# Game Theory

**Lecture Notes By**

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## Chapter 5: Pure Strategy Nash Equilibrium

**Note**: *This is a only a draft version, so there could be flaws. If you find any errors, please do send email to* `hari@csa.iisc.ernet.in`. *A more thorough version would be available soon in this space.*

In this chapter, we introduce the all important notion of pure strategy Nash equilibrium. We provide several examples to illustrate this notion. We show that pure strategy Nash equilibrium may not exist in many cases while in many other cases, there could exist multiple Nash equilibria. We also show that the payoffs that players get in a Nash equilibrium may not be socially optimal.

## 1 Nash Equilibrium

Dominant strategy equilibria (strongly dominant, weakly dominant, very weakly dominant), if they exist, are very desirable but rarely do they exist because the conditions to be satisfied are too demanding. A dominant strategy equilibrium requires that each player's choice be a best response against all possible choices of all the other players. If we only insist that each player's choice is a best response against the best response strategies of the other players, we get the notion of Nash equilibrium. This solution concept is named after John Nash, one of the most celebrated game theorists of our times. In this section, we introduce and discuss the notion of pure strategy Nash equilibrium. In the following section, we discuss the notion of mixed strategy Nash equilibrium.

**Definition 1 (Pure Strategy Nash Equilibrium.)** *Given a strategic form game $\Gamma = \langle N, (S_i), (u_i) \rangle$, the strategy profile $s^* = (s_1^*, s_2^*, \ldots, s_n^*)$ is said to be a pure strategy Nash equilibrium of $\Gamma$ if,*

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*), \quad \forall s_i \in S_i, \quad \forall i = 1, 2, \ldots, n.$$

That is, each player's Nash equilibrium strategy is a best response to the Nash equilibrium strategies of the other players.

**Definition 2 (Best Response Correspondence.)** *Given a game $\Gamma = \langle N, (S_i), (u_i) \rangle$, the best response correspondence for player $i$ is the mapping $B_i : S_{-i} \to 2^{S_i}$ defined by*

$$B_i(s_{-i}) = \{s_i \in S_i : u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \ \ \forall s_i' \in S_i\}.$$

That is, given a profile $s_{-i}$ of strategies of the other players, $B_i(s_{-i})$ gives the set of all best response strategies of player $i$.

It can be seen that the strategy profile $(s_1^*, \ldots, s_n^*)$ is a Nash equilibrium iff,

$$s_i^* \in B_i(s_{-i}^*), \quad \forall i = 1, \ldots, n.$$

Given a strategic form game $\Gamma = \langle N, (S_i), (u_i) \rangle$, a strongly (weakly) (very weakly) dominant strategy equilibrium $(s_1^*, \ldots s_n^*)$ is also a Nash equilibrium. This can be shown in a straightforward way (see problem 1 at the end of the chapter). The intuitive explanation for this is as follows. In a dominant strategy equilibrium, the equilibrium strategy of each player is a best response irrespective of the strategies of the rest of the players. In a pure strategy Nash equilibrium, the equilibrium strategy of each player is a best response against the Nash equilibrium strategies of the rest of the players. Thus, the Nash equilibrium is a much weaker version of a dominant strategy equilibrium. It is also fairly obvious to note that a Nash equilibrium need not be a dominant strategy equilibrium.

# 2 Examples of Pure Strategy Nash Equilibrium

## 2.1 The BOS Game

Recall the two player BOS game with the following payoff matrix:

| | 2 | |
|---|---|---|
| 1 | A | B |
| A | 2,1 | 0, 0 |
| B | 0,0 | 1,2 |

There are two Nash equilibria here, namely $(A, A)$ and $(B, B)$. The profile $(A, A)$ is a Nash equilibrium because

$$\begin{aligned} u_1(M, M) &> u_1(F, M) \\ u_2(M, M) &> u_2(M, F) \end{aligned}$$

The profile $(F, F)$ is a Nash equilibrium because

$$\begin{aligned} u_1(F, F) &> u_1(M, F) \\ u_2(F, F) &> u_2(F, M) \end{aligned}$$

The best response sets are given by:

$$B_1(M) = \{M\}; \;\; B_1(F) = \{F\}; \;\; B_2(M) = \{M\}; \;\; B_2(F) = \{F\}$$

Since $M \in B_1(M)$ and $M \in B_2(M)$, $(M, M)$ is a Nash equilibrium. Similarly since $F \in B_1(F)$ and $F \in B_2(F)$, $(F, F)$ is a Nash equilibrium. The profile $(M, F)$ is not a Nash equilibrium since

$$M \notin B_1(F); \;\; F \notin B_2(M)$$

### 2.2 Prisoner's Dilemma

We consider the prisoner's dilemma problem which has the following payoff matrix:

| | 2 | |
|---|---|---|
| 1 | NC | C |
| NC | $-2, -2$ | $-10, -1$ |
| C | $-1, -10$ | $-5, -5$ |

Note that $(C, C)$ is the unique Nash equilibrium here. To see why, we have to just look at the best response sets:

$$B_1(C) = \{C\};\ \ B_1(NC) = \{C\};\ \ B_2(C) = \{C\};\ \ B_2(NC) = \{C\}$$

Since $s_i^* \in B_1(s_2^*)$ and $s_2^* \in B_2(s_1^*)$ for a Nash equilibrium, the only possible Nash equilibrium here is $(C, C)$. In fact as already seen, this is a strongly dominant strategy equilibrium.

### 2.3 Tragedy of the the Commons

Recall that $N = \{1, 2, \ldots, n\}$ is a set of farmers and the strategy sets are $S_1 = S_2 = \cdots = S_n = \{0, 1\}$. A 1 corresponds to keeping a sheep, and a 0 corresponds to not keeping a sheep. Keeping a sheep gives a benefit of 1. However, when a sheep is kept, damage to the environment is 5. This damage is equally shared by all the farmers. Note that for $i = 1, 2, \ldots, n$

$$u_i(s_1, \ldots, s_n) = s_i - \frac{5}{n}\sum_{j=1}^{n} s_j = \left(\frac{n-5}{n}\right) s_i - \frac{5}{n}\sum_{j\neq i} s_j$$

When $n < 5$, we have shown in the previous chapter that $(0, 0, \ldots, 0)$ is a strongly dominant strategy equilibrium. That is, there is no incentive for any farmer to keep a sheep. When $n > 5$, we have shown that $(1, 1, \ldots, 1)$ is a strongly dominant strategy equilibrium. That is, keeping a sheep is a strongly dominant strategy for each farmer. Let us look at the case $n = 5$. Here,

$$u_i(0, s_{-i}) = -\frac{5}{n}\sum_{j\neq i} s_j$$
$$u_i(1, s_{-i}) = -\frac{5}{n}\sum_{j\neq i} s_j$$

Thus

$$u_i(0, s_{-i}) = u_i(1, s_{-i}), \quad \forall s_{-i} \in S_{-i}$$

This implies

$$B_i(s_{-i}) = \{0, 1\} \quad \forall s_{-i} \in S_{-i}$$

It can be seen that all the strategy profiles are Nash Equilibria here. Also note that they are neither weakly dominant nor strongly dominant strategy equilibria.

If the Government decides to impose a pollution tax of 5 units for each sheep kept, we have

$$u_i(s_1, \ldots, s_n) = s_i - 5s_i - \frac{5}{n}\sum_{j=1}^{n} s_j = -4s_i - \frac{5}{n}s_i - \frac{5}{n}\sum_{j\neq i} s_j$$

Here

$$
\begin{aligned}
u_i(0, s_{-i}) &= -\frac{5}{n}\sum_{j\neq i} s_j \\
u_i(1, s_{-i}) &= -4 - \frac{5}{n} - \frac{5}{n}\sum_{j\neq i} s_j \\
\therefore B_i(s_{-i}) &= \{0\} \quad \forall i \in N
\end{aligned}
\tag{1}
$$

This means whatever the value of $n$, $(0, 0, \ldots, 0)$ is a strongly dominant strategy equilibrium.

### 2.4 Bandwidth Sharing Problem

Recall the bandwidth sharing game discussed in [1] and also presented in Chapter 3. We compute a Nash equilibrium for this game in the following way. Let $x_i$ be the amount of flow that player $i$ $(i = 1, 2, \ldots, n)$ wishes to transmit on the channel and assume that

$$\sum_{i\in N} x_i < 1.$$

Consider player $i$ and define:

$$t = \sum_{j\neq i} x_j.$$

The payoff for the player $i$ is equal to

$$x_i(1 - t - x_i).$$

In order to maximize the above payoff, we have to choose

$$
\begin{aligned}
x_i^* &= \arg\max_{x_i\in[0,1]} x_i(1 - t - x_i) \\
&= \frac{1-t}{2} \\
&= \frac{1 - \sum_{j\neq i} x_j^*}{2}.
\end{aligned}
$$

If this has to be satisfied for all $i \in N$, then we end up with $n$ simultaneous equations

$$x_i^* = \frac{1 - \sum_{j\neq i} x_j^*}{2} \quad i = 1, 2, \ldots, n.$$

A Nash equilibrium of this game is any solution to the above $n$ simultaneous equations. It can be shown that the above set of simultaneous equations has the unique solution:

$$x_i^* = \frac{1}{1+n} \quad i = 1, 2, \ldots, n.$$

The profile $(x_1^*, \ldots, x_n^*)$ is thus a Nash equilibrium. The payoff for player $i$ in the above Nash equilibrium

$$= \left(\frac{1}{n+1}\right)\left(\frac{1}{n+1}\right).$$

Therefore the total payoff to all players combined

$$= \frac{n}{(n+1)^2}.$$

As shown below, the above is not a very happy situation. Consider the following non-equilibrium profile

$$\left(\frac{1}{2n}, \frac{1}{2n}, \ldots, \frac{1}{2n}\right).$$

This profile gives each player a payoff

$$\begin{aligned} &= \frac{1}{2n}\left(1 - \frac{n}{2n}\right) \\ &= \frac{1}{4n}. \end{aligned}$$

Therefore the total payoff to all the players

$$= \frac{1}{4} > \frac{n}{(n+1)^2}.$$

Thus a non-equilibrium payoff $\left(\frac{1}{2n}, \frac{1}{2n}, \ldots, \frac{1}{2n}\right)$ provides more payoff than a Nash equilibrium payoff. In general, like in the prisoner's dilemma problem, the equilibrium payoffs may not be the best possible outcome for the players individually and also collectively. This is a property that Nash equilibrium payoffs often suffer from and illustrates the conflict between individual and social good.

## 2.5 Duopoly Pricing Game

Recall the pricing game discussed in Chapter 3. There are two companies 1 and 2 that wish to maximize their profits by choosing their prices $p_1$ and $p_2$. The utilities of the two companies are:

$$\begin{aligned} u_1(p_1, p_2) &= (p_1 - c)\, x_1(p_1, p_2) \\ u_2(p_1, p_2) &= (p_2 - c)\, x_2(p_1, p_2). \end{aligned}$$

Note that $u_1(c, c) = 0$ and $u_2(c, c) = 0$. Also, it can be easily noted that

$$u_1(c, c) \geq u_1(p_1, c) \ \ \forall p_1 \in S_1$$

$$u_2(c, c) \geq u_2(c, p_2) \ \ \forall p_2 \in S_2.$$

Therefore the strategy profile $(c, c)$ is a pure strategy Nash equilibrium. The implication of this result is that in the equilibrium, the companies set their prices equal to the marginal cost. The intuition behind this result is to imagine what would happen if both the companies set equal prices above marginal cost. Then the two companies would get half the market at a higher than marginal cost price. However, by lowering prices just slightly, a firm could capture the whole market, so both firms are tempted to lower prices as much as they can. It would not make sense to price below marginal cost, because the firm would make a loss. Therefore, both firms will lower prices until they reach the marginal cost limit.

### 2.6 Braess Paradox Game

Here we consider the Braess paradox game which we discussed in Chapters 3 and 4. First consider that there is no link from A to B. Suppose $(s_1, \ldots, s_n)$ is any strategy profile such that

$$n_A(s_1, \ldots, s_n) = n_B(s_1, \ldots, s_n) = 500$$

That is, of the 1000 vehicles, exactly 500 take the route via A while the rest of the 500 vehicles take the route via B. Clearly, for such a strategy profile, $u_i(s_1, s_2, \ldots, s_n) = -35$ for all vehicles $i \in N$. Suppose vehicle $i$ deviates from $s_i$ with the rest of the vehicles retaining their strategies. The utility of vehicle $i$ now becomes $-25 - \frac{501}{50}$ which is less than 35. In fact, because of the unilateral deviation by vehicle $i$, the utility of all the 499 vehicles which were following the same route as vehicle $i$ will now be better off whereas vehicle $i$ and the rest of the 500 vehicles will be worse off. Thus all strategy profiles satisfying the above condition will be pure strategy Nash equilibria.

### 2.7 Games without Pure Strategy Nash Equilibria

Recall the matching pennies game discussed in Chapter 3 and the payoff matrix for this game:

| | 2 | |
|---|---|---|
| 1 | A | B |
| A | 1, −1 | −1, 1 |
| B | −1, 1 | 1, −1 |

It is easy to see that this game does not have a pure strategy Nash equilibrium. This shows that there is no guarantee that a pure strategy Nash equilibrium will exist. In Chapter 9, we will state sufficient conditions under which a given strategic form game is guaranteed to have a pure strategy Nash equilibrium. In the next chapter (chapter 6), we will show that this game has a mixed strategy Nash equilibrium.

Another game that does not have a pure strategy Nash equilibrium is the rock-paper-scissors game. This game also has a mixed strategy Nash equilibrium.

We will now study another example that does not have a pure strategy Nash equilibrium. It turns out that this game does not even have a mixed strategy Nash equilibrium.

### 2.8 Procurement Exchange Game

Recall the above example discussed in citeTARDOS07 and presented in Chapter 3. We will show that this game does not have a pure strategy Nash equilibrium. First, we explore whether the strategy profile $(1, s_2)$ is a Nash equilibrium for any $s_2 \in [0, 1]$. Note that

$$\begin{aligned} u_1(1, s_2) &= 2 \text{ if } s_2 = 1 \\ &= 1 \text{ if } s_2 < 1 \\ u_2(1, s_2) &= 1 \text{ if } s_2 = 1 \\ &= 2s_2 \text{ if } s_2 < 1. \end{aligned}$$

It is easy to observe that $u_2(1, s_2)$ has a value $2s_2$ for $0 \leq s_2 < 1$. Therefore $u_2(1, s_2)$ increases when $s_2$ increases from 0, until $s_2$ reaches 1 when it suddenly drops to 1. Thus it is clear that a profile of the form $(1, s_2)$ cannot be a Nash equilibrium for any $s_2 \in [0, 1]$. Similarly, no profile of the form $(s_1, 1)$ can be a Nash equilibrium for any $s_1 \in [0, 1]$.

We now explore if there exists any Nash equilibrium $(s_1^*, s_2^*)$, with $s_1^*, s_2^* \in [0, 1)$. There are two cases here.

- **Case 1**: If $s_1^* \leq \frac{1}{2}$, then the best response for player 2 would be to bid $s_2 = 1$ since that would fetch him the maximum payoff. However bidding $s_2 = 1$ is not an option here since the range of values for $s_2$ is $[0, 1)$.

- **Case 2**: If $s_1^* > \frac{1}{2}$, there are two cases: (1) $s_1^* \leq s_2^*$ (2) $s_1^* > s_2^*$. Suppose $s_1^* \leq s_2^*$. Then

$$\begin{aligned} u_1(s_1^*, s_2^*) &= 2s_1^* \\ u_2(s_1^*, s_2^*) &= s_2^*. \end{aligned}$$

  Choose $s_2$ such that $\frac{1}{2} < s_2 < s_1^*$. Then

$$\begin{aligned} u_2(s_1^*, s_2) &= 2s_2 \\ &> s_2^* \text{ since } 2s_2 > 1 \text{ and } s_2^* < 1 \\ &= u_2(s_1^*, s_2^*). \end{aligned}$$

  Thus we can improve upon $(s_1^*, s_2^*)$ and hence $(s_1^*, s_2^*)$ is not a Nash equilibrium.

  Now, suppose, $s_1^* > s_2^*$. Then

$$\begin{aligned} u_1(s_1^*, s_2^*) &= s_1^* \\ u_2(s_1^*, s_2^*) &= 2s_2^*. \end{aligned}$$

  Now let us choose $s_1$ such that $1 > s_1 > s_1^*$. Then

$$u_1(s_1, s_2^*) = s_1 > s_1^* = u_1(s_1^*, s_2^*).$$

  Thus we can always improve upon $(s_1^*, s_2^*)$. Therefore this game does not have a pure strategy Nash equilibrium.

# 3 Interpretations of Nash Equilibrium

Nash equilibrium is one of the most extensively discussed and debated topics in game theory. Many interpretations have been provided. Note that a Nash equilibrium is a profile of strategies of the $n$ players, such that each player's choice is the player's best response given that the rest of the players play their Nash equilibrium strategies. By deviating from a Nash equilibrium strategy, a player will not be better off given that the other players stick to their Nash equilibrium strategies. In the following, we provide several interpretations put forward by game theorists.

A popular interpretation views a Nash equilibrium as a *prescription*. An adviser or a consultant to the $n$ players would essentially prescribe a Nash equilibrium strategy profile to the players. If the adviser recommends strategies that do not constitute a Nash equilibrium, then some players would find that it would be better for them to do differently than advised. If the adviser prescribes strategies that do constitute a Nash equilibrium, then the players are not unhappy because playing the equilibrium strategy is best under the assumption that the other players will play their equilibrium strategies.

Thus a logical, rational, adviser would recommend a Nash equilibrium profile to the players. There is a caveat however: A Nash equilibrium is an insurance against only unilateral deviations (that is, only one player at a time deviating from the equilibrium strategy). Two or more players deviating might result in players improving their payoffs compared to their equilibrium payoffs. For example, in the prisoner's dilemma problem, $(C, C)$ is a Nash equilibrium. If both the players decide to deviate, then the resulting profile is $(NC, NC)$, which is better for both the players. Note that $(NC, NC)$ is not a Nash equilibrium.

Another popular interpretation of Nash equilibrium is that of *prediction*. If the players are rational and intelligent, then a Nash equilibrium is a good prediction for the game. For example, a systematic elimination of strongly dominated strategies will lead to a reduced form that will include a Nash equilibrium. Often, iterated elimination of strongly dominated strategies leads to a unique prediction which would be invariably a Nash equilibrium.

An appealing interpretation of Nash equilibrium is that of *self-enforcing agreement*. A Nash equilibrium can be viewed as an implicit or explicit agreement between the players. Once this agreement is reached, it does not need any external means of enforcement because it is in the self-interest of each player to follow this agreement if the others do. In a non-cooperative game, agreements cannot be enforced, hence, Nash equilibrium agreements are the only ones sustainable.

A natural, easily understood interpretation for Nash equilibrium has to do with *Evolution and Steady-State*. A Nash equilibrium is a potential stable point of a dynamic adjustment process in which players adjust their behavior to that of other players in the game, constantly searching for strategy choices that will give them the best results. This interpretation has been used to explain biological evolution. In this interpretation, Nash equilibrium is the outcome that results over time when a game is played repeatedly. A Nash equilibrium is like a stable social convention that people are happy to maintain forever.

Common knowledge was usually a standard assumption in determining conditions leading to a Nash equilibrium. More recently, it has been shown that the common knowledge assumption may not be required; instead, mutual knowledge is adequate. Suppose that each player is rational, knows his own payoff function, and knows the strategy choices of the others; then the strategy choices of the players will constitute a Nash equilibrium.

### 3.1 Existence of Multiple Nash Equilibria

We have seen several examples of strategic form games where multiple Nash equilibria exist. If a game has multiple Nash equilibria, then a fundamental question to ask is which of these would be implemented by the players? This question has been addressed by numerous game theorists, in particular, Thomas Schelling, who proposed the *focal point effect*. According to Schelling, anything that tends to focus the player's attention on one equilibrium may make them all expect it and hence fulfill it, like a self-fulfilling prophecy. Such a Nash equilibrium, which has some property that distinguishes it from all other equilibria is called a *focal equilibrium* or a *Schelling Point*.

As an example, consider the BOS game that we discussed in Example **??**. Recall the payoff matrix of this game:

| | 2 | |
|---|---|---|
| 1 | A | B |
| A | 2,1 | 0,0 |
| B | 0,0 | 1,2 |

Here $(A, A)$ and $(B, B)$ are both Nash equilibria. If there is a special interest (or hype) created about product A, then $(A, A)$ may become the focal equilibrium. On the other hand, if there is a marketing blitz on product B, then $(B, B)$ may become the focal equilibrium.

### To Probe Further

The material discussed in this chapter draws upon mainly from the the books by Myerson [2] and Osborne and Rubinstein [3]. The paper by Tardos and Vazirani [1] is a fine introduction to concepts in game theory. In fact, we have taken many examples from their paper.

The books by Osborne [4], Straffin [5], and Binmore [6] contain very interesting discussion on Nash equilibrium.

As is well known, the notion of Nash equilibrium was proposed by John Nash as part of his doctoral work which was published in [7, 8]. Holt and Roth [9] have recently published an insightful perspective on the notion of Nash equilibrium.

## 4 Problems

1. Show in a strategic form game that any strongly (weakly) (very weakly) dominant strategy equilibrium is also a pure strategy Nash equilibrium.

2. Assume that two merchants A and B have the option of selling any one of three products X, Y, and Z. If A decides to sell X and B decides to sell X, then A makes a profit of 100 and B makes a loss of 100. We represent this as $U_A(X, X) = 100$ and $U_B(X, X) = -100$. With this notation, following is the list of all utilities:

$$\begin{array}{ll}
U_A(X,X) = 100 & U_B(X,X) = -100 \\
U_A(X,Y) = 100 & U_B(X,Y) = 100 \\
U_A(X,Z) = 100 & U_B(X,Z) = -50 \\
U_A(Y,X) = -100 & U_B(Y,X) = -100 \\
U_A(Y,Y) = -100 & U_B(Y,Y) = 100 \\
U_A(Y,Z) = -50 & U_B(Y,Z) = -50 \\
U_A(Z,X) = 50 & U_B(Z,X) = -50 \\
U_A(Z,Y) = -50 & U_B(Z,Y) = 100 \\
U_A(Z,Z) = 100 & U_B(Z,Z) = 100
\end{array}$$

   Compute all the Nash equilibria for this problem. How many of these are dominant strategy equilibria?

3. Find the pure strategy Nash equilibria in the following games discussed in Chapter 3: coordination game; hawk-dove; cold war; ISP routing; pollution control; and cournot model.

4. Find the pure strategy Nash equilibrium of the following game.

| | X | Y | Z |
|---|---|---|---|
| A | 6, 6 | 8, 20 | 0, 8 |
| B | 10, 0 | 5, 5 | 2, 8 |
| C | 8, 0 | 20, 0 | 4, 4 |

5. Find the pure strategy Nash equilibria for the following two player game.

| | A | B | C | D |
|---|---|---|---|---|
| A | 5, 2 | 2, 6 | 1, 4 | 0, 4 |
| B | 0, 0 | 3, 2 | 2, 1 | 1, 1 |
| C | 7,0 | 2, 2 | 1, 5 | 5, 1 |
| D | 9, 5 | 1, 3 | 0, 2 | 4, 8 |

6. Find the pure strategy Nash equilibrium of the following game.

| | X | Y | Z |
|---|---|---|---|
| A | 6, 6 | 8, 20 | 0, 8 |
| B | 10, 0 | 5, 5 | 2, 8 |
| C | 8, 0 | 20, 0 | 4, 4 |

7. An $m \times m$ matrix is called a latin square if each row and each column is a permutation of $(1, \ldots, m)$. Compute pure strategy Nash equilibria, if they exist, of a two person game for which a latin square is the payoff matrix. Generalize the result.

8. In the tragedy of the commons example with $n = 5$ and no pollution tax, all profiles have been shown to be pure strategy Nash equilibria. Are these also very weakly dominant strategy equilibria?

9. In the Braess paradox game without the link from A to B, we have derived certain Nash equilibria (namely strategy profiles where 500 vehicles follow route $A$ and the other 500 vehicles follow route $B$). Are these the only Nash equilibria. Also, in the extended game with a link from A to B, are there equilibria other than the profile corresponding to all vehicles following the route $AB$?

10. Give examples of two player pure strategy games for the following situations
    (a) The game has a unique Nash equilibrium which is not a weakly dominant strategy equilibrium
    (b) The game has a unique Nash equilibrium which is a weakly dominant strategy equilibrium but not a strongly dominant strategy equilibrium
    (c) The game has one strongly dominant or one weakly dominant strategy equilibrium and a second one which is only a Nash equilibrium

11. *First Price Auction.* Assume two bidders with valuations $v_1$ and $v_2$ for an object. Their bids are in multiples of some unit (that is, discrete). The bidden with higher bid wins the auction and pays the amount that he has bid. If both bid the same amount, one of them gets the object with equal probability $\frac{1}{2}$. In this game, compute a pure strategy Nash equilibrium of the game.

12. Compute a Nash equilibrium for the two person game with

$$S_1 = \{0, 1\} \qquad S_2 = \{3, 4\}$$

$$u_1(x, y) = -u_2(x, y) = |x - y| \quad \forall (x, y) \in [0, 1] \times [3, 4]$$

13. Consider the game $(N, (S_i), (u_i))$ when $N = \{1, \ldots, n\}$ and $S_i = \{1, \ldots, 1n\} \quad \forall i \in N$.

$$\begin{aligned} u_1(s_i, \ldots, s_n) &= a_{ik} > 0 \qquad \text{if } s_1 = \cdots = s_n = k \\ &= 0 \qquad\qquad \text{otherwise} \end{aligned}$$

Show that the only pure strategy profiles which are not equilibrium points are those with exactly $(n-1)$ of the $s_i$ equal.

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