Game Theory

Lecture Notes By

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Chapter 12: Matrix Games

Note: *This is a only a draft version, so there could be flaws. If you find any errors, please do send email to* `hari@csa.iisc.ernet.in`. *A more thorough version would be available soon in this space.*

Two player zerosum games, also called matrix games, are interesting in many ways and their analysis is tractable due to their simplicity. Von Neumann and Morgenstern showed that linear programming can be used to solve these games. In this chapter, we provide an overview of well known results for matrix games and prove the key results.

A two person zerosum game is of the form $\langle \{1,2\}, S_1, S_2, u_1, -u_1 \rangle$. Note that when a player tries to maximize her payoff, she is also simultaneously minimizing payoff of the other player. For this reason, these games are also called *strictly competitive games*. Player 1 is usually called the *row player* and player 2 is called the *column player*.

Let $S_1 = \{s_{11}, s_{12}, \ldots, s_{1m}\}$ and $S_2 = \{s_{21}, s_{22}, \ldots, s_{2n}\}$. Without any confusion, we will assume from now on that $S_1 = \{1, 2, \ldots, m\}$ and $S_2 = \{1, 2, \ldots, n\}$. Since the payoffs in a finite two person zerosum game can be completely described by a single matrix, namely the matrix that represents $u_1(i,j) \;\; \forall i \;\; \forall j$, such a game is aptly called a *matrix game*.

Since the payoffs of one player are just the negative of the payoffs of the other player, these games can be represented by a matrix with m rows and n columns. For this reason, these games are also called matrix games.

1 Examples of Matrix Games

Example 1: Matching Pennies

Consider the standard matching pennies game, whose payoff matrix is given by the following payoff matrix, assuming that strategy 1 corresponds to *heads* and strategy 2 corresponds to *tails*:

	2	
1	1	2
1	1,−1	−1,+1
2	−1, +1	1, −1

The above payoff matrix can also be specified by a simpler matrix A where $a_{ij} = u_1(i, j)$ represents the payoff for player 1 (row player) when the row player plays strategy i and the column player plays strategy j. The resulting matrix will be:

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Example 2: Rock-Paper-Scissors

We have already seen in Chapter 3 the rock-paper-scissors game where there are two players and each player has three possible strategies: 1 (rock); 2 (paper); and 3 (scissors). This is a matrix game with the following matrix:

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

Example 3: Product Prediction Game

Assume that there are two strictly competitive companies 1 and 2 capable of producing one of three products at a time, call the products A, B, and C. A company can only produce one product at a time and the payoff to the company depends on the products being produced by the two companies. In all outcomes of this game, one company gets profit while the other company makes an equal amount of loss. Assuming A, B, and C as strategies 1, 2, and 3, respectively, we have $S_1 = S_2 = \{1, 2, 3\}$. Suppose the payoff matrix for player 1 is given by

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ -1 & 0 & -2 \end{bmatrix}$$

The companies have to decide simultaneously which product to produce. This leads to a matrix game. We would be interested in predicting which products the two companies will produce.

Example 4: Constant Sum Game

An immediate generalization of a zerosum game is a constant sum game: $(\{1, 2\}, S_1, S_2, u_1, u_2)$ such that $u_1(s_1, s_2) + u_2(s_1, s_2) = C,\ \forall s_1 \in S_1; s_2 \in S_2$ where C is a known constant. Constant sum games can always be transformed into a zerosum game using a straightforward transformation (subtract the constant from each payoff of the row player, for example) and are equivalent to zerosum games.

2 Saddle Points

In the case of the product prediction game, it is easy to see that the profile (1, 1) is a pure strategy Nash equilibrium. In fact, this is the only pure strategy Nash equilibrium for this game. Clearly, strategy 1 is a best response strategy for the row player if the column player is playing strategy 1 and vice-versa. Also, we make the following important observations:

- By playing strategy 1, the row player can assure herself a payoff of at least 1. By playing strategy 1, the column player can assure himself that the row player will get a payoff of at most 1.

- If the row player gets a payoff of less than 1, then the row player could have done better by playing strategy 1. If the row player gets a payoff of greater than 1, then the column player could have done better by playing strategy 1.
- The payoff 1 obtained by player 1 in the above outcome is simultaneously a minimum in its row and a maximum in its column.

Such points are called *saddle points* of the matrix and correspond to pure strategy Nash equilibria of the game.

Saddle Point of a Matrix

Given a matrix $A = [a_{ij}]$, the element a_{ij} is called a saddle point of A if

$$\begin{aligned} a_{ij} &\leq a_{il} \quad \forall l = 1, \ldots, n \\ a_{ij} &\geq a_{kj} \quad \forall k = 1, \ldots, m \end{aligned}$$

That is, the element a_{ij} is simultaneously a minimum in its row and a maximum in its column.

If (i, j) is a saddle point of a given matrix game, then the payoff that the row player gets in the saddle point is called the *value* of the game.

Proposition: For a matrix game with payoff matrix A, a_{ij} is a saddle point if and only if the outcome (i, j) is a pure strategy Nash equilibrium.

Proof: Let a_{ij} be a saddle point.

$\Leftrightarrow$ a_{ij} is a row minimum and a_{ij} is a column maximum

$\Leftrightarrow$ $-a_{ij}$ is a row maximum and $+a_{ij}$ is a column maximum

$\Leftrightarrow$ The column player is playing a best response w.r.t. strategy i of the row player and the row player is playing a best response with respect to strategy j of the column player.

$\Leftrightarrow$ (i, j) is a Nash equilibrium.

Thus saddle points and pure strategy Nash equilibria are one and the same. The following theorem gives a necessary and sufficient condition for the existence of a pure strategy Nash equilibrium or saddle point.

Theorem: In a matrix $A = [a_{ij}]$, let

$$\begin{aligned} u_R &= \max_i \min_j a_{ij} \\ u_C &= \min_j \max_i a_{ij} \end{aligned}$$

Then the matrix A has a saddle point if and only if $u_R = u_C$.

The proof of the above is left as an exercise. The following proposition gives a useful property of saddle points.

Proposition: If in a matrix A, the elements a_{ij} and a_{hk} are both saddle points, then a_{ik} and a_{hj} are also saddle points. Also, any two saddle points in the game will have the same value.

Examples: Saddle Points

For the matching pennies game,

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The maxmin and minmax values are given by

$$\begin{aligned} u_R &= \max_i \ \min_j \ a_{ij} = \max\{-1, -1\} = -1 \\ u_C &= \min_j \ \max_i \ a_{ij} = \min\{+1, +1\} = +1 \end{aligned}$$

Thus the game does not have a saddle point.

For the rock-paper-scissors game,

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

The maxmin and minmax values are given by

$$\begin{aligned} u_R &= \max_i \ \min_j \ a_{ij} = \max\{-1, -1, -1\} = -1 \\ u_C &= \min_j \ \max_i \ a_{ij} = \min\{+1, +1, +1\} = +1 \end{aligned}$$

This game again does not have a saddle point.

For the product prediction game, we have

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ -1 & 0 & -2 \end{bmatrix}$$

$$\begin{aligned} u_R &= \max_i \ \min_j \ a_{ij} = \max\{1, -1, -2\} = 1 \\ u_C &= \min_j \ \max_i \ a_{ij} = \min\{1, 2, 2\} = 1 \end{aligned}$$

Therefore $u_R = u_C$ and a_{11} is a saddle point. In fact, this is the only saddle point for this game.

As another example, let us look at a matrix game with the following matrix:

$$A = \begin{bmatrix} 5 & 3 & 5 & 3 \\ 2 & 1 & -1 & -2 \\ 4 & 3 & 5 & 3 \end{bmatrix}$$

$$
\begin{aligned}
u_R &= \max_i \; \min_j \; a_{ij} = \max\{3, -2, 3\} = 3 \\
u_C &= \min_j \; \max_i \; a_{ij} = \min\{5, 3, 5, 3\} = 3
\end{aligned}
$$

The above example has four saddle points, namely $(1, 2), (1, 4), (3, 2),$ and $(3, 4)$. Note that all of them have the same value.

3 Mixed Strategies in Matrix Games

After studying pure strategy Nash equilibria in games, we now turn to mixed strategy Nash equilibria. Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ be the mixed strategies of the row player and the column player respectively. Note that a_{ij} is the payoff of the row player (player 1) when the row player chooses row i and column player chooses column j with probability 1. The corresponding payoff for the column player is $-a_{ij}$. The expected payoff to the row player with the above mixed strategies x and y is given by:

$$
\begin{aligned}
&= u_1(x, y) \\
&= \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i y_j \\
&= xAy \quad \text{where} \quad x = (x_1, \ldots, x_m);\; y = (y_1, \ldots, y_n)^T; A = [a_{ij}]
\end{aligned}
$$

The expected payoff to column player $= -xA_y$. When the row player plays x, she assures herself of an expected payoff

$$\min_{y \in \Delta(S_2)} xAy$$

The row player should therefore look for a mixed strategy x that maximizes the above. That is, an x such that

$$\max_{x \in \Delta(S_1)} \; \min_{y \in \Delta(S_2)} \; xAy$$

In other words, an optimal strategy for the row player is to do *maxminimization*. Note that the row player chooses a mixed strategy that is best for her on the assumption that whatever she does, the column player will choose an action that will hurt her (row player) as much as possible. This is a a direct consequence of rationality and the fact that the payoff for each player is the negative of the other player's payoff.

Similarly, when the column player plays y, he assures himself of a payoff

$$
\begin{aligned}
&= \min_{x \in \Delta(S_1)} -xAy \\
&= -\max_{x \in \Delta(S_1)} xAy
\end{aligned}
$$

That is, he assures himself of losing no more than

$$\max_{x \in \Delta(S_1)} xAy$$

The column player's optimal strategy should be to minimize this loss:

$$\min_{y \in \Delta(S_2)} \max_{x \in \Delta(S_1)} xAy$$

This is called *minmaximization*.

3.1 An Important Lemma

This lemma asserts that when the row player plays x, among the most effective strategies y of the column player, there is always at least one pure strategy. Symbolically,

$$\min_{y \in \Delta(S_2)} xAy = \min_j \sum_{i=1}^{m} a_{ij} x_i$$

This can be proved as follows. For a given j, the summation

$$\sum_{i=1}^{m} a_{ij} x_i$$

gives the payoff to the row player when she plays $x = (x_1, \ldots, x_m)$ and the column player player the pure strategy y_j. That is,

$$\sum_{i=1}^{m} a_{ij} x_i = u_1(x, y_j)$$

Therefore

$$\min_j \sum_{i=1}^{m} a_{ij} x_i$$

gives the minimum payoff that the row player gets when she plays x and when the column player plays only pure strategies. Since a pure strategy is a special case of mixed strategies, we have

$$\min_j \sum_{i=1}^{m} a_{ij} x_i \geq \min_{y \in \Delta(S_2)} xAy \tag{1}$$

On the other hand,

$$\begin{aligned} xAy &= \sum_{j=1}^{n} y_j \left(\sum_{i=1}^{m} a_{ij} x_i \right) \\ &\geq \sum_{j=1}^{n} y_j \left(\min_j \sum_{i=1}^{m} a_{ij} x_i \right) \\ &= \min_j \sum_{i=1}^{m} a_{ij} x_i \quad \text{since} \quad \sum_{j=1}^{n} y_j = 1 \end{aligned}$$

Therefore, we have:

$$xAy \geq \min_j \sum_{i=1}^{m} a_{ij} x_i \ \forall y \in \Delta(S_2); \ \forall x \in \Delta(S_1)$$

This implies that

$$\min_{y \in \Delta(S_2)} xAy \geq \min_j \sum_{i=1}^{m} a_{ij} x_i \tag{2}$$

From (1) and (2), we have,

$$\min_{y \in \Delta(S_2)} xAy = \min_j \sum_{i=1}^{m} a_{ij} x_i$$

Similarly, it can be shown that

$$\max_{x \in \Delta(S_1)} xAy = \max_i \sum_{j=1}^{n} a_{ij} y_j$$

From the above lemma, we can describe the optimization problems of the row player and column players as follows.

3.2 Row Player's Optimization Problem (Maxminimization)

The optimization problem facing the row player can be expressed as

$$\text{maximize } \min_j \sum_{i=1}^{m} a_{ij} x_i$$

subject to

$$\begin{aligned} &\sum_{i=1}^{m} x_i = 1 \\ &x_i \geq 0 \quad i = 1, 2, \ldots, m \end{aligned}$$

Call the above problem P_1. Note that this problem is equivalent to

$$\max_{x \in \Delta(S_1)} \min_{y \in \Delta(S_2)} xAy$$

3.3 Column Player's Optimization Problem (Minmaximization)

The optimization problem facing the column player can be expressed as

$$\text{minimize } \max_i \sum_{j=1}^{n} a_{ij} y_j$$

subject to

$$\begin{aligned} &\sum_{j=1}^{n} y_j = 1 \\ &y_j \geq 0 \quad j = 1, \ldots, n \end{aligned}$$

Call the above problem P_2. Note that this is equivalent to

$$\min_{y \in \Delta(S_2)} \max_{x \in \Delta(S_1)} xAy$$

The following proposition shows that the problems P_1 and P_2 are equivalent to appropriate linear programs.

Proposition: The following problems are equivalent.

$$\begin{array}{lll} \text{Maximize} & \min_j & \displaystyle\sum_{i=1}^m a_{ij}x_i \\ \text{subject to} & & \\ \displaystyle\sum_{i=1}^m x_i & = 1 & P_1 \\ x_i \geq 0 \quad i & = & 1,\ldots,m \end{array}$$

$$\begin{array}{lllll} \text{Maximize} & & z & & \\ \text{subject to} & & & & \\ z - \displaystyle\sum_{i=1}^m a_{ij}x_i & \leq & 0 & j = 1,\ldots,n & \\ \displaystyle\sum_{i=1}^m x_i = 1 & & LP_1 & & \\ x_i \geq 0 & i = & 1,\ldots,m & & \end{array}$$

Proof: Note that P_1 is a maximization problem and therefore by looking at the constraints

$$z - \sum_{i=1}^m a_{ij}x_i \leq 0 \quad j = 1,2,\ldots,n$$

any optimal solution z^* will satisfy the equality in the above constraint. That is,

$$z^* = \sum_{i=1}^m a_{ij}x_i^* \quad \text{for some} \quad j \in \{1,\ldots,n\}$$

Let j^* be one such value of j. Then

$$z^* = \sum_{i=1}^m a_{ij^*}x_i^*$$

Because z^* is a feasible solution of LP_1, we have

$$\sum_{i=1}^m a_{ij^*}x_i^* \leq \sum_{i=1}^m a_{ij}x_i^* \quad \forall j = 1,\ldots,n$$

This means

$$\sum_{i=1}^m a_{ij^*}x_i^* = \min_j \sum_{i=1}^m a_{ij}x_i^*$$

If not, we have

$$z^* < \sum_{i=1}^m a_{ij}x_i \; \forall j = 1,2,\ldots,n$$

If this happens, we can find a feasible solution $\hat{z}$ such that $\hat{z} > z^*$. Such a $\hat{z}$ is precisely the one for which equality will hold. But since z^* is a maximal value, the existence of $\hat{z} > z^*$ is a contradiction!

Thus the following two linear programs describe the optimization problems facing the row player and the column player.

Row Player's Linear Program (LP1)

$$\begin{aligned}
&\text{maximize} \quad z \\
&\text{subject to} \\
&z - \sum_{i=1}^{m} a_{ij} x_i \leq 0 \quad j = 1, \ldots, n \\
&\sum x_i = 1 \quad x_i \geq 0 \quad \forall i
\end{aligned}$$

Column Player's Linear Program (LP2)

$$\begin{aligned}
&\text{minimize} \quad w \\
&\text{subject to} \\
&w - \sum_{j=1}^{n} a_{ij} x_i \geq 0 \quad i = 1, \ldots, m \\
&\sum y_j = 1 \quad y_j \geq 0 \quad \forall j = \qquad 1, \ldots, n
\end{aligned}$$

Example: Rock-Paper-Scissors Game

For the rock-paper-scissors game, recall the matrix of payoffs of row player:

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

The problem P_1 would be:

$$\text{maximize } \min \ \{x_2 - x_3, \ -x_1 + x_3, \ x_1 - x_2\}$$

subject to

$$x_1 + x_2 + x_3 = 1$$

$$x_1 \geq 0; \ x_2 \geq 0; \ x_3 \geq 0$$

The above problem is equivalent to the linear program (LP_1):

$$\text{maximize } z$$

subject to

$$z \leq x_2 - x_3$$

$$z \leq -x_1 + x_3$$

$$z \leq x_1 - x_2$$

$$x_1 + x_2 + x_3 = 1$$

$$x_1 \geq 0;\ x_2 \geq 0;\ x_3 \geq 0$$

Corresponding to the column player, the problem P_2 would be:

$$\text{minimize } \max\ \{-y_2 + y_3,\ y_1 - y_3,\ -y_1 + y_2\}$$

subject to

$$y_1 + y_2 + y_3 = 1$$

$$y_1 \geq 0;\ y_2 \geq 0;\ y_3 \geq 0$$

The above problem is equivalent to the linear program (LP_2):

$$\text{minimize } w$$

subject to

$$w \geq -y_2 + y_3$$

$$w \geq y_1 - y_3$$

$$w \leq -y_1 + y_2$$

$$y_1 + y_2 + y_3 = 1$$

$$y_1 \geq 0;\ y_2 \geq 0;\ y_3 \geq 0$$

4 Minimax Theorem

This result is one of the important landmarks in the initial decades of game theory. This result was proved by von Neumann in 1928 using the Brouwer's fixed point theorem. Later, he and Morgenstern provided an elegant proof of this theorem using linear programming duality. The key implication of the minimax theorem is the existence of a mixed strategy Nash equilibrium in any matrix game.

Theorem: For every $(m \times n)$ matrix A, there is a stochastic row vector $x^* = (x_1^*, \ldots, x_m^*)$ and a stochastic column vector $y^* = (y_1^*, \ldots, y_n^*)^T$ such that

$$\min_{y \in \Delta(S_2)} x^* A y = \max_{x \in \Delta(S_1)} x A y^*$$

Proof: Given a matrix A, we have derived linear programs LP_1, LP_2. LP_1 represents the optimal strategy of the row player while LP_2 represents the optimal strategy of the column player. First we make the observation that the linear program LP_2 is the dual of the linear program LP_1. We now invoke the strong duality theorem which states: *If an LP has an optimal solution, then its dual also has an optimal solution; moreover the optimal value of the dual is the same as the optimal value of the original (primal) LP*. See the appendix for a quick primer on LP duality.

To apply the strong duality theorem in the current context, we first observe that the problem P_1 has an optimal solution by the very nature of the problem. Since LP_1 is equivalent to the problem P_1, the immediate implication is that LP_1 has an optimal solution. Thus we have two linear programs LP_1 and LP_2 which are duals of each other and LP_1 has an optimal solution. Then by the strong duality theorem, LP_2 also has an optimal solution and the optimal value of LP_2 is the same as the optimal value of LP_1.

Let $z^*, x_1^*, \ldots, x_m^*$ be an optimal solution of LP_1. Then, we have

$$z^* = \sum_{i=1}^{m} a_{ij^*} x_i^* \quad \text{for some} \quad j^* \in \{1, \ldots, n\}$$

By the feasibility of the optimal solution in LP_1, we have

$$\sum_{i=1}^{m} a_{ij^*} x_i^* \leq \sum_{i=1}^{m} a_{ij} x_i^* \quad \text{for} \quad j = 1, \ldots, n$$

This implies that

$$\begin{aligned} \sum_{i=1}^{m} a_{ij^*} x_i^* &= \min_j \sum_{i=1}^{m} a_{ij} x_i^* \\ &= \min_{y \in \Delta(S_2)} x^* A y \quad \text{(by the lemma)} \end{aligned}$$

Thus

$$z^* = \min_{y \in \Delta(S_2)} x^* A y$$

Similarly, let $w^*, y_1^*, \ldots, y_n^*$ be an optimal solution of LP_2. Then

$$w^* = \sum_{j=1}^{n} a_{i^*j} y_j^* \quad \text{for some} \quad j^* \in \{1, \ldots, m\}$$

By the feasibility of the optimal solution in LP_2, we have

$$\sum_{j=1}^{m} a_{i^*j} y_j^* \geq \sum_{j=1}^{n} a_{ij} y_j^* \quad \text{for} \quad j = 1, 2, \ldots, m$$

$$\begin{aligned} \Rightarrow \sum_{j=1}^{n} a_{i^*j} y_j^* &= \max_i \sum_{j=1}^{n} a_{ij} y_j^* \\ &= \max_{x \in \Delta(S_1)} x A y^* \quad \text{(by Lemma)} \end{aligned}$$

Therefore

$$w^* = \max_{x \in \Delta(S_1)} x A y^*$$

By the strong duality theorem, the optimal values of the primal and the dual are the same and therefore $z^* = w^*$. This means

$$\min_{y \in \Delta(S_2)} x^* A y = \max_{x \in \Delta(S_1)} x A y^*$$

This proves the minimax theorem.

We now show that the mixed strategy profile (x^*, y^*) is in fact a mixed strategy Nash equilibrium of the matrix game with matrix A. For this, consider

$$x^* A y^* \;\; \geq \;\; \min_{y \in \Delta(S_2)} x^* A y$$

$$
\begin{aligned}
&= \max_{x \in \Delta(S_1)} xAy^* \\
&\geq xAy^* \quad \forall x \in \Delta(S_1)
\end{aligned}
$$

That is, $x^*Ay^* \geq xAy^* \ \forall x \in \Delta(S_1)$. This implies

$$u_1(x^*, y^*) \geq u_1(x, y^*) \ \ \forall x \in \Delta(S_1)$$

Further

$$
\begin{aligned}
x^*Ay^* &\leq \max_{x \in \Delta(S_1)} xAy^* \\
&= \min_{x \in \Delta(S_2)} x^*Ay \\
&\geq x^*Ay \quad \forall y \in \Delta(S_2)
\end{aligned}
$$

That is, $x^*Ay^* \leq x^*Ay \ \forall y \in \Delta(S_2)$. This implies

$$u_2(x^*, y^*) \geq u_2(x^*, y) \ \ \forall y \in \Delta(S_2)$$

Thus (x^*, y^*) is a mixed strategy Nash equilibrium or a randomized saddle point. This means the minimax theorem guarantees the existence of a mixed strategy Nash equilibrium for any matrix game.

Example: Rock-Paper-Scissors

For the rock-paper-scissors game, it is easy to see that the linear programs LP_1 and LP_2 are duals of each other. Moreover, the optimal solution of LP_1 can be seen to be

$$x_1^* = \frac{1}{3};\ x_2^* = \frac{1}{3};\ x_3^* = \frac{1}{3};\ z^* = 0$$

The optimal solution of LP_2 can be seen to be

$$y_1^* = \frac{1}{3};\ y_2^* = \frac{1}{3};\ y_3^* = \frac{1}{3};\ w^* = 0$$

4.1 A Necessary and Sufficient Condition for a Nash Equilibrium

We now discuss a key theorem that provides necessary and sufficient conditions for a mixed strategy profile to be a Nash equilibrium in matrix games.

Theorem: Given a two player zerosum game

$$(\{1,2\}, S_1, S_2, u_1, -u_1)$$

a mixed strategy profile (x^*, y^*) is a Nash equilibrium if and only if

$$x^* \in \begin{matrix}\text{argmax} \\ x \in \Delta(S_1)\end{matrix} \ \min_{y \in \Delta(S_2)} xAy$$

and

$$y^* \in \begin{matrix}\text{argmin} \\ y \in \Delta(S_2)\end{matrix} \ \max_{x \in \Delta(S_1)} xAy$$

Furthermore

$$\begin{aligned}
u_1(x^*, y^*) &= -u_2(x^*, y^*) \\
&= \max_{x \in \Delta(S_1)} \min_{y \in \Delta(S_2)} xAy \\
&= \min_{y \in \Delta(S_2)} \max_{x \in \Delta(S_1)} xAy
\end{aligned}$$

Proof: First we prove the necessity. Suppose (x^*, y^*) is a Nash equilibrium. Then

$$u_1(x^*, y^*) \geq u_1(x, y^*) \quad \forall x \in \Delta(S_1)$$

$$\Rightarrow u_1(x^*, y^*) = \max_{x \in \Delta(S_1)} u_1(x, y^*) \tag{3}$$

Also, note that

$$u_1(x, y^*) \geq \min_{y \in \Delta(S_2)} u_1(x, y) \; \forall x \in \Delta(S_1)$$

$$\Rightarrow \max_{x \in \Delta(S_1)} u_1(x, y^*) \geq \max_{x \in \Delta(S_1)} \left\{ \min_{y \in \Delta(S_2)} u_1(x, y) \right\} \tag{4}$$

since $f(x) \geq g(x) \; \forall x \Rightarrow \max_x f(x) \geq \max_x g(x)$. From (3) and (4), we have

$$u_1(x^*, y^*) \geq \max_{x \in \Delta(S_1)} \min_{y \in \Delta(S_2)} u_1(x, y) \tag{5}$$

On similar lines, using $u_1(x^*, y^*) = -u_2(x^*, y^*)$,, we can show that

$$u_1(x^*, y^*) \leq \min_{y \in \Delta(S_2)} \max_{x \in \Delta(S_1)} u_1(x, y) \tag{6}$$

We have

$$\begin{aligned}
u_1(x^*, y^*) &= -u_2(x^*, y^*) \\
&= -\{\max_{y \in \Delta(S_2)} u_2(x^*, y)\} \\
&= \min_{y \in \Delta(S_2)} \{-u_2(x^*, y)\} \\
&= \min_{y \in \Delta(S_2)} u_1(x^*, y) \\
u_1(x^*, y^*) &= \min_{y \in \Delta(S_2)} u_1(x^*, y)
\end{aligned}$$

We know that

$$\begin{aligned}
\max_{x \in \Delta(S_1)} \min_{y \in \Delta(S_2)} u_1(x, y) &\geq \min_{y \in \Delta(S_2)} u_1(x^*, y) \\
&= u_1(x^*, y^*) \quad \text{by (5)}
\end{aligned}$$

Similarly we know that

$$\begin{aligned}
\min_{y \in \Delta(S_2)} \max_{x \in \Delta(S_1)} u_1(x, y) &\geq \max_{x \in \Delta(S_1)} u_1(x, y^*) \\
&= u_1(x^*, y^*)
\end{aligned}$$

(3) and (6) imply that

$$u_1(x^*, y^*) = \max_{x \in \Delta(S_1)} \min_{y \in \Delta(S_2)} u_1(x, y)$$

(4) and (7) imply that

$$u_1(x^*, y^*) = \min_{y \in \Delta(S_2)} \max_{x \in \Delta(S_1)} u_1(x, y)$$

From the above two expressions, we have

$$\begin{array}{rcl} x^* & \in & \begin{array}{c}\text{argmax} \\ x \in \Delta(S_1)\end{array} \min_{y \in \Delta(S_2)} u_1(x, y) \\ y^* & \in & \begin{array}{c}\text{argmin} \\ y \in \Delta(S_2)\end{array} \max_{x \in \Delta(S_1)} u_1(x, y) \end{array}$$

This completes the necessity part of the proof. To prove the sufficiency, we are given that (8) and (9) are satisfied and we have to show that (x^*, y^*) is a Nash equilibrium. This is left as an exercise. The crucial fact which is required for proving sufficiency is the existence of a mixed strategy Nash equilibrium, which is guaranteed by the minimax theorem.

5 Appendix: A Quick Primer on LP Duality

First we consider an example of an LP in canonical form:

$$\text{minimize } \; 6x_1 + 8x_2 - 10x_3$$

subject to

$$\begin{array}{rcl} 3x_1 + x_2 - x_3 & \geq & 4 \\ 5x_1 + 2x_2 - 7x_3 & \geq & 7 \\ x_1, x_2, x_3 & \geq & 0 \end{array}$$

The dual of this is the LP is given by

$$\text{maximize } 4w_1 + 7w_2$$

subject to

$$\begin{array}{rcl} 3w_1 + 5w_2 & \leq & 6 \\ w_1 + 2w_2 & \leq & 8 \\ -w_1 - 7w_2 & \leq & -10 \\ w_1, w_2 & \geq & 0 \end{array}$$

In general, given

$$\begin{array}{rcl} c & = & [c_1 \ldots c_n] \quad x = [x_1 \cdots x^n]^T \\ A & = & [a_{ij}]_{m \times n} \quad b = [b_1 \cdots b_m]^T \\ w & = & [w_1 \cdots w_m] \end{array}$$

the primal LP in canonical form is:

$$\begin{array}{ll} \text{minimize} & cx \\ \text{subject to} & Ax \geq b \\ & x \geq 0. \end{array}$$

The dual of the above primal is given by

$$\begin{array}{ll} \text{maximize} & wb \\ \text{subject to} & wA \leq c \\ & w \geq 0. \end{array}$$

A primal LP in standard form is

$$\begin{array}{ll} \text{minimize} & cx \\ \text{subject to} & Ax = b \\ & x \geq 0. \end{array}$$

The dual of the above primal is:

$$\begin{array}{ll} \text{maximize} & wb \\ \text{subject to} & wA \leq c \\ & w \quad \text{unrestricted} \end{array}$$

If we consider a maximization problem, then corresponding to the primal:

$$\begin{array}{ll} \text{maximize} & cx \\ \text{subject to} & Ax \leq b \\ & x \geq 0. \end{array}$$

we have the dual given by

$$\begin{array}{ll} \text{maximize} & wb \\ \text{subject to} & wA \geq c \\ & w \geq 0 \end{array}$$

It is a simple matter to show that the dual of the dual of a (primal) problem is the original (primal) problem itself. We now state a few important results concerning duality, which are relevant to the current context.

- **Weak Duality Theorem**: If the primal is a maximization problem, then the value of any feasible primal solution is greater than or equal to the value of any feasible dual solution. If the primal is a minimization problem, then the value of any feasible primal solution is less than or equal to the value of any feasible dual solution.

- If x_0 is a feasible primal solution and w_0 is a feasible dual solution, and $cx_0 = w_0 b$, then x_0 is an optimal solution of the primal problem and w_0 is an optimal solution of the dual problem.

- **Strong Duality Theorem**: Between a primal and its dual, if one of them has an optimal solution then the other also has an optimal solution and the values of the optimal solutions are the same. Note that this is the key result which was used in proving the minimax theorem.

- **Fundamental Theorem of Duality**: Given a primal and its dual, exactly one of the following statements is true.

  1. Both possess optimal solution x^* and w^* with $cx^* = w^* b$.
  2. One problem has unbounded objective value in which case the other must be infeasible.
  3. Both problems are infeasible.

Problems

1. Given a matrix $A = [a_{ij}]$, define

$$u_R = \max_i \; \min_j \; a_{ij}$$

$$u_C = \min_j \; \max_i \; a_{ij}$$

Show that A has a saddle point if and only if $u_R = u_C$.

2. In a matrix $A = [a_{ij}]$, if two elements a_{ij} and a_{hk} are saddle points, then show that a_{ik} and a_{hj} are also saddle points.

3. Consider the following game.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Derive the conditions on the values of a, b, c, d for which the game is guaranteed to have a saddle point. Also, compute all mixed strategy Nash equilibria for the game.

4. Given a two player zero sum game with 3 pure strategies for each player, which numbers among $\{0, 1, \ldots, 9\}$ cannot be the total number of *pure* strategy Nash equilibria for the game? Justify your answer.

5. (Jones [1]). Construct a two player zero sum game with $S_1 = \{A, B, C\}$, $S_2 = \{X, Y, Z\}$ with value $= \frac{1}{2}$ and such that the set of optimal strategies for the row player is exactly the set

$$\left\{ (\alpha, 1 - \alpha, 0); \quad \frac{3}{8} \leq \alpha \leq \frac{5}{8} \right\}$$

6. (Osborne and Rubinstein [2]). Let G be a two player zero sum game that has a pure strategy Nash equilibrium.

   (a) Show that if some of the player 1's payoffs in G are increased in such a way that the resulting game G' is strictly competitive then G' has no equilibrium in which player 1 is worse off than she was in an equilibrium of G. (Note that G' may have no equilibrium at all.)

   (b) Show that the game that results if player 1 is prohibited from using one of her actions in G does not have an equilibrium in which player 1's payoffs is higher than it is in an equilibrium of G.

   (c) Give examples to show that neither of the above properties necessarily holds for a game that is not strictly competitive.

7. (Osborne and Rubinstein [2]). Army A has a single plane with which it can strike one of three possible targets. Army B has one anti-aircraft gun that can be assigned to one of the targets. The value of target k is v_k, with $v_1 > v_2 > v_3 > 0$. Army A can destroy a target only if the target is undefended and A attacks it. Army A wishes to maximize the expected value of the damage and army B wishes to minimize it. Formulate the situation as a (strictly competitive) strategic game and find its mixed strategy Nash equilibria.

8. For the following two player zero sum game, write down the primal and dual LPs and compute all Nash equilibria.

	A	B
A	2, -2	3,-3
B	4,-4	1, -1

9. For the following two player zero sum game, write down the primal and dual LPs and compute all Nash equilibria.

	A	B	C
A	2, -2	3,-3	1,-1
B	4,-4	1, -1	2,-2
C	4,-4	1, -1	3,-3

10. For the following matrix game, formulate an appropriate LP and compute all mixed strategy equilibria.

$$A = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & 0 \\ -\frac{1}{2} & 1 \\ 0 & 0 \end{bmatrix}$$

11. Show that the following holds for any two player game.

$$\max_{x \in \Delta(s_1)} \min_{y \in \Delta(s_2)} xAy \leq \min_{y \in \Delta(s_2)} \max_{x \in \Delta(s_1)} xAy$$

12. Show that the payoffs in Nash equilibrium of a symmetric matrix game (matrix game with symmetric payoff matrix) will be equal to zero for each player.

13. Complete the sufficiency part of the theorem that provides a necessary and sufficient condition for a mixed strategy profile (x^*, y^*) to be a Nash equilibrium in a matrix game.

To Probe Further

Two person zerosum games provide, perhaps, the simplest class of games which were studied during the initial years of game theory. John von Neumann is credited with the minimax theorem, which he proved in 1928 [3] by invoking the Brouwer's fixed point theorem. The classic book by Neumann and Morgenstern [4] contained a detailed exposition of matrix games, including the LP duality based approach to the minimax theorem.

The book by Myerson [5] and the book on linear programming by Chavatal [6] have inspired the exposition in this chapter. Other books which can be consulted are the ones by Osborne [7], by Rapoport [8], and by Straffin [9].

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