Game Theory

Lecture Notes By

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July 2012

Chapter 4: Dominant Strategy Equilibria

Note: *This is a only a draft version, so there could be flaws. If you find any errors, please do send email to* `hari@csa.iisc.ernet.in`*. A more thorough version would be available soon in this space.*

In this chapter, we start analyzing strategic form games by defining the notion of dominant strategies and dominant strategy equilibria.

There are three notions of dominance that are aptly called strong dominance, weak dominance, and very weak dominance. First we introduce strong dominance and weak dominance and provide several examples. We introduce very weak dominance towards the end of the chapter.

1 Strong Dominance

Strongly Dominated Strategy

Given a game $\Gamma = \langle N, (S_i), (u_i) \rangle$, a strategy $s_i \in S_i$ is said to be *strongly dominated* if there exists another strategy $s_i' \in S_i$ such that

$$u_i(s_i', s_{-i}) > u_i(s_i, s_{-i}) \;\; \forall s_{-i} \in S_{-i}$$

In such a case, we say strategy s_i' *strongly dominates* strategy s_i.

Strongly Dominant Strategy

A strategy $s_i^* \in S_i$ is said to be a strongly dominant strategy for player i if it strongly dominates every other strategy $s_i \in S_i$. That is, $\forall s_i \neq s_i^*$,

$$u_i(s_i^*, s_{-i}) > u_i(s_i, s_{-i}) \;\; \forall s_{-i} \in S_{-i}$$

Strongly Dominant Strategy Equilibrium

A profile of strategies $(s_1^*, s_2^*, \ldots, s_n^*)$ is called a strongly dominant strategy equilibrium of the game $\Gamma = \langle N, (S_i), (u_i) \rangle$ if $\forall i = 1, 2, \ldots, n$, the strategy s_i^* is a strongly dominating strategy for player i.

Example: Prisoner's Dilemma

Recall the prisoner's dilemma problem where $N = \{1, 2\}$ and $S_1 = S_2 = \{C, NC\}$ and the payoff matrix is given by:

	2	
1	NC	C
NC	$-2, -2$	$-10, -1$
C	$-1, -10$	$-5, -5$

Note that the strategy NC is strongly dominated by strategy C for player 1 since

$$\begin{aligned} u_1(C, NC) &> u_1(NC, NC) \\ u_1(C, C) &> u_1(NC, C) \end{aligned}$$

Similarly, the strategy NC is strongly dominated by strategy C for player 2 since

$$\begin{aligned} u_2(NC, C) &> u_2(NC, NC) \\ u_2(C, C) &> u_2(C, NC) \end{aligned}$$

Thus C is a strongly dominant strategy for player 1 and also for player 2. Therefore (C, C) is a strongly dominant strategy equilibrium for this game.

Note that if a (rational) player has a strongly dominating strategy then we should expect the player to choose that strategy. On the other hand, if a player has a strongly dominated strategy, then we should expect the player not to play it.

2 Weak Dominance

Given a game $\Gamma =< N, (S_i), (u_i) >$, a strategy $s_i \in S_i$ is said to be *weakly dominated* by a strategy $s'_i \in S_i$ for player i if for all $s_i \in S_i$,

$$u_i(s'_i, s_{-i}) \geq u_i(s_i, s_{-i}) \;\; \forall s_{-i} \in S_{-i} \;\text{ and }\; u_i(s'_i, s_{-i}) > u_i(s_i, s_{-i}) \;\text{ for some }\; s_{-i} \in S_{-i}$$

Note that strict inequality is satisfied for at least one s_{-i}. The strategy s'_i is said to *weakly dominate* strategy s_i.

Weakly Dominant Strategy

A strategy s^*_i is said to be a weakly dominant strategy for player i if it weakly dominates every other strategy $s_i \in S_i$.

Weakly Dominant Strategy Equilibrium

Given a game $\Gamma = \langle N, (S_i), (u_i) \rangle$, a strategy profile $(s^*_1, \ldots, s^*_n)$ is called a weakly dominant strategy equilibrium if for $i = 1, \ldots, n$, the strategy s^*_i is a weakly dominant strategy for player i.

Example: Modified Prisoner's Dilemma

Consider the following payoff matrix of a slightly modified version of the prisoner's dilemma problem.

	2	
1	NC	C
NC	$-2, -2$	$-10, -2$
C	$-2, -10$	$-5, -5$

It is easy to note that C is a weakly dominant strategy for player 1 and also for player 2. Therefore the strategy profile (C, C) is a weakly dominant strategy equilibrium.

3 Examples

3.1 Example: Tragedy of the Commons

Recall that

$$N = \{1, 2, \ldots, n\} \text{ is a set of farmers}$$
$$S_1 = S_2 = \cdots = S_n = \{0, 1\}$$

1 corresponds to keeping a sheep, and 0 corresponds to not keeping a sheep. Keeping a sheep gives a benefit of 1. However, when a sheep is kept, damage to the environment is 5. This damage is equally shared by all the farmers.

For $i = 1, 2, \ldots, n$

$$u_i(s_1, \ldots, s_n) = s_i - \frac{5}{n} \sum_{j=1}^{n} s_j = \left(\frac{n-5}{n} \right) s_i - \frac{5}{n} \sum_{j \neq i} s_j$$

Case 1: $n < 5$.

Given any $s_{-i} \in S_{-i}$,

$$u_i(0, s_{-i}) = -\frac{5}{n} \sum_{j \neq i} s_j$$
$$u_i(1, s_{-i}) = \left(\frac{n-5}{n} \right) - \frac{5}{n} \sum_{j \neq i} s_j$$

since $n < 5$, $\left(\frac{n-5}{n}\right) < 0$, and therefore, $u_i(0, s_{-i}) > u_i(1, s_{-i}) \quad \forall s_{-i} \in S_{-i}$. This implies that

$$B_i(s_{-i}) = \{0\} \quad \forall i \in N$$

This means $(0, 0, \ldots, 0)$ is a strongly dominant strategy equilibrium. That is, there is no incentive for any farmer to keep a sheep.

Case 2: $n = 5$.

Here

$$u_i(0, s_{-i}) = -\frac{5}{n} \sum_{j \neq i} s_j$$
$$u_i(1, s_{-i}) = -\frac{5}{n} \sum_{j \neq i} s_j$$

Thus

$$u_i(0, s_{-i}) = u_i(1, s_{-i}), \quad \forall s_{-i} \in S_{-i}$$

It is easy to see that none of the strategies here is a weakly dominant strategy or a strongly dominant strategy.

Case 3: $n > 5$. Here

$$u_1(0, s_{-i}) = -\frac{5}{n}\sum_{j \neq i} s_j$$

$$u_i(1, s_{-i}) = \frac{n-5}{n} - \frac{5}{n}\sum_{j \neq i} s_j$$

Thus

$$u_i(1, s_{-i}) > u_i(0, s_{-i}) \quad \forall s_{-i} \in S_{-i}$$

Hence $(1, 1, \ldots, 1)$ is a strongly dominant strategy equilibrium. Thus if $n > 5$, it is good for all the farmers to keep a sheep.

Now if the Government decides to impose a pollution tax of 5 units for each sheep kept, we have

$$u_i(s_1, \ldots, s_n) = s_i - 5s_i - \frac{5}{n}\sum_{j=1}^{n} s_j = -4s_i - \frac{5}{n}s_i - \frac{5}{n}\sum_{j \neq i} s_j$$

Here

$$u_i(0, s_{-i}) = -\frac{5}{n}\sum_{j \neq i} s_j$$

$$u_i(1, s_{-i}) = -4 - \frac{5}{n} - \frac{5}{n}\sum_{j \neq i} s_j$$

This means whatever the value of n, $(0, 0, \ldots, 0)$ is a strongly dominant strategy equilibrium. This is bad news for the farmers.

3.2 Example: Braess Paradox Game

Recall the Braess paradox game with additional capacity introduced from A to B. In this game, it can be shown, for every player i, that

$$u_i(AB, s_{-i}) > u_i(A, s_{-i}) \ \forall \ s_{-i} \ \in \ S_{-i}$$

$$u_i(AB, s_{-i}) > u_i(B, s_{-i}) \ \forall \ s_{-i} \ \in \ S_{-i}$$

This shows that $(AB, AB, \ldots, AB)$ is a strongly dominant strategy equilibrium. Note that the above equilibrium profile leads to a total delay of 40 minutes. On the other hand, if 500 vehicles use the strategy A and and the other 500 vehicles use the strategy B, the total delay for each vehicle is only 35 minutes. The paradox here is that the introduction of an additional link forces the strategy of AB on every vehicle (AB being a strongly dominant strategy for each vehicle) thereby leading to a delay that is higher than what it would be for a non-equilibrium profile.

3.3 Example: Second Price Sealed Bid Auction with Complete Information

Consider the second price sealed bid auction for selling a single indivisible item discussed Example ... Let $b_1, b_2, \ldots, b_n$ be the bids (strategies) and we shall denote a bid profile (strategy profile) by $b = (b_1, b_2, \ldots, b_n)$. Assume that $v_i, b_i \in (0, \infty)$ for $i = 1, 2, \ldots, n$. Recall that the item is awarded to the bidder who has the lowest index among all the highest bidders. Recall the allocation function:

$$\begin{aligned} y_i(b_1, \ldots, b_n) \quad &= 1 \text{ if} \quad b_i > b_j \text{ for } j = 1, 2, \ldots, i-1 \text{ and} \\ & \qquad\quad\; b_i \geq b_j \text{ for } j = i+1, \ldots, n \\ &= 0 \text{ else.} \end{aligned}$$

The payoff for each bidder is given by:

$$u_i(b_1, \ldots, b_n) = y_i(b_1, \ldots, b_n)(v_i - t_i(b_1, \ldots, b_n))$$

where $t_i(b_1, \ldots, b_n)$ is the amount paid by the winning bidder. Being second price auction, the winner pays only the next highest bid. We now show that the strategy profile $(b_1, \ldots, b_n) = (v_1, \ldots, v_n)$ is a weakly dominant strategy equilibrium for this game.

Proof: Consider bidder 1. His value is v_1 and bid is b_1. The other bidders have bids $b_2, \ldots, b_n$ and valuations $v_2, \ldots, v_n$. We consider the following cases.

> **Case 1**: $v_1 \geq \max(b_2, \ldots, b_n)$. There are two sub-cases here: $b_1 \geq \max(b_2, \ldots, b_n)$ and $b_1 < \max(b_2, \ldots, b_n)$.
>
> **Case 2**: $v_1 < \max(b_2, \ldots, b_n)$. There are two sub-cases here: $b_1 \geq \max(b_2, \ldots, b_n)$ and $b_1 < \max(b_2, \ldots, b_n)$.

We analyze these cases separately below.

Case 1: $v_1 \geq \max(b_2, \ldots, b_n)$.

We look at the following scenarios.

- Let $b_1 \geq \max(b_2, \ldots, b_n)$. This implies that bidder 1 is the winner, which implies that $u_1 = v_1 - \max(b_2, \ldots, b_n) \geq 0$.
- Let $b_1 < \max(b_2, \ldots, b_n)$. This means that bidder 1 is not the winner, which in turn means $u_1 = 0$.
- Let $b_1 = v_1$, then since $v_1 \geq \max(b_2, \ldots, b_n)$, we have $u_1 = v_1 - \max(b_2, \ldots, b_n)$.

Therefore, if $b_1 = v_1$, the utility u_1 is greater than or equal to the maximum utility obtainable. Thus, whatever the values of $b_2, \ldots, b_n$, it is a best response for player 1 to bid v_1. Thus $b_1 = v_1$ is a weakly dominant strategy for a bidder 1.

Case 2: $v_1 < \max(b_2, \ldots, b_n)$.

As before, we look at the following scenarios.

- Let $b_1 \geq \max(b_2, \ldots, b_n)$. This implies that bidder 1 is the winner and the payoff is given by:

$$u_1 = v_1 - \max(b_2, \ldots, b_n) \quad < 0.$$

- Let $b_1 < \max(b_2, \ldots, b_n)$. This means bidder 1 is not the winner. Therefore $u_1 = 0$.
- If $b_1 = v_1$, then bidder 1 is not the winner and therefore $u_1 = 0$.

From the above analysis, it is clear that $b_1 = v_1$ is a best response strategy for player 1 in Case 2 also. Combining our analysis of Case 1 and Case 2, we have that

$$u_1(v_1, b_2, \ldots, b_n) \geq u_1(\hat{b_1}, b_2, \ldots, b_n) \;\; \forall \, \hat{b_1} \in S_1 \quad \forall \, b_2 \in S_2, \;\; \ldots, \;\; b_n \in S_n$$

Also, we can show (and this is left as an exercise) that, for any $b_1' \neq v_1$, we can always find $b_2 \in S_2, b_3 \in S_3, \ldots, b_n \in S_n$, such that

$$u_1(v_1, b_2, \ldots b_n) > u_1(b_1', b_2, \ldots, b_n).$$

Thus $b_1 = v_1$ is a weakly dominant strategy for a bidder 1. Using almost similar arguments, we can show that $b_i = v_i$ is a weakly dominant strategy for bidder i where $i = 2, 3, \ldots, n$. Therefore $(v_1, \ldots, v_n)$ is a weakly dominant strategy equilibrium.

4 Very Weak Dominance

Given a game $\Gamma =< N, (S_i), (u_i) >$, a strategy $s_i \in S_i$ is said to be *very weakly dominated* by a strategy $s_i' \in S_i$ for player i if for all $s_i \in S_i$,

$$u_i(s_i', s_{-i}) \geq u_i(s_i, s_{-i}) \;\; \forall s_{-i} \in S_{-i}$$

Note that strict inequality need not be satisfied for any s_{-i} as in the case of weak dominance. The strategy s_i' is said to *very weakly dominate* strategy s_i.

Very Weakly Dominant Strategy

A strategy s_i^* is said to be a very weakly dominant strategy for player i if it weakly dominates every other strategy $s_i \in S_i$.

Example: Modified Prisoner's Dilemma - Version 2

Consider the following payoff matrix of another modified version of the prisoner's dilemma problem.

	2	
1	NC	C
NC	$-2, -2$	$-5, -2$
C	$-2, -10$	$-5, -5$

It is easy to note that C is a very weakly dominant strategy for player 1 while C is a weakly dominant strategy for player 2. The strategy profile (C, C) now consists of a very weakly dominant strategy (for player 1) and a weakly dominant strategy (for player 2).

We often use the notion of very weak dominance in mechanism design settings (Part 2 of the book).

5 To Probe Further

The material discussed in this chapter is mainly taken from the books by Myerson [1]; Mascolell, Whinston, and Green [2]; Shoham and Leyton-Brown [3].

6 Problems

1. Consider the following instance of the prisoners' dilemma problem.

	2	
1	NC	C
NC	$-4, -4$	$-2, -x$
C	$-x, -2$	$-x, -x$

   Find the values of x for which:

   (a) the profile (C,C) is a strongly dominant strategy equilibrium.

   (b) the profile (C,C) is a weakly dominant strategy equilibrium but not a strongly dominant strategy equilibrium.

   (c) the profile (C,C) is a not even a weakly dominant strategy equilibrium.

   In each case, say whether it is possible to find such an x. Justify your answer in each case.

2. *First Price Auction.* Assume two bidders with valuations v_1 and v_2 for an object. Their bids are in multiples of some unit (that is, discrete). The bidden with higher bid wins the auction and pays the amount that he has bid. If both bid the same amount, one of them gets the object with equal probability $\frac{1}{2}$. In this game,

   (a) Are any strategies strongly dominated?

   (b) Are any strategies weakly dominated?

3. There are n departments in I.I.Sc. Each department can try to convince the Director to get a certain budget. If h_i is the number of hours of work put in by a department to make the proposal and $c_i = w_i h_i^2$ is cost of this effort to the department, where w_i is a constant. When the effort levels of the departments are $(h_1, h_2, \ldots, h_n)$, the total budget that gets allocated to all the departments is:

$$\alpha \sum_{i=1}^{n} h_i + \beta \prod_{i=1}^{n} h_i$$

   where α and β are constants. Consider a game where the departments simultaneously and independently decide how many hours to spend on this effort. Show that a strictly dominant strategy equilibrium exists iff $\beta = 0$. Compute this equilibrium.

4. Complete the proof that reporting true values in Vickrey auction is a weakly dominant strategy equilibrium.

5. In Case 2 ($n = 5$) of the tragedy of the commons game, investigate whether any of the strategies is a very weakly dominant strategy.

6. Compute strongly or weakly dominant strategy equilibria of the Braess paradox game when the number 25 is replaced by the number 20.

References

[1] Roger B. Myerson. *Game Theory: Analysis of Conflict*. Harvard University Press, Cambridge, Massachusetts, USA, 1997.

[2] Andreu Mas-Colell, Michael D. Whinston, and Jerry R. Green. *Micoreconomic Theory*. Oxford University Press, 1995.

[3] Yoam Shoham and Kevin Leyton-Brown. *Multiagent systems: Algorithmic, Game-Theoretic, and Logical Foundations*. Cambridge University Press, New York, USA, 2009, 2009.