Lecture Notes By

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2. Extensive Form Games

Note: *This is a only a draft version, so there could be flaws. If you find any errors, please do send email to* `hari@csa.iisc.ernet.in`. *A more thorough version would be available soon in this space.*

In this chapter, we study *extensive form games* which provide a more detailed representation than *strategic form games*. We explain the all important notion of a *strategy* and describe how an extensive form game can be transformed into a strategic form game.

In the previous chapter, we have got introduced to strategic form games. In this chapter, we introduce extensive form games. The extensive form of a game represents a detailed and richly structured way to describe a game. This form was first proposed by von Neumann and Morgenstern [1] and was later refined by Kuhn [2]. The extensive form captures complete sequential play of a game. Specifically it captures (1) who moves when (2) what actions each player may play (3) what the players know before playing at each stage (4) what the outcomes are as a function of the actions, and (5) payoffs that players obtain from each outcome. Extensive form games are depicted graphically using game trees. We first present a few simple examples.

1 Extensive Form Games: Examples

1.1 Matching Pennies with Observation

In the matching pennies game, there are two players, 1 and 2, who each has a rupee coin. One of the players puts down his rupee coin heads up or tails up. The other player sees the outcome and puts down her coin heads up or tails up. If both the coins show heads or both the coins show tails, player 2 gives one rupee to player 1 who becomes richer by one rupee. If one of the coins shows heads and the other coin shows tails, then player 1 pays one rupee to player 2 who becomes richer by one rupee. Depending on whether player 1 moves first or player 2 moves first, there are two versions of this game. Figure 1 shows the game tree when player 1 moves first while Figure 2 shows the game tree when player 2 moves first.

In the game tree representation, the nodes are of three types: (1) root node (initial decision node); (2) decision nodes (which are the internal nodes); and (3) terminal nodes (or leaf nodes). Each possible sequence of events that could occur in the game is captured by a path of links from the root node to

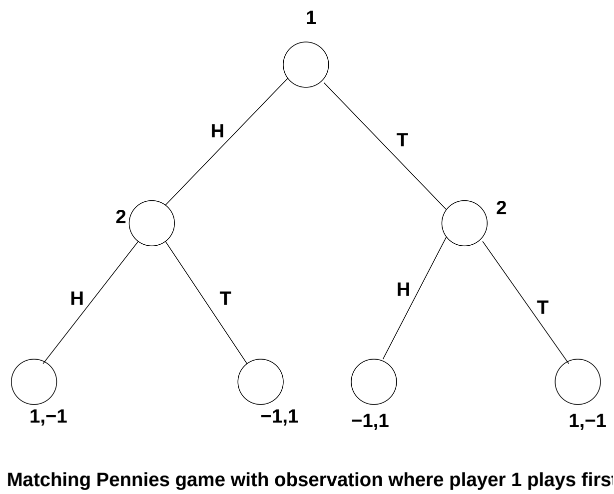

Figure 1: Matching pennies games with observation when player 1 moves first

one of the terminal nodes. When the game actually takes place, the path that represents the sequence of events is called the *path of play*. Each decision node is labeled with the player who takes a decision at that node. The links that are outgoing at the decision node are labeled with the actions the player may select at that node. Note that each node represents not only the current position in the game but also how it was reached. The terminal nodes are labeled with the payoffs that the players would get in the outcomes corresponding to those nodes. The game trees shown in Figures 1 and 2 are self-explanatory.

1.2 Matching Pennies without Observation

In this case, one of the players puts down his rupee coin heads up or tails up. The other player *does not observe the outcome* and only puts down her rupee coin heads up or tails up. Depending on whether player 1 moves first or player 2 moves first, we obtain the game tree of Figure 3 or Figure 4 respectively. Note that these game trees are similar to the ones corresponding to the game with observation except that the two decision nodes corresponding to player 2 in Figure 3 are connected with dotted lines. Similarly the two decision nodes corresponding to player 1 in Figure 4 are connected with dotted lines. A set of nodes that are connected with dotted lines is called an *information set*. When the game reaches one of the decision nodes in an information set, the player who is supposed to make a move at that node does not know in which of the nodes of that information set she is in. The reason for this is that the player does not observe something about the game that has previously occurred in the game.

Definition 1 (Information Set). *An information set of a player gives a set of that player's decision nodes which are indistinguishable to the player.*

The information sets of a player describe a collection of all possible distinguishable circumstances in which the player is called upon to move. Clearly, in every node within a given information set, the corresponding player must have the same set of possible actions.

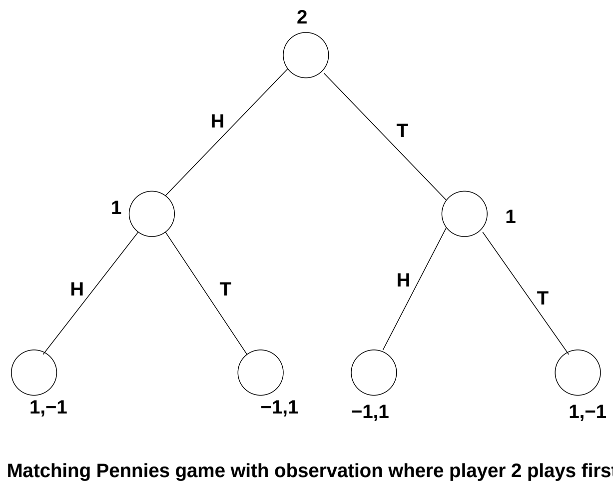

Figure 2: Matching pennies games with observation when player 2 moves first

1.3 Matching Pennies with Simultaneous Play

In this version of the game, the two players put down their rupee coins simultaneously. Clearly, each player has no opportunity to observe the outcome of the move of the other player. The order of play is obviously not relevant here. Thus both the game trees depicted in Figure 3 and Figure 4 provide a valid representation of this version of the game.

2 Games with Perfect Information

An extensive form game with *perfect information* is one which all the information sets are singletons. This implies that each player is able to observe all previous moves or the entire history thus far. Each player knows precisely where she is currently and also knows precisely how she has reached that node. If at least one information set of at least one player has two or more elements, the game is said to be of *imperfect information*. As immediate examples, the games depicted in Figures 1 and 2 are games with perfect information while the games shown in Figures 3 and 4 are games with imperfect information. The matching pennies game with simultaneous play is obviously a game with imperfect information.

3 Extensive Form Games: Definition

We now formally define an extensive form game with perfect information. This definition follows closely the one given by Osborne [3].

Definition 2 (Extensive Form Game). *An extensive form game* Γ *with perfect information consists of a tuple* $\Gamma = \langle N, (A_i), \mathcal{H}, P, (u_i) \rangle$ *where*

- $N = \{1, 2, \ldots, n\}$ *is a finite set of players*
- A_i *for* $i = 1, 2, \ldots, n$ *is the set of actions available to player* i

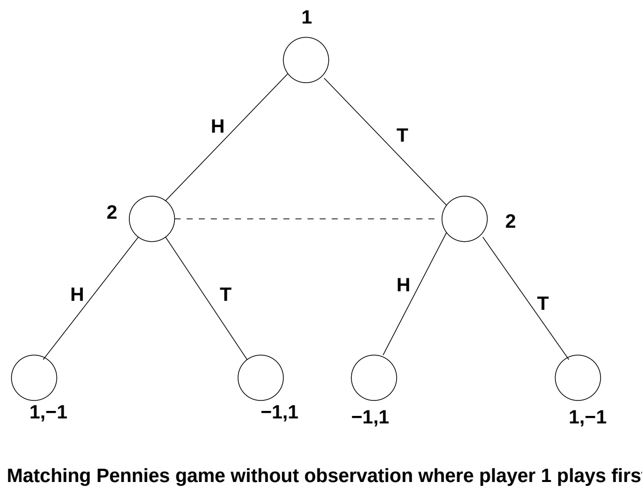

Figure 3: Matching pennies games without observation when player 1 moves first

- $\mathcal{H}$ *is the set of all* terminal histories *where a terminal history is a path of actions such that it is not a proper subhistory of any other terminal history.*
- $P : S_{\mathcal{H}} \rightarrow N$ *is a mapping that associates each proper subhistory to a certain player; the set* $S_{\mathcal{H}}$ *is the set of all proper subhistories (including empty history) of all terminal histories.*
- $u_i : \mathcal{H} \rightarrow \mathcal{R}$ *for* $i = 1, 2, \ldots, n$ *gives the utility of player* i *corresponding to each terminal history.*

We illustrate the above definition for the matching pennies game shown in Figure 1.

$$N = \{1, 2\}$$
$$A_1 = A_2 = \{H, T\}$$
$$\mathcal{H} = \{HH, HT, TH, TT\}$$
$$S_{\mathcal{H}} = \{\epsilon, H, T\} \text{ where } \epsilon \text{ represents the empty history}$$
$$P(\epsilon) = 1;\ P(H) = 2;\ P(T) = 2$$
$$u_1(HH) = 1;\ u_1(HT) = -1;\ u_1(TH) = -1;\ u_1(TT) = 1$$
$$u_2(HH) = -1;\ u_2(HT) = 1;\ u_2(TH) = 1;\ u_2(TT) = -1$$

To define an extensive form game with imperfect information, we need to additionally specify the set of all information sets for each player. Note that each information set of a player consists of all proper subhistories relevant to that player which are indistinguishable to that player. In the matching pennies game shown in Figure 3, the only information set of a player 1 is the singleton $\{\epsilon\}$ consisting of the empty history. The information set of player 2 is the set $\{H, T\}$ that consists of the proper histories H and T which are indistinguishable to player 2. On the other hand, in the game with perfect information shown in Figure 1, player 1 has only one information set namely $\{\epsilon\}$ whereas player 2 has two information sets $\{H\}$ and $\{T\}$ because these two proper subhistories are distinguishable to player 2.

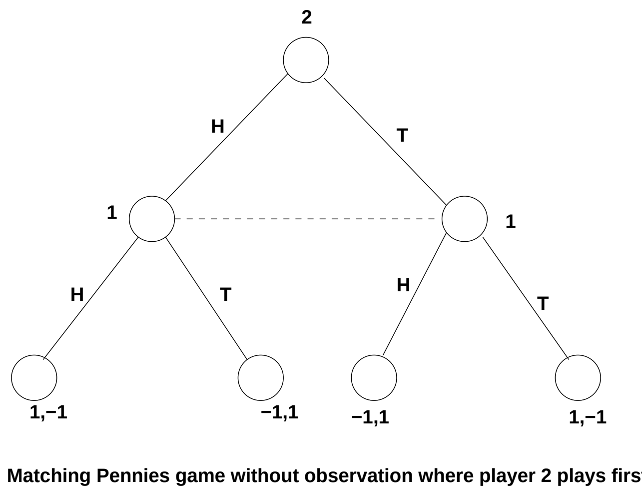

Figure 4: Matching pennies games without observation when player 1 moves first

4 The Notion of a Strategy

The notion of a strategy is one of the most important notions in game theory. A *strategy* can be described as a complete contingent plan which specifies what a player will do at each of the information sets where the player is called upon to play. A strategy of a player completely specifies the action the player chooses to play in each of her information sets if and when it is reached during play of the game.

Suppose $\mathcal{I}_i$ denotes the set of all information sets of player i in the given game. Let A_i as usual denote the actions available to player i. Given an information set $J \in \mathcal{I}_i$, let $C(J) \subseteq A_i$ be the set of all actions possible to player i in the information set J. Then we define a strategy of a player formally as follows.

Definition 3 (Strategy). *A strategy s_i of player i is a mapping $s_i : \mathcal{I}_i \to A_i$ such that $s_i(J) \in C(J) \ \forall \ J \in \mathcal{I}_i$.*

The meaning of strategy s_i for player i is that it is a complete contingent plan by specifying an action for every information set of the player. A strategy thus determines the action the player is going to choose in every stage or history of the game the player is called upon to play.

In fact, the player can prepare a look-up table with two columns, one for her information sets and the other for corresponding actions; an agent of the player can then take over and play the game using table look-up. Different strategies of the player correspond to different contingent plans of actions. We illustrate the notion of strategy through an example.

4.1 Strategies in Matching Pennies with Observation

Consider the game shown in Figure 1. We have $\mathcal{I}_1 = \{\{\epsilon\}\}$; $\mathcal{I}_2 = \{\{H\}, \{T\}\}$. Player 1 has the two strategies:

$$s_{11} : \{\epsilon\} \to H$$

$$s_{12} : \{\epsilon\} \to T$$

Player 2 has the following four strategies:

$$s_{21} : \{H\} \to H; \quad \{T\} \to H$$
$$s_{22} : \{H\} \to H; \quad \{T\} \to T$$
$$s_{23} : \{H\} \to T; \quad \{T\} \to H$$
$$s_{24} : \{H\} \to H; \quad \{T\} \to H$$

The payoffs obtained by the players 1 and 2 can now be described by the following payoff matrix. Note that when the strategy of player 1 is s_{11}, the player plays H and when the strategy of player 2 is s_{21}, the player 2 plays H, leading to the payoffs 1, -1.

	2			
1	s_{21}	s_{22}	s_{23}	s_{24}
s_{11}	$1, -1$	$1, -1$	$-1, 1$	$-1, 1$
s_{12}	$-1, 1$	$1, -1$	$-1, 1$	$1, -1$

The above game is a *strategic form game* equivalent of the original extensive form game. For the game shown in Figure 2, player 2 will have two strategies and player 1 will have four strategies and a payoff matrix such as above can be easily derived.

4.2 Strategies in Matching Pennies without Observation

Consider the game shown in Figure 3. It is easy to see that $\mathcal{I}_1 = \{\{\epsilon\}\}$ and $\mathcal{I}_2 = \{\{H, T\}\}$. Here player 1 has exactly two strategies and player 2 has two strategies as shown below.

$$s_{11} : \{\epsilon\} \to H$$
$$s_{12} : \{\epsilon\} \to T$$
$$s_{21} : \{H, T\} \to H$$
$$s_{22} : \{H, T\} \to T$$

The payoff matrix corresponding to all possible strategies that can be played by the players can be easily derived as follows.

	2	
1	s_{21}	s_{22}
s_{11}	1, -1	-1, 1
s_{12}	-1, 1	1, -1

Clearly, the matching pennies game with simultaneous moves also will have the same strategies and payoff matrix as above.

It is to be noted that every extensive form game has a unique strategic form representation. The uniqueness is up to renaming or renumbering of strategies. We can also immediately observe that a given normal form game may correspond to multiple extensive form games. For example, the extensive form game in Figure 5 has the same normal form representation as that of matching pennies with observation.

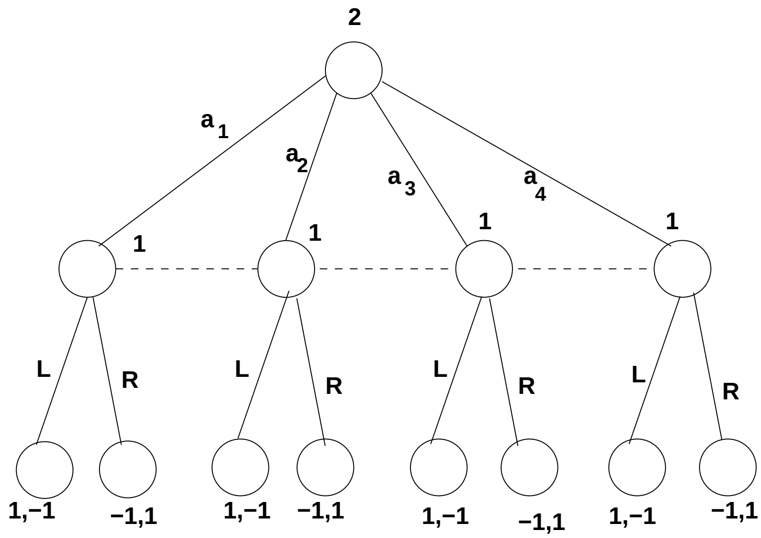

Figure 5. An extensive form game which has the same normal form as the game in Figure 1.

Figure 5: An extensive form game having the same strategic form as matching pennies with observation

5 To Probe Further

Much of the material in this chapter is based on relevant discussions in the books by Osborne [3] and by Mas-Colell, Whinston, and Green [4]. Chapter 14 in this book covers more material on extensive form games.

6 Problems

1. You might know the tick-talk-toe game. Sketch a game tree for this game.
2. Imagine the game of chess as an extensive form game and attempt to write a game tree. Do you think it can be expressed as a strategic form game?
3. ([4]). In a game, a certain player has m information sets indexed by $j = 1, 2, \ldots, m$. There are k_j possible actions for information set j. How many strategies does the player have?
4. For the extensive form game (Figure 6), write down for each player, all the applicable information sets and strategies.
5. For games shown in Figures 7,8, and 9, write down the terminal histories, proper subhistories, information sets, strategic form game representation.

```
                PLAYER 1
                  / \
                A/   \B
                /     \
          PLAYER 2     (2,0)
              /\
            C/  \D
            /    \
      PLAYER 1    (3,1)
          /\
        E/  \F
        /    \
    (1,2)    (0,0)
```

Figure 6: An extensive form game

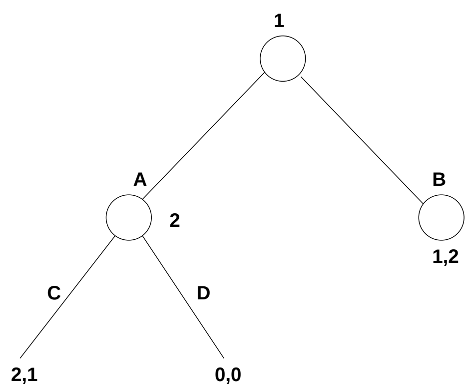

Figure 7: An extensive form game

References

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[2] H.W. Kuhn. Extensive form games and the problem of information. In *Contributions to the Theory of Games II*, pages 193–216. Princeton University Press, 1953.

[3] Martin J. Osborne. *An Introduction to Game Theory*. The MIT Press, 2003.

[4] Andreu Mas-Colell, Michael D. Whinston, and Jerry R. Green. *Micoreconomic Theory*. Oxford University Press, 1995.

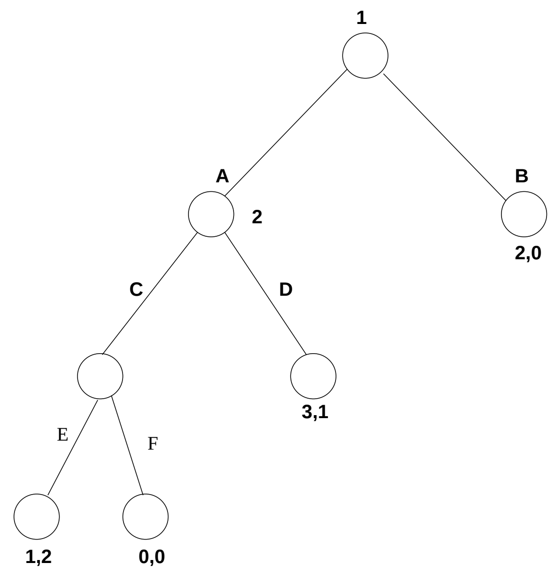

Figure 8: An extensive form game

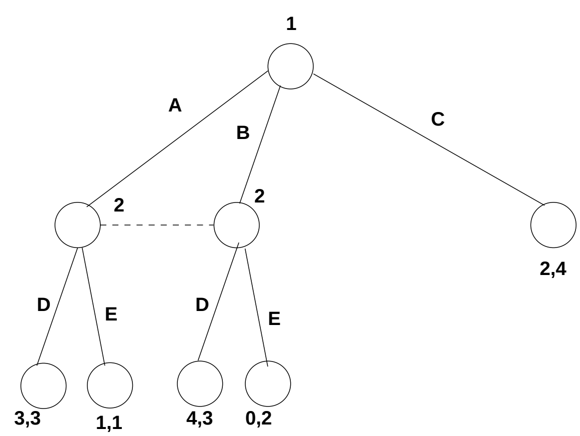

Figure 9: An extensive form game