times \{B\}$; $\{B\} \times \{A\}$; $\{A\} \times \{A, B\}$; $\{B\} \times \{A, B\}$; $\{A, B\} \times \{A\}$; $\{A, B\} \times \{B\}$ do not lead to any Nash equilibrium. we now investigate if there exists a Nash equilibrium with the support $\{A, B\} \times \{A, B\}$. Let $\sigma_1^* = (x, 1-x)$; $\quad \sigma_2^* = (y, 1-y)$ with $x \neq 0,\ x \neq 1, y \neq 0, y \neq 1$ be a Nash equilibrium. Then condition (2) is trivially satisfied (since the support in each case is the entire strategy set). Let us check condition (1) which leads to:

$$
\begin{aligned}
u_1(A, \sigma_2^*) &= u_1(B, \sigma_2^*) \\
u_2(\sigma_1^*, A) &= u_2(\sigma_1^*, B)
\end{aligned}
$$

The above equations are equivalent to

$$
\begin{aligned}
10y &= 1 - y \\
10x &= 1 - x
\end{aligned}
$$

This leads to: $y = \frac{1}{11}$; $\quad x = \frac{1}{11}$. This means the mixed strategy profile $(\sigma_1^* = (\frac{1}{11}, \frac{10}{11})), \sigma_2^*) = (\frac{1}{11}, \frac{10}{11})$ is also a Nash equilibrium. This indeed explains why students could be found going to a movie with a high probability rather than studying in the college! It is interesting that though staying in college gives more pay off, the friends are more likely meet in a movie if this is the equilibrium that is selected. Note that the players have no real preference over the probabilities that they play their strategies with. What actually determines these probabilities is the Nash equilibrium consideration namely the need to make the other player indifferent over his strategies. This has prompted many game theorists to question the usefulness of mixed strategy Nash equilibria. There are mainly two concerns.

1. Concern 1: If players have a choice of pure strategies that give them the same payoff as an equilibrium mixed strategy, why should they randomize at all? The explanation is that the players do not randomize but choose the pure strategy that they play based on some private information.

2. Concern 2: If players randomize, they must randomize with exact values of the probabilities as even small changes in these probabilities can disturb the Nash equilibrium.

6 To Probe Further

The material discussed in this chapter draws upon mainly from the the books by Myerson [1] and Osborne and Rubinstein [2]. Many of the problems below are also taken from the above books.

In this chapter, we have made an implicit assumption that the strategy sets are all finite and mixed strategies have been defined for only such games. However, mixed strategies can be naturally extended to infinite strategy sets by defining probability distributions over those sets.

The celebrated result by John Nash states that every finite strategic form game (that is with finite number of players and finite strategy sets) will surely have at least one mixed strategy Nash equilibrium. We will be proving the result in Chapter 9.

Computation of Nash equilibria is an issue of intense interest. We will be covering that in Chapter 10.

7 Problems

1. Let S be any finite set with n elements. Show that the set $\Delta(S)$, the set of all probability distributions over S, is a convex set.

2. Given a normal form game $(N, (\Delta(S_i)), (u_i))$, show for any two mixed strategies, ${\sigma_i}^*, \sigma_i$ that

$$u_i({\sigma_i}^*, \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i}) \quad \forall \ \sigma_{-i} \in \Delta(S_{-i})$$

   if and only if

$$u_i({\sigma_i}^*, s_{-i}) > u_i(\sigma_i, s_{-i}) \quad \forall \ s_{-i} \in S_{-i}$$

3. Show that any strictly dominant strategy in the game $(N, \Delta(S_i), (u_i))$ must be a pure strategy.

4. Find the mixed strategy Nash equilibria for the matching pennies game:

	H	T
H	1, -1	-1, 1
T	-1, 1	1, -1

5. Find the mixed strategy Nash equilibria for the rock-paper-scissors game:

	3		
1	Rock	Paper	Scissors
Rock	$0,0$	$-1,1$	$1,-1$
Paper	$1,-1$	$0,0$	$-1,1$
Scissors	$-1,1$	$1,-1$	$0,0$

6. Find the mixed strategy Nash equilibria for the following game.

	H	T
H	1, 1	0, 1
T	1, 0	0, 0

7. Find the mixed strategy Nash equilibria for the following game.

	A	B
A	6, 2	0, 0
B	0, 0	2, 6

If all these numbers are multiplied by 2, will the equilibria change?

8. Consider any arbitrary two player game of the following type (with a,b,c,d any arbitrary real number):

	A	B
A	a,a	b,c
B	c,b	d,d

It is known that the game has a strongly dominant strategy equilibrium. Now prove or disprove: The above strongly dominant strategy equilibrium is the only possible mixed strategy equilibrium of the game.

9. There are two sellers 1 and 2 and there are there buyers A, B, and C.

- A can only buy from seller 1.
- C can only buy from seller 2.
- B can buy from either seller 1 or seller 2.
- Each buyer has a budget (maximum willingness to pay) of 1 and wishes to buy one item.
- The sellers have enough items to sell.
- Each seller announces a price as a real number in the range $[0, 1]$. Let s_1 and s_2 be the prices announced by sellers 1 and 2, respectively..
- Naturally, buyer A will buy an item from seller 1 at price s_1 and buyer C will buy an item from seller 2 at price s_2.
- In the case of buyer B, if $s_1 \leq s_2$, then he will buy an item from seller 1, otherwise he will buy from seller 2.

We have shown in Chapter 5 that the above game does not have pure strategy Nash equilibrium. Does this game have a mixed strategy Nash equilibrium?

10. Consider an n player game with $S_i = \{1, 2\}$ $\forall_i$. The payoff is

$$u_i(s_1, \ldots, s_n) = s_i \prod_{j \neq i} (1 - \delta(s_i, s_j))$$

where δ is the kronecker δ given by

$$\begin{aligned} \delta(s_i, s_j) &= 1 \qquad \text{if } s_i = s_j \\ &= 0 \qquad \text{otherwise} \end{aligned}$$

If player i uses a mixed strategy in which pure strategy 1 is chosen with probability p_i, prove that $(p_1, p_2, \ldots p_n)$ defines an equilibrium point iff

$$\prod_{j \neq i} (1 - p_j) = 2 \prod_{j \neq i} p_j \qquad \forall i \in N$$

Deduce that a mixed strategy equilibrium is given by

$$p_i = \frac{1}{1 + 2^{\frac{1}{n-1}}} \qquad \forall i \in N$$

and that for $n = 2, 3$ this is the only equilibrium points.

11. *(A war of attrition)* Two players are involved in a dispute over an object. The value of the object to player i is $v_i > 0$. Time is modeled as a continuous variable that starts at 0 and runs indefinitely. Each player chooses when to concede the object to the other player; if the first player to concede does so at time t, the other player obtains the object at that time. If both players concede simultaneously, the object is split equally between them, player i receiving a payoff of $v_i/2$. Time is valuable: until the first concession each player loses one unit of payoff per unit time. Formulate this situation as a strategic game and show that in all Nash equilibria, one of the players concedes immediately.

12. *(A location game)* Each of n people chooses whether or not to become a political candidate, and if so which position to take. There is a continuum of citizens, each of whom has a favorite position; the distribution of favorite positions is given by a density function f on $[0, 1]$ with $f(x) > 0$ for all $x \in [0, 1]$. A candidate attracts the votes of those citizens whose favorite positions are closer to his position than to the position-of any other candidate; if k candidates choose the same position then each receives the fraction $1/k$ of the votes that the the position attracts. The winner of the competition is the candidate who receives the most votes. Each person prefers to be the unique winning candidate than to tie for the first place, prefers to tie for the first place than to stay out of the competition, and prefers to stay out of the competition than to enter and lose. Formulate this situation as a strategic game, find the set of Nash equilibria when $n = 2$, and show that there is no Nash equilibrium when $n = 2$, and show that there is no Nash equilibrium when $n = 3$.

13. *(An exchange game)* Each of two players receives a ticket on which there is a number in some finite subset S of the interval $[0, 1]$. The number on a player's ticket is the size of a prize that he may receive. The two prizes are identically and independently distributed, with distribution function$F$. Each player is asked independently and simultaneously whether he wants to exchange his prize for the other player's prize. If both players agree then the prizes are exchanged; otherwise each player receives his own prize. Each player's objective is to maximize his expected payoff. Model this situation as a Bayesian game and show that in any Nash equilibrium the highest prize that either player is willing to exchange is the smallest possible prize.

14. *(Guess the average)* Each of n people announces a number in the set $\{1, ..., K\}$. A prize of \$1 is split equally between all the people whose number is closest to $\frac{2}{3}$ of the average number. Show that the game has a unique mixed strategy Nash equilibrium, in which each player's strategy is pure.

15. *(An investment race)* Two investors are involved in a competition with a prize of \$1. Each investor can spend any amount in the interval $[0, 1]$. The winner is the investor who spends the most; in the event of a tie each investor receives \$0.50. Formulate this situation as a strategic game and find its mixed strategy Nash equilibria. (Note that the player's payoff functions are discontinuous, so that Glicksberg's result does not apply; nevertheless the game has a mixed strategy Nash equilibrium.)

16. Each of n people announces a number in the set $\{1, 2, \ldots, m\}$. A prize of Rs 10000 is split equally between all the people whose number is closest to $\frac{2}{3}$ of the average number. Show that the game has a unique mixed strategy Nash equilibrium, in which each player's strategy is pure.

17. Consider a single player game with $N = \{1\}$ and $S_1 = [0, 1]$ (compact). Define the utility function as a discontinuous map:

$$\begin{aligned} u_1(s_1) &= s_1 \quad \text{if } 0 \le s_1 < 1 \\ &= 0 \quad \text{if } s_1 = 1 \end{aligned}$$

Show that the above game does not have a mixed strategy equilibrium.

18. Consider again a single player game with $N = \{1\}$ but with $S_1 = [0, 1]$ (not compact). Define the utility function as a continuous map:

$$u_1(s_1) = s_1 \;\; \forall s_1 \in [0, 1]$$

Show that this game also does not have a mixed strategy equilibrium.

19. Write down the necessary and sufficient conditions for a mixed strategy Nash equilibrium and using those, compute all mixed strategy Nash equilibria of the following problem:

	A	B
A	20, 0	0, 10
B	0, 90	20, 0

References

[1] Roger B. Myerson. *Game Theory: Analysis of Conflict*. Harvard University Press, Cambridge, Massachusetts, USA, 1997.

[2] Martin J. Osborne and Ariel Rubinstein. *A Course in Game Theory*. Oxford University Press, 1994.