Game Theory

Lecture Notes By

Y. Narahari

Department of Computer Science and Automation

Indian Institute of Science

Bangalore, India

July 2012

Chapter 3. Strategic Form Games: Examples

Note: *This is a only a draft version, so there could be flaws. If you find any errors, please do send email to* `hari@csa.iisc.ernet.in`. *A more thorough version would be available soon in this space.*

In this chapter, we provide a variety of examples to help improve our intuitive understanding of strategic form games. We present examples of finite games as well as infinite games.

1 Introduction

We have seen in Chapter 1 that a strategic form game Γ is a tuple $\Gamma = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$ where $N = \{1, 2, \ldots, n\}$ is a finite set of players; $S_1, S_2, \ldots, S_n$ are the strategy sets of the players; and $u_i : S_1 \times S_2 \times \cdots \times S_n \to \mathbb{R}$ for $i = 1, 2, \ldots, n$ are utility functions.

The idea behind the strategic form representation is that a player's decision problem is to simply choose a strategy that will counter best the strategies adopted by the other players. Each player is faced with this problem and the players can be thought of as simultaneously choosing their strategies from the respective sets $S_1, S_2, \ldots, S_n$. An intuitive view the play of a strategic game is as follows: each player simultaneously writes down a chosen strategy on a piece of paper and hands it over to a referee who then computes the outcome and the utilities.

We have seen in the previous chapter that any given extensive form game has an equivalent strategic form game. An important question is: does the equivalent strategic form representation summarize all of the strategically relevant information? This question has been debated quite intensely by game theorists. In games where all the players statically choose their actions at the same time (without observing the choices of other players), the strategic form and the extensive form have the same representational power.

1.1 Interpretations of Strategic Form Games

Osborne and Rubinstein [1] provide the following two interpretations for the strategic form representation.

Interpretation 1

A strategic game is a model of an event that occurs only once. Each player knows the details of the game and the fact that all players are rational. The players choose their strategies simultaneously and independently. Each player is unaware of the choices being made by the other players.

Interpretation 2

In this interpretation, a player can form an expectation of the other players' behavior on the basis of information about the way that the game or a similar game was played in the past. A strategic form game models a sequence of plays of the game under the condition that there is no strategic link between the plays of the game. That is, a player who plays the game many times should only worry about his own instantaneous payoff and ignore the effects of his current action on the future behavior of the other players. A class of games called *repeated games* are relevant if there is a strategic link between plays of a game.

2 Matching Pennies with Simultaneous Moves

We have already studied this game in Chapter 1. Recall that in this game, two players 1 and 2 put down their respective rupee coins, heads up or tails up. If both the coins match (both heads or both tails), then player 1 pays 1 Rupee to player 2. Otherwise, player 2 pays 1 Rupee to player 1. Let us say A denotes *heads* and B denotes *tails*. It is easy to see that:

$$\begin{aligned} N &= \{1,2\} \\ S_1 &= S_2 = \{A,B\} \\ S &= S_1 \times S_2 = \{(A,A),(A,B),(B,A),(B,B)\} \end{aligned}$$

The payoff matrix is given by

	2	
1	H	T
H	$1,-1$	$-1,1$
T	$-1,1$	$1,-1$

This is a perhaps the simplest example of a two player game. This belongs to a class of games called two player zero sum games (so called because the sum of utilities in every outcome is equal to zero).

To understand why a simple game such as the above is important to study, let us consider the following situation. There are two companies, call them 1 and 2. Each company is capable of producing two products A and B, but at any given time, a company can only produce one product, due to high setup and switchover costs. Company 1 is known to produce superior quality products but company 2 scores over company 1 in terms of marketing and advertising. If both the companies produce the same product (A or B), it turns out that company 1 makes all the profits and company 2 loses out, because of the superior quality of products produced by company 1. This is reflected in our model with a payoff of $+1$ for company 1 and a payoff of -1 for company 2, corresponding to the strategy profiles (A,A) and (B,B).

On the other hand, if one company produces product A and the other company produces product B, it turns out that because of aggressive marketing by company 2 in differentiating the product offerings A and B, company 2 captures all the market, resulting in a payoff of +1 for company 2 and a payoff of −1 for company 1.

The two companies have to simultaneously decide (each one does not know the decision of the other) which product to produce. This is the strategic decision facing the two companies. This situation is captured by a strategic form game $\Gamma = \langle N, S_1, S_2, u_1, u_2 \rangle$, where $N = \{1, 2\}$; $S_1 = S_2 = \{A, B\}$, and the utility functions are as described in the above table.

3 Rock-Paper-Scissors Game

This is an example of another two player zero sum game where each player has three strategies, called *rock*, *paper*, and *scissors*. Actually, this is a popular hand game played by two people. Another name for this game is *roshambo*. Two players simultaneously display one of three symbols: a *rock*, a *paper*, or *scissors*. The *rock* symbol beats *scissors* symbol; *scissors* symbol beats *paper* symbol; *paper* symbol beats *rock* symbol (symbolically, scissors cannot cut a rock; scissors can cut paper; and paper can cover rock). The payoff matrix for this game is given as follows.

	3		
1	Rock	Paper	Scissors
Rock	$0, 0$	$-1, 1$	$1, -1$
Paper	$1, -1$	$0, 0$	$-1, 1$
Scissors	$-1, 1$	$1, -1$	$0, 0$

4 BOS Game (Bach or Stravinsky)

This game is named after the famous musicians Bach or Stravinsky. This is also called the *Battle of Sexes* game. This is a game where the players want to cooperate with each other, however they have a disagreement about which of the outcomes is better. Let us say two players 1 and 2 wish to go out together to an event A or to an alternative event B. Player 1 prefers to go to event A and player 2 prefers to go to event B. The payoff matrix is as shown.

	2	
1	A	B
A	2,1	0,0
B	0,0	1,2

Clearly, this game captures a situation where the players want to coordinate but they have conflicting interests. The outcomes (A,B) and (B,A) are unfavorable to either player. The choice is essentially between (A,A) and (B,B). Recalling the company analogy, suppose we have two companies, 1 and 2. Each company can produce two products A and B, but at any given time, a company can only produce one type of product, The products A and B are competing products. Product A is a niche product of company 1 while product B is a niche product of company 2. If both the companies produce product A, the consumers are compelled to buy product A and would naturally prefer to buy it from company 1 rather than from 2. Assume that company 1 will capture two thirds of the market. We will reflect this fact by saying that the payoff to company 1 is twice as much as for company 2. If both the companies produce product B, the reverse situation will prevail and company 2 will make twice as much payoff as company 1.

On the other hand, if the two companies decide to produce different products, then the market gets segmented and each company tries to outwit the other through increased spending on advertising. In fact, their competition may actually benefit a third company and effectively, neither company 1 nor company 2 makes any payoff. The above table depicts the payoff structure for this game.

5 A Coordination Game

This game is similar to the BOS game but the two players now have preference for the same option, namely event A. The payoff matrix is as shown. In this case, the outcomes (A, A) and (B, B) in that order are preferred.

	2	
1	A	B
A	(2,2)	(0,0)
B	(0,0)	(1,1)

Continuing our analogy of companies, the above game corresponds to a situation where, when the two companies produce the same product, they have equal market share. This market share is twice as much for product A than for product B. On the other hand, if the two companies produce different products, a third company may capture all the market share leaving nothing for companies 1 and 2.

One more analogy would be the following. There are two students 1 and 2 who are friends and would derive some utility by spending time together. To spend time together, they have two options: option A (college) and option B (movie). If both of them are in college (and study together), they get a payoff of 2 each. If both of them go to a movie, each gets a payoff of 1 each. If one remains in college and the other goes to movie, the payoff is 0 for each. The two friends have to make the decision simultaneously independent of each other.

Since the payoffs are the same for both the players in all outcomes, such games are also called *common payoff games*.

6 Hawk-Dove (Chicken)

There are two players who are fighting over a property or a prey or a disputed item. Each player can behave like a hawk or a dove.

	2	
1	Hawk	Dove
Hawk	3,3	4,1
Dove	1,4	0,0

The best outcome happens for a player who acts as a hawk while the other acts like a dove. When both are hawks, the outcome is least desirable since nobody wins. When both are hawks, the outcome is better than if both were doves. It would be interesting to analyze what the most likely outcome for this game would be.

7 Cold War

Two countries A and B have to decide whether they should spend more on defense or on healthcare. Consider the following payoff matrix.

	B	
A	Healthcare	Defense
Healthcare	10,10	-10, 20
Defense	20, -10	0,0

The first observation is that each player finds that "Defense" is the best response whatever the other player plays. The second observation is that healthcare is good only if the other player plays healthcare. We will see in a future chapter that the predicted outcome for the game is (Defense, Defense) but (healthcare, healthcare) is socially optimal (Pareto efficient). If the players can cooperate, the outcome will be socially optimal.

8 Prisoner's Dilemma

This is one of the most extensively studied problems in game theory, with many interesting interpretations in a wide variety of situations. Two individuals are arrested for allegedly committing a crime and are lodged in separate prisons. The district attorney interrogates them separately. The attorney privately tells each prisoner that if he is the only one to confess, he will get a light sentence of 1 year in jail while the other would be sentenced to 10 years in jail. If both players confess, they would get 5 years each in jail. If neither confesses, then each would get 2 years in jail. The attorney also informs each prisoner what has been told to the other prisoner. Thus the payoff matrix is common knowledge.

	2	
1	NC	C
NC	-2, -2	-10, -1
C	-1, -10	-5, -5

How would the prisoners behave in such a situation? They would like to play a strategy that is best response to a (best) response strategy that the other player may adopt, the latter player also would like to play a best response to the other player's best response, and so on. First observe that C is each player's best strategy regardless of what the other player plays:

$$u_1(C,C) > u_1(NC,C); \quad u_1(C,NC) > u_1(NC,NC)$$

$$u_2(C,C) > u_2(C,NC); \quad u_2(NC,C) > u_2(NC,NC)$$

Thus (C,C) is a natural prediction for this game. However, the outcome (NC, NC) is the best outcome jointly for the players. Prisoner's Dilemma is a classic example of a game where rational, intelligent behavior does not lead to a socially optimal result. Also, each prisoner has a negative effect or externality on the other. When a prisoner moves away from (NC, NC) to reduce his jail term by 1 year, the jail term of the other player increases by 8 years.

9 Company's Dilemma: An Example of Prisoner's Dilemma

This game develops a company game situation on the lines of the prisoner's dilemma problem. Here again, we have two companies 1 and 2, each of which can produce two competing products A and B, but only one at a time. The companies are known for product A than for product B. Environmentalists have launched a negative campaign on product A branding it as non-eco friendly.

If both the companies produce product A, then, in spite of the negative campaign, their payoff is quite high since product A happens to be a niche product of both the companies. On the other hand, if both the companies produce product B, they still make some profit, but not as much as they would if they both produced product A.

On the other hand, if one company produces product A and the other company produces product B, then because of the negative campaign about product A, the company producing product A makes zero payoff while the other company captures all the market and makes a high payoff.

The table below depicts the payoff structure for this game.

	2	
1	A	B
A	6,6	0,8
B	8,0	3,3

10 ISP Routing Dilemma: Another Example of Prisoner's Dilemma

This is based on an example discussed in the paper by Tardos and Vazirani [2]. Consider two ISPs (Internet Source Providers) A and B. See Figure 1. There are two kinds of traffic.

- The first type of traffic emanates from node a_1 belonging to ISP A and is targeted towards node b_1 belonging to ISP B.
- The second type of traffic originates at node b_2 belonging to ISP B and destined towards node a_2 belonging to ISP A.

C and NC are two transit points or peering points or hand-off points or peering points or hand-off point S between the two ISPs. That is the traffic between the two ISP networks can only flow either through C or NC.

The two source points, a_1 and b_2, are closer to the transit point C while the destination points a_2 and b_1 are closer to the transit point NC. ISP A faces the problem of routing the traffic from a_1 and has two strategies: (1) first route the traffic from a_1 to C and let ISP B takes care of the traffic from b_2 to b_1. (call this strategy C) (2) transmit the traffic from a_1 internally to the transit point NC and then let ISP B route it from NC to b_1 (call this strategy NC). The strategy C minimizes the work for ISP A but this strategy makes ISP B work hard. On the other hand, strategy NC involves more work for ISP A and less work for ISP B. Similarly, ISP B also has two strategies C and NC for routing the traffic from b_2 to a_2.

The C and NC are intuitively similar to the strategies C and NC in the prisoner's dilemma problem.

Let the cost of routing traffic from a_1 to C, b_2 to C, NC to a_1, and NC to a_2 be 1 unit each. On the other hand, let the cost of routing traffic from a_1 to NC and b_2 to NC be 3 units each. Then the following matrix provides a reasonable representation for the payoff matrix of this game.

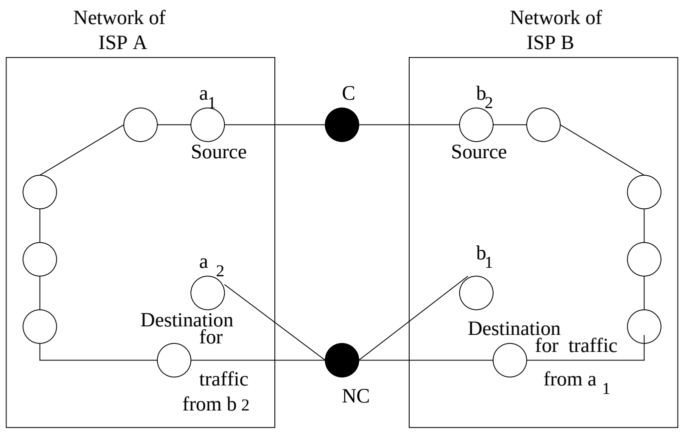

Figure 1: A network with two ISPs

	B	
A	NC	C
NC	$-4, 4$	$-6, -2$
C	$-2, -6$	$-5, -5$

11 Pollution Control Game

This is again based on an example discussed in the paper by Tardos and Vazirani [2]. First we discuss the case of two players and show its similarity to the prisoner's dilemma problem. Next we extend it to multiple players.

Consider two neighboring countries 1 and 2 which are grappling with the problem of pollution. Each country has to decide whether or not to pass a legislation for controlling pollution. The two strategies are 0 and 1, where "0" stands for undertaking pollution control measures and "1" indicates not undertaking pollution control measure.

Assume that the cost of pollution control is 5 units while the cost of mitigating pollution effects is 1 unit. If a country decides to play the strategy "0", then the cost incurred for this country is 5 units. On the other hand, if the country decides to play "1", then it incurs a cost of 1 unit for mitigating the effects of pollution it causes. In addition, even the other country also incurs a cost of 1 unit for mitigating the effects of pollution caused by the first country.

Under this situation, note that the strategy profile (0, 0) will mean that both countries will spend 5 units on pollution control and remain pollution free. The strategy profile (1, 1) will mean that both countries will pollute and hence incur a cost of 2 units each (1 unit for pollution caused by the other country). The payoff matrix will be as shown.

	2	
1	0	1
0	$-5, -5$	$-6, -1$
1	$-1, -6$	$-2, -2$

We can generalize this to the case of n countries as follows.

$$\begin{aligned} N &= \{1, 2, \ldots, n\} \\ S_1 &= S_2 = \ldots, = S_n = \{0, 1\} \\ u_i(s_i, s_{-i}) &= -5(1 - s_i) - \sum_{j \in N} s_j \end{aligned}$$

Note that

$$u_i(0, s_{-i}) = -5 - \sum_{j \neq i} s_j$$

while

$$u_i(1, s_{-i}) = -1 - \sum_{j \neq i} s_j$$

Thus

$$u_i(1, s_{-i}) > u_i(0, s_{-i}) \;\; \forall s_{-i} \in S_{-i}$$

12 Tragedy of the Commons

The Tragedy of the Commons represents a type of social paradox or social tragedy. The problem involves a conflict over resources between individual interests and social interests. A village has n farmers $N = \{1, 2, \ldots, n\}$. Each farmer has the option of keeping a sheep or not. $S_1 = S_2 = \cdots = S_n = \{0, 1\}$. The utility from keeping a sheep (that arises because of milk, wool, etc.) is equal to 1 unit. The village has a limited stretch of grassland and when a sheep grazes on this, there is a damage to the environment, equal to 5 units. This damage to the environment is to be shared equally by the farmers.

Let s_i be the strategy of each farmer. Then $s_i = 0, 1$. The payoff to farmer i is given by:

$$\begin{aligned} &= u_i(s_1, s_2, \ldots s_i, \ldots, s_n) \\ &= s_i - \left[\frac{5(s_1 + \cdots + s_n)}{n}\right] \end{aligned}$$

For the case $n = 2$, the payoff matrix would be:

	2	
1	0	1
0	0,0	-2.5,-1.5
1	-1.5, -2.5	-4, -4

We make the following observations:

- If $n > 5$, keeping a sheep would add more utility to a farmer from milk/wool than subtract utility from him due to environmental damage. If $n < 5$, then the farmer gets less utility from keeping than from not keeping a sheep. If $n = 5$, the farmer has equal benefit and loss.
- If $n > 5$, then every farmer would like to keep a sheep. How about $n \leq 5$?

- If the Government now imposes a pollution tax of 5 units for every sheep kept, the payoff becomes:
$$u_1(s_1, \ldots, s_n) = s_i - 5s_i - \frac{5(s_1 + \cdots + s_n)}{n}$$
Now every farmer will prefer not to keep a sheep.

13 Bandwidth Sharing Game

This problem is based on an example presented by Tardos and Vazirani [2]. There is a shared communication channel of maximum capacity 1. There are n users of this channel, and user i wishes to send x_i units of flow, where $x_i \in [0, 1]$. We have

$$\begin{aligned} N &= \{1, 2, \ldots, n\} \\ S_1 &= S_2 = \ldots = S_n = [0, 1]. \end{aligned}$$

If $\sum_{i \in N} x_i \geq 1$, then the transmission cannot happen since the capacity is exceeded, and the payoff to each player is zero. If $\sum_{i \in N} x_i < 1$, then assume that the following is the payoff to user i:

$$u_i = x_i(1 - \sum_{j \in N} x_j)$$

The above expression models the fact that the payoff to a player is proportional to the flow sent by the player but is negatively impacted by the total flow. The second term captures the fact that the quality of transmission deteriorates with the total bandwidth used. The above defines an n-player infinite game. We will show in Chapter 5 that an equilibrium outcome here is not socially optimal.

14 A Sealed Bid Auction

There is a seller who wishes to allocate an indivisible item to one of n prospective buyers in exchange for a payment. Here, $N = \{1, 2, \ldots, n\}$ represents the set of buying agents. Let $v_1, v_2, \ldots, v_n$ be the valuations of the players for the object. The n buying agents submit sealed bids and these bids need not be equal to the valuations. Assume that the sealed bid from player i ($i = 1, \ldots, n$) could be any real number greater than 0. Then the strategy sets of the players are: $S_i = (0, \infty)$ for $i = 1, \ldots, n$. Assume that the object is awarded to the agent with the lowest index among those who bid the highest. Let $b_1, \ldots, b_n$ be the bids from the n players. Then the allocation function will be:

$$\begin{aligned} y_i(b_1, \ldots, b_n) \quad &= 1 \text{ if} \quad b_i > b_j \text{ for } j = 1, 2, \ldots, i-1 \text{ and} \\ & \qquad\quad\; b_i \geq b_j \text{ for } j = i+1, \ldots, n \\ &= 0 \text{ else} \end{aligned}$$

In the first price sealed bid auction, the winner pays an amount equal to his bid, and the losers do not pay anything. In the second price sealed bid auction, the winner pays an amount equal to the highest bid among the players who do not win, and as usual the losers do not pay anything. The payoffs or utilities to the bidders in these two auctions are of the form:

$$u_i(b_1, \ldots, b_n) = y_i(b_1, \ldots, b_n)(v_i - t_i(b_1, \ldots, b_n))$$

where $t_i(b_1, \ldots, b_n)$ is the amount to be paid by bidder i in the auction. Suppose $n = 4$, and suppose the values are $v_1 = 20;\ v_2 = 20;\ v_3 = 16;\ v_4 = 16$, and the bids are $b_1 = 10;\ b_2 = 12;\ b_3 = 8;\ b_4 = 14$. Then for both first price and second price auctions, we have the allocation $y_1(.) = 0;\ y_2(.) = 0;\ y_3(.) = 0;\ y_4(.) = 1$. The payments for the first price auction are $t_1(.) = 0;\ t_2(.) = 0;\ t_3(.) = 0;\ t_4(.) = 14$ whereas the payments for the second price auction would be: $t_1(.) = 0;\ t_2(.) = 0;\ t_3(.) = 0;\ t_4(.) = 12$. The utilities can be easily computed from the values and the payments.

An important question is: What will the strategies of the bidders be in these two auctions. This question will be discussed at length in forthcoming chapters.

15 A Duopoly Pricing Game (Bertrand Model)

This is due to Bertrand (1883). There are two companies 1 and 2 which wish to maximize their profits. The demand for a price p is given by a continuous and strictly decreasing function $x(p)$. The cost for producing each unit of product $= c > 0$. The companies simultaneously choose their prices p_1 and p_2. The amount of sales for each company is given by:

$$
\begin{aligned}
x_1(p_1, p_2) &= x(p_1) && \text{if} \quad p_1 < p_2 \\
&= \tfrac{1}{2}x(p_1) && \text{if} \quad p_1 = p_2 \\
&= 0 && \text{if} \quad p_1 > p_2
\end{aligned}
$$

$$
\begin{aligned}
x_2(p_1, p_2) &= x(p_2) && \text{if} \quad p_2 < p_1 \\
&= \tfrac{1}{2}x(p_2) && \text{if} \quad p_1 = p_2 \\
&= 0 && \text{if} \quad p_2 > p_1
\end{aligned}
$$

It is assumed that the firms incur production costs only for an output level equal to their actual sales. Given prices p_1 and p_2, the utilities of the two companies are:

$$
\begin{aligned}
u_1(p_1, p_2) &= (p_1 - c)x_1(p_1, p_2) \\
u_2(p_1, p_2) &= (p_2 - c)x_2(p_1, p_2)
\end{aligned}
$$

Note that for this game, $N = \{1, 2\}$ and $S_1 = S_2 = (0, \infty)$.

16 A Duopoly Pricing Game (Cournot Model)

Here, the two companies simultaneously decide how much to produce. This model is due to Cournot (1838). An example of this is provided by agriculturists deciding how much of a perishable crop to pick each morning to send to a market. Here $p(\cdot) = \bar{x}^1(\cdot)$ is the inverse demand function and gives the price that clears the market. Here $N = \{1, 2\}$ and $S_1 = S_2 = \{1, 2, \ldots, Q\}$, where Q is some maximum quantity. The utility functions are given by:

$$
\begin{aligned}
u_1(q_1, q_2) &= p(q_1, q_2)q_1 - cq_1 \\
u_2(q_1, q_2) &= p(q_1, q_2)q_2 - cq_2.
\end{aligned}
$$

17 A Procurement Exchange Game

This example is adapted from an example presented by by Tardos and Vazirani [2]. Imagine a procurement exchange where buyers and sellers meet to match supply and demand for a particular product. Suppose that there are two sellers 1 and 2 and three buyers A, B, and C. Because of certain constraints such as logistics, assume that

- A can only buy from seller 1.
- C can only buy from seller 2.
- B can buy from either seller 1 or seller 2.
- Each buyer has a maximum willingness to pay of 1 and wishes to buy one item.
- The sellers have enough items to sell.
- Each seller announces a price in the range $[0, 1]$.

Let s_1 and s_2 be the prices announced. It is easy to see that buyer A will buy an item from seller 1 at price s_1 and buyer C will buy an item from seller 2 at price s_2. If $s_1 < s_2$, then buyer B will buy an item from seller 1; otherwise buyer B will buy from seller 2. Assume that buyer B will buy from seller 1 if $s_1 = s_2$. The game can now be defined as follows:

$$\begin{aligned}
N &= \{1, 2\} \\
S_1 &= S_2 = [0, 1] \\
u_1(s_1, s_2) &= 2s_1 \ \text{ if } s_1 \leq s_2 \\
&= s_1 \ \text{ if } s_1 > s_2 \\
u_2(s_1, s_2) &= 2s_2 \ \text{ if } s_1 > s_2 \\
&= s_2 \ \text{ if } s_1 \leq s_2.
\end{aligned}$$

18 Braess Paradox Game

This game is developed on the lines of the game presented in the book by Easley and Kleinberg [3]. This game illustrates the Braess paradox which is named after the German mathematician Dietrich Braess. This paradox is usually associated with transportation networks and brings out the counter-intuitive fact that a transportation network with extra capacity added may actually perform worse (in terms of time delays) than when the extra capacity did not exist. Figure shows a network that consists of a source S and a destination T, and two intermediate hubs A and B. It is required to travel from S to T. One route is via the hub A and the other route goes via the hub B.

Regardless of the number of vehicles on the route, it takes 25 minutes to travel from S to B or from A to T. On the other hand, the travel time from S to A takes time $\frac{m}{50}$ minutes where m is the number of vehicles on that link. Similarly, the travel time from B to T takes time $\frac{m}{50}$ minutes where m is the number of vehicles on that link. Assume that there are $n = 1000$ vehicles that wish to move from S to T. This means $N = \{1, 2, \ldots, 1000\}$. The strategy sets are $S_1 = \ldots, = S_n = \{A, B\}$. Given a strategy profile $(s_1, \ldots, s_n)$, let $n_A(s_1, \ldots, s_n)$ $(n_B(s_1, \ldots, s_n))$ denote the number of vehicles taking the route via A (B). It is easy to note that $n_A(s_1, \ldots, s_n) + n_B(s_1, \ldots, s_n) = n$. If utility of a player

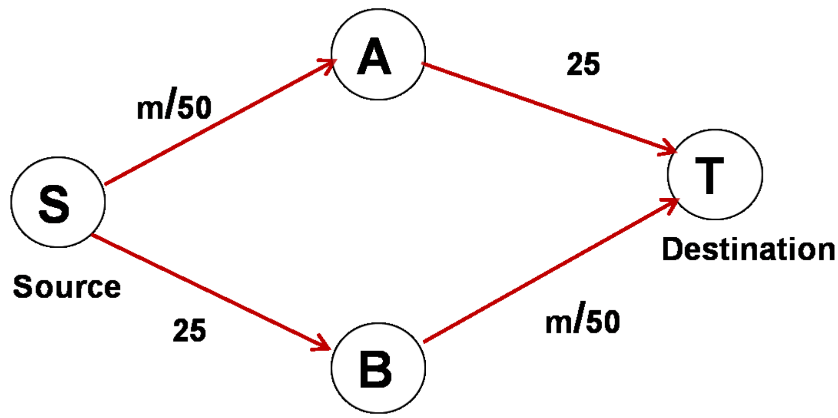

Figure 2: A transportation network with with four nodes

(in this case vehicle) is defined as the negative of the travel time for that player from S to T, then it is easy to see that

$$\begin{aligned} u_i(s_1, \ldots, s_n) &= -25 - \frac{n_A(s_1, \ldots, s_n)}{50} \quad (s_i = A) \\ &= -25 - \frac{n_B(s_1, \ldots, s_n)}{50} \quad (s_i = B) \end{aligned}$$

This defines a strategic form game. Note that

$$u_i(A, A, \ldots, A) = -45$$

$$u_i(B, B, \ldots, B) = -45$$

$$u_i(s_1, s_2, \ldots, s_n) = -35 \quad \text{whenever} \quad n_A(s_1, \ldots, s_n) = n_B(s_1, \ldots, s_n) = 500$$

In Chapter 5, we will be showing that any profile that satisfies the last condition above will be a Nash equilibrium profile.

Let us say we now introduce a fast link from A to B to ease the congestion in the network (as a degenerate case, we will assume the travel time from A to be to be zero). Figure depicts this new network with extra capacity added from A to B. Now the strategies available to each vehicle are to go from S to A to T (call this strategy A); S to B to T (call this strategy B); and S to A to B to T (call this strategy AB). So we have $S_1 = \ldots, = S_n = \{A, B, AB\}$. Defining $n_A(s_1, \ldots, s_n), n_B(s_1, \ldots, s_n), n_{AB}(s_1, \ldots, s_n)$ on the same lines as before, we get

$$\begin{aligned} u_i(s_1, \ldots, s_n) &= -25 - \frac{n_A(s_1, \ldots, s_n)}{50} \quad (s_i = A) \\ &= -25 - \frac{n_B(s_1, \ldots, s_n)}{50} \quad (s_i = B) \\ &= -\frac{n_A(s_1, \ldots, s_n) + n_{AB}(s_1, \ldots, s_n)}{50} \\ &\quad -\frac{n_B(s_1, \ldots, s_n) + n_{AB}(s_1, \ldots, s_n)}{50} \quad (s_i = AB) \end{aligned}$$

We will analyze the above two games in Chapters 4 and 5 and illustrate the Braess paradox.

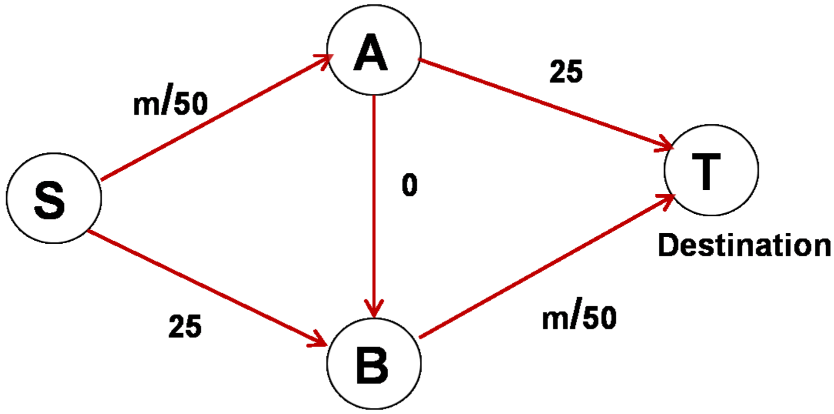

Figure 3: Transportation network with an extra link from A to B

19 Conclusion

From the next chapter onwards, we start analyzing strategic form games by looking at their equilibrium behavior. In the next chapter, we discuss the notion of dominant strategy equilibria. In the sections following that, we introduce the notion of Nash equilibrium. The notation we use is summarized in Table 1.

N	A set of players, $\{1, 2, \ldots, n\}$
S_i	Set of actions or pure strategies of player i
S	Set of all action profiles $S_1 \times \ldots \times S_n$
s	An action profile, $s = (s_1, \ldots, s_n) \quad \in \ S$
S_{-i}	Set of action profiles of all agents other than $i = S_1 \times \ldots S_{i-1} \times S_{i+1} \times \ldots \times S_n$
s_{-i}	An action profile of agents other than i, $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n) \quad \in \ S_{-i}$
(s_i, s_{-i})	Another representation for strategy profile $(s_1, \ldots, s_n)$
s_i^*	Equilibrium strategy of player i
u_i	Utility function of player i; $u_i : S \to \mathbb{R}$

Table 1: Notation for strategic form games

Problems

1. (Mascolell, Whinston, Green [4]). The following payoff matrix corresponds to a modified version of the Prisoner's Dilemma problem called the DA's brother problem. In this problem prisoner 1 is related to the District Attorney. How is this problem different?

	2	
1	NC	C
NC	$0, -2$	$-10, -1$
C	$-1, -10$	$-5, -5$

2. (Attrition Game - Osborne and Rubinstein [1]) Two players are involved in a dispute over an object. The value of the object to player i is $v_i > 0$. Time is modeled as a continuous variable that starts at 0 and runs indefinitely. Each player chooses when to concede the object to the other player; if the first player to concede does so at time t, the other player obtains the object at that time. If both players concede simultaneously, the object is split equally between them, player i receiving a payoff of $v_i/2$. Time is valuable: until the first concession each player loses one unit of payoff per unit time. Formulate this situation as a strategic game.

3. (A location game -Osborne and Rubinstein [1]) Each of n people chooses whether or not to become a political candidate, and if so which position to take. There is a continuum of citizens, each of whom has a favorite position; the distribution of favorite positions is given by a density function f on $[0, 1]$ with $f(x) > 0$ for all $x \in [0, 1]$. A candidate attracts the votes of those citizens whose favorite positions are closer to his position than to the position-of any other candidate; if k candidates choose the same position then each receives the fraction $1/k$ of the votes that the the position attracts. The winner of the competition is the candidate who receives the most votes. Each person prefers to be the unique winning candidate than to tie for the first place, prefers to tie for the first place than to stay out of the competition, and prefers to stay out of the competition than to enter and lose. Formulate this situation as a strategic game.

4. (Guess the Average - Osborne and Rubinstein [1]) Each of n people announces a number in the set $\{1, ..., K\}$. A prize of \$1 is split equally between all the people whose number is closest to $\frac{2}{3}$ of the average number. Formulate this as a strategic form game.

5. (An investment race - Osborne and Rubinstein [1]) Two investors are involved in a competition with a prize of \$1. Each investor can spend any amount in the interval $[0, 1]$. The winner is the investor who spends the most; in the event of a tie each investor receives \$0.50. Formulate this situation as a strategic game.

To Probe Further

The material discussed in this chapter draws upon mainly from three sources, namely the books by Myerson [5], Mascolell, Whinston, and Green [4], and Osborne and Rubinstein [1]. The paper by Tardos and Vazirani [2] is a fine introduction to concepts in game theory. In fact, we have taken many examples from their paper.

The following books also contain illustrative examples of strategic form games: Osborne [1], Straffin [6], and Binmore [7].

References

[1] Martin J. Osborne and Ariel Rubinstein. *A Course in Game Theory.* Oxford University Press, 1994.

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