Game Theory

Lecture Notes By

Y. Narahari

Department of Computer Science and Automation

Indian Institute of Science

Bangalore, India

August 2012

Chapter 10: Computation of Nash Equilibria

Note: *This is a only a draft version, so there could be flaws. If you find any errors, please do send email to* `hari@csa.iisc.ernet.in`. *A more thorough version would be available soon in this space.*

Computing Nash equilibria is one of the fundamental computational problems in game theory. In fact, this is one of the extensively investigated problems in theoretical computer science in recent times. In this chapter, we will provide some insights into this problem. In the next chapter, we look into the computational complexity of this problem.

1 Supports and Nash Equilibria

1.1 Support of a Mixed Strategy Profile

Consider the game $\Gamma = \langle N, (\Delta(S_i)), (u_i)) \rangle$. Given a mixed strategy σ_i of player i, recall that the support of σ_i, denoted by $\delta(sigma_i)$ is defined as the set of all pure strategies of player i which have a non-zero probability in σ_i:

$$\delta(\sigma_i) = \{s_i \in S_i \ : \ \delta(s_i) > 0\}$$

Given a mixed strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$, the support of σ is defined in a natural way as the Cartesian product of all the individual supports:

$$\delta(\sigma_1, \ldots, \sigma_n) = \delta(\sigma_1) \times \ldots \times \delta(\sigma_n)$$

This is the set of all pure strategy profiles that would have positive probability if the players chose their strategies according to σ. This is denoted by $\delta(\sigma)$. We make the important observation that every Nash equilibrium is associated with a support. For a finite game, we have a finite number of supports and each support can be investigated for possible Nash equilibria.

1.2 Example: BOS Game

Consider the following version of the BOS game:

	2	
1	A	B
A	3, 1	0, 0
B	0, 0	1, 3

For the above game, the set of all possible supports is given by: $\{A\} \times \{A\}$, $\{A\} \times \{B\}$, $\{B\} \times \{A\}$, $\{B\} \times \{B\}$, $\{A\} \times \{A, B\}$, $\{B\} \times \{A, B\}$, $\{A, B\} \times \{A\}$, $\{A, B\} \times \{B\}$, $\{A, B\} \times \{A, B\}$.

For this game, we have already seen that (A, A) and (B, B) are pure strategy Nash equilibria. These correspond to the supports $\{A\} \times \{A\}$ and $\{B\} \times \{B\}$, respectively. We now compute a third equilibrium which in this case has the support $\{A, B\} \times \{A, B\}$. To do this, we use the necessary and sufficient condition that we proved in the Chapter 6: A mixed strategy profile is a Nash equilibrium is a Nash equilibrium if and only if:

1. for each player i, all pure strategies having positive probabilities in player i's equilibrium strategy will give the player the same payoff
2. for each player, the above payoff will be greater than or equal to the payoff the player would get with any other pure strategies.

If (σ_1^*, σ_2^*) is a Nash equilibrium with support $\{A, B\} \times \{A, B\}$, this would then mean that

$$\begin{aligned} u_1(A, \sigma_2^*) &= u_1(B, \sigma_2^*) \\ u_2(\sigma_1^*, A) &= u_2(\sigma_1^*, B) \end{aligned}$$

Note that

$$\begin{aligned} u_1(A, \sigma_2^*) &= 3\sigma_2^*(A) \\ u_1(B, \sigma_2^*) &= \sigma_2^*(B) \\ u_2(\sigma_1^*, A) &= \sigma_1^*(A) \\ u_2(\sigma_1^*, B) &= 3\sigma_1^*(B) \end{aligned}$$

We therefore have

$$\begin{aligned} 3\sigma_2^*(A) &= \sigma_2^*(B) \\ \sigma_1^*(A) &= 3\sigma_1^*(B) \end{aligned}$$

Since $\sigma_1^*(A) + \sigma_1^*(B) = \sigma_2^*(A) + \sigma_2^*(B) = 1$, we get

$$\sigma_1^* = \left(\frac{3}{4}, \frac{1}{4}\right) \quad \text{and} \quad \sigma_2^* = \left(\frac{1}{4}, \frac{3}{4}\right)$$

Note that the above strategy profile trivially satisfies condition (2) above and therefore the profile is a Nash equilibrium. We now generalize the above process of finding a Nash equilibrium.

2 A General Algorithm for Finding Nash Equilibria of Finite Strategic Form Games

The first observation we make is that although there are infinitely many (in fact uncountably so) mixed strategy profiles, there are only finitely many subsets of $S_1 \times S_2 \times \ldots S_n$ that can be supports of Nash equilibria. Note that the number of supports of a mixed strategy of a player i is precisely

$$= \text{number of non-empty subsets of } S_i = 2^{|S_i|} - 1$$

Therefore the total number of supports of mixed strategy profiles would be

$$(2^{|S_1|} - 1) \times (2^{|S_2|} - 1) \ldots \times (2^{|S_n|} - 1)$$

One can sequentially consider one support at a time and search for Nash equilibria with that support. In doing this, the following characterization of Nash equilibrium will be extremely useful: Given a game $\langle N, \Delta(S_i), (u_i) \rangle$, the mixed strategy profile $(\sigma_i, \ \ldots \ , \sigma_n)$ is a Nash equilibrium iff $\forall i \in N$,

(1) $u_i(s_i, \sigma_{-i})$ is the same for all $s_i \in \delta(\sigma_i)$

(2) $u_i(s_i, \sigma_{-i}) \geq u_i(s_i', \sigma_{-i}) \ \forall s_i \in \delta(\sigma_i) \ \ \forall s_i' \in \delta(\sigma_i)$.

2.1 Equations to be Solved

Let $X_i \subset S_i$ be a non-empty subset of S_i which will represent our current guess as to which strategies of player i have positive probability in Nash equilibrium. That is, our current guess of a support for Nash equilibrium is $X_1 \times X_2 \times \ldots \times X_n$. If there exists a Nash equilibrium σ with this support, then, by the above result, there must exist numbers $w_1, \ldots, w_n$ (where $w_i = u_i(s_i, \sigma_{-i})$ for $i = 1, 2, \ldots, n$ and mixed strategies $\sigma_1, \ldots, \sigma_n$ such that

$$w_i == \sum_{s_{-i} \in S_{-i}} \left(\prod_{j \neq i} \sigma_j(s_j) \right) u_i(s_i, s_{-i}) \ \ \forall s_i \in X_i \ \ \forall i \in N \tag{1}$$

The above condition asserts that each player i must get the same payoff, denoted by w_i, by choosing any of the pure strategies having positive probability in the mixed strategy σ_i.

$$w_i \geq \sum_{s_{-i} \in S_{-i}} \left(\prod_{j \neq i} \sigma_j(s_j) \right) u_i(s_i, s_{-i}) \quad \forall s_i \in S_i \setminus X_i \quad \forall i \in N \tag{2}$$

The above condition ensures that the pure strategies in X_i are no worse than pure strategies in $S_i \setminus X_i$.

$$\sigma_i(x_i) > 0 \quad \forall x_i \in X_i \quad \forall i \in N \tag{3}$$

The condition above states that the probability of all pure strategies of a player in the support of the mixed strategy must be greater than zero.

$$\sigma_i(s_i) = 0 \quad \forall s_i \in S_i \setminus X_i \quad \forall i \in N \tag{4}$$

The above condition asserts that the probability of all pure strategies of a player not in the support of the mixed strategy must be zero.

$$\sum_{x_i \in S_i} \sigma_i(x_i) = 1 \quad \forall i \in N \tag{5}$$

The above ensures that each σ_i is a probability distribution over S_i.

We need to find $w_1, w_2, \ldots, w_n$ and $\sigma_1(s_1) \forall s_1 \in S_1$, $\sigma_2(s_2) \forall s_2 \in S_2$; …, and $\sigma_n(s_n) \forall s_n \in S_n$, such that the above equations (1) – (5) are satisfied. Then $(\sigma_1, \ldots, \sigma_n)$ is a Nash equilibrium and w_i is the expected payoff to player i in that Nash equilibrium. On the other hand, if there is no solution that satisfies (1)–(5), then there is no equilibrium with support $X_1 \times \ldots \times X_n$. The number of unknowns in the above is $n + |S_1| + \cdots + |S_n|$, where n corresponds to the variables $w_1, w_2, \ldots, w_n$ while $|S_i|$ corresponds to the variables $\sigma_i(s_i)$, $s_i \in S_i$.

(1) leads to $|X_1| + |X_2| + \cdots + |X_n|$ equations.

(2) leads to $|S_1 \setminus X_1| + \cdots + |S_n \setminus X_n|$ equations.

(3) leads to $|X_1| + \cdots + |X_n|$ equations

(4) leads to $|S_1 \setminus X_1| + \cdots + |S_n \setminus X_n|$ equations.

(5) leads to n equations.

Thus we have a total of

$$n + 2\sum_{i \in N} |S_i|$$

equations. For example, if we have 2 players with 3 strategies each, we will have 14 equations. We make the following observations:

- Note from (1) and (2) that the equations are in general non-linear because of the term

$$\Pi_{j \neq i} \sigma_j(x_j)$$

- If there are only two players, then we will have only linear equations. The number of these equations will be $2 + 2|S_1| + 2|S_2|$.
- The number above will be the number of equations for each support. The maximum number of supports to be examined is:

$$\Pi_{i \in N} \left(2^{|S_i|} - 1 \right)$$

- So even for a two player game, the number of equations to be solved can explode.
- If the number of players exceeds 2, then not only do we have a huge number of equations, we also have to deal with non-linearity.
- For two player games, the resulting equations are said to constitute so called *linear complementarity problem* (LCP).

2.2 Linear Complementarity Problem

A linear complementarity problem (LCP) is a general problem that unifies linear programming, quadratic programming, and bimatrix games. Complementary pivot algorithm is a popular algorithm for solving LCPs. This algorithm has been generalized to yield efficient algorithms for

- Computing Brouwer and Kakutani fixed points
- Computing economic equilibria
- solving systems of nonlinear equations
- solving non-linear programming problems

Let M be a given square matrix of order n and q an n-dimensional column vector of real numbers. In an LCP, there is no objective function to be optimized. The problem is to find $w = (w_1, \cdots, w_n)^T$ and $z = (z_1, \cdots, z_n)^T$, satisfying:

$$\begin{aligned} w - Mz &= q \\ w &\geq 0 \\ z &\geq 0 \\ w_i z_i &= 0 \quad \forall i \end{aligned}$$

An example of an LCP would be:

$$n = 2; \quad M = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}; \quad q = \begin{bmatrix} -5 \\ -6 \end{bmatrix}$$

Find w_1, w_2, z_1, z_2 satisfying

$$\begin{aligned} &w_1 - 2z_1 - z_2 = -5 \\ &w_2 - z_1 - 2z_2 = -6 \\ &w_1, w_2, z_1, z_2 \geq 0 \\ &w_1 z_1 = w_2 z_2 = 0 \end{aligned}$$

It is to be noted that:

- Solving an LP can be cast as solving an LCP
- Necessary conditions for optimality of quadratic programming problems lead to LCPs
- Computing Nash equilibria in bimatrix games (two person non-zero sum games) leads to LCP.

For more details about LCPs, the excellent book by Katta Murthy [1] may be consulted.

2.3 Non-Linear Complementarity Problem

For $i = 1, \ldots, n$, let $f_i(z)$ be a real valued function on $\mathbb{R}^n$. Let $f(z) = (f_1(z), \ldots, f_n(z))$. The problem of finding $z \in \mathbb{R}$ satisfying

$$\begin{aligned} &z \geq 0 \\ &f(z) \geq 0 \\ &z_i f_i(z) = 0 \qquad f_n \quad j = 1, \ldots, n \end{aligned}$$

is known as a non-linear complementarity problem (NLCP). Computing Nash equilibria for strategic form games with three or more players leads to NLCP problems. Obviously, NLCP problems are much harder to solve than LCPs.

3 An Example for Computing Nash Equilibrium

This highly illustrative example is taken from Myerson's book [2]. This is a two player game with payoff matrix as shown.

	2		
1	L	M	R
T	7,2	2,7	3,6
B	2,7	7,2	4,5

$$S_1 = \{T, B\} \quad S_2 = \{L, M, R\}$$

A support for this game is of the form $X_1 \times X_2$ where $X_1 \subseteq S_1$, $X_2 \subseteq S_2$, $X_1 \neq \phi$, $X_2 \neq \phi$. Number of such supports is equal to $(2^2 - 1)(2^3 - 1)$, which is 21. The possible supports are $\{T\} \times \{L\}$, $\{T\} \times \{M\}$, $\{T\} \times \{R\}$, $\{T\} \times \{L, M\}$, $\{T\} \times \{L, R\}$, $\{T\} \times \{M, R\}$, $\{T\} \times \{L, M, R\}$, $\{B\} \times \{L\}$, $\{B\} \times \{M\}$, $\{B\} \times \{R\}$, $\{B\} \times \{L, M\}$, $\{B\} \times \{L, R\}$, $\{B\} \times \{M, R\}$, $\{B\} \times \{L, M, R\}$, $\{T, B\} \times \{L\}$, $\{T, B\} \times \{M\}$, $\{T, B\} \times \{R\}$, $\{T, B\} \times \{L, M\}$, $\{T, B\} \times \{L, R\}$, $\{T, B\} \times \{M, R\}$, $\{T, B\} \times \{L, M, R\}$. Our analysis proceeds as follows.

- Let us look for a Nash equilibrium in which player 1 plays pure strategy T. Player 2's best response for this is the pure strategy M. Note that player 2 cannot choose any other non-pure strategy also. If player 2 plays M, player 1's best response is B. Thus there is no Nash equilibrium in which player 1 plays pure strategy T and this rules out the first 7 supports.

- Now let us look for a Nash equilibrium in which player 1 plays pure strategy B. If player 1 chooses B, player 2 would choose L. Player 1's best response to L is T. This immediately implies that there is no Nash equilibrium in which player 1 plays pure strategy B. Thus the second set of 7 supports can be ruled out.

- As a consequence of the above two facts, in any Nash equilibrium, player 1 must randomize between T and B with positive probabilities for both T and B.

- Let us see what happens if player 2 chooses a pure strategy. If player 2 chooses L, player 1 chooses T; If player 2 chooses M, 1 chooses B; If player 2 chooses R, player 1 chooses B. Thus when player 2 plays a pure strategy, the best response of player 1 is also a pure strategy. However we have seen that in any Nash equilibrium, player 1 has to give positive probability to both the strategies T and B. Therefore, there is no Nash equilibrium in which player 2 plays a pure strategy and supports $\{T, B\} \times \{L\}$, $\{T, B\} \times \{M\}$, $\{T, B\} \times \{R\}$ can be dropped from the potential list of Nash equilibria.

- The summary so far is: (a) The game does not have any pure strategy Nash equilibria (b) The game does not have any Nash equilibria in which a player plays only a pure strategy (a) This leaves only the following supports for further exploration

  1. $\{T, B\} \times \{L, M, R\}$
  2. $\{T, B\} \times \{M, R\}$
  3. $\{T, B\} \times \{L, M\}$
  4. $\{T, B\} \times \{L, R\}$

Candidate Support 1: $\{T, B\} \times \{L, M, R\}$

Player 1 must get the same payoff from T and B. This leads to

$$w_1 = 7\sigma_2(L) + 2\sigma_2(M) + 3\sigma_2(R) \tag{6}$$
$$w_1 = 2\sigma_2(L) + 7\sigma_2(M) + 4\sigma_2(R) \tag{7}$$

Similarly, player 2 must get the same payoff from each of L, M, R:

$$w_2 = 2\sigma_1(T) + 7\sigma_1(B) \tag{8}$$
$$w_2 = 7\sigma_1(T) + 2\sigma_1(B) \tag{9}$$
$$w_2 = 6\sigma_1(T) + 5\sigma_1(B) \tag{10}$$

In addition, we have

$$\sigma_1(T) + \sigma_1(B) = 1 \tag{11}$$
$$\sigma_2(L) + \sigma_2(M) + \sigma_2(R) = 1 \tag{12}$$

we have 7 equations in 7 unknowns. However,

$$(8), (9), (10) \Rightarrow \sigma_1(T) = \sigma_1(B) = \frac{1}{2}$$

whereas

$$(9), (10), (11) \Rightarrow \sigma_1(T) = \frac{3}{4}; \quad \sigma_1(B) = \frac{1}{4}$$

Thus this system of equations does not even have a solution and surely will not lead to a Nash equilibrium profile.

Candidate Support 2: $\{T, B\} \times \{M, R\}$

Here we get the equations

$$\begin{aligned}
w_1 &= 2\sigma_2(M) + 3\sigma_2(R) \\
w_1 &= 7\sigma_2(M) + 4\sigma_2(R) \\
w_2 &= 7\sigma_1(T) + 2\sigma_1(B) \\
w_2 &= 6\sigma_1(T) + 5\sigma_1(B) \\
\sigma_1(T) &+ \sigma_1(B) = 1 \\
\sigma_2(L) &+ \sigma_2(M) + \sigma_2(R) = 1 \\
\sigma_2(L) &= 0
\end{aligned}$$

The solution is

$$\begin{aligned}
\sigma_1(T) = \frac{3}{4}; &\quad \sigma_1(B) = \frac{1}{4} \\
\sigma_2(M) = -\frac{1}{4}; &\quad \sigma_2(R) = \frac{5}{4}
\end{aligned}$$

but the solution leads to negative numbers and thus is not valid.

Candidate Support 3: $\{T, B\} \times \{L, M\}$

We get the equations

$$\begin{aligned}
w_1 &= 7\sigma_2(L) + 2\sigma_2(M) \\
w_1 &= 2\sigma_2(L) + 7\sigma_2(M) \\
w_2 &= 2\sigma_1(T) + 7\sigma_1(B) \\
w_2 &= 7\sigma_1(T) + 2\sigma_1(B) \\
\sigma_1(T) &+ \sigma_1(B) = 1 \\
\sigma_2(L) &+ \sigma_2(M) = 1 \\
\sigma_2(R) &= 0
\end{aligned}$$

These equations have a unique solution.

$$\begin{aligned}
&\sigma_1(T) = \sigma_1(B) = \frac{1}{2} \\
&\sigma_2(L) = \sigma_2(M) = \frac{1}{2} \quad \sigma_2(R) = 0 \\
&w_1 = w_2 = 4.5
\end{aligned}$$

Before we can declare this as a Nash equilibrium, we need to do one more check. Note that $\sigma_2(R) = 0$. So we have to check whether player 2 actually prefers L and M over R. We have to check what payoff player 2 would get when he plays R against player 1 playing σ_1.

$$\begin{aligned}
u_2(\sigma_1, R) &= \sigma_1(T)u_2(T, R) + \sigma_1(B)u_2(B, R) \\
&= \frac{1}{2} \times 6 + \frac{1}{2} \times 5 \\
&= 5.5 \\
&> 4.5
\end{aligned}$$

This means player 2 would not be willing to choose σ_2 when player 1 plays σ_1; player 2 would prefer to play pure strategy R instead. Thus this solution is also not a Nash equilibrium.

Candidate Support 4: $\{T, B\} \times \{L, R\}$

The equations here are:

$$\begin{aligned}
&w_1 = 7\sigma_2(L) + 3\sigma_2(R) \\
&w_1 = 2\sigma_2(L) + 4\sigma_2(R) \\
&w_2 = 2\sigma_1(T) + 7\sigma_1(B) \\
&w_2 = 6\sigma_1(T) + 5\sigma_1(B) \\
&\sigma_1(T) + \sigma_1(B) = 1 \\
&\sigma_2(L) + \sigma_2(R) + \sigma_2(M) = 1 \\
&\sigma_2(M) = 0 \\
&\sigma_1(T), \sigma_1(B), \sigma_2(L), \sigma_2(R) \geq 0 \\
&w_2 \geq \sigma_1(T)u_2(T, M) + \sigma_2(B)u_2(B, M)
\end{aligned}$$

The unique solution of the above system of linear complementary equations is

$$\sigma_1(T) = \frac{1}{3} \quad \sigma_1(B) = \frac{2}{3}$$
$$\sigma_2(L) = \frac{1}{6} \quad \sigma_2(M) = 0 \quad \sigma_2(R) = \frac{5}{6}$$
$$w_1 = \frac{8}{3} \quad w_2 = \frac{16}{3}$$

Moreover,

$$u_2(\sigma_1, M) = 7(\frac{1}{3}) + 2(\frac{2}{3}) = \frac{11}{3} \leq \frac{16}{3}$$

This is certainly a Nash equilibrium. Thus the mixed profile

$$\left(\left(\frac{1}{3}, \frac{2}{3}\right), \left(\frac{1}{6}, 0, \frac{5}{6}\right)\right)$$

is the unique mixed strategy Nash equilibrium of the given game. Note that

$$u_2(\sigma_1, M) = \sigma_1(T)u_2(T, M) + \sigma_1(B)u_2(B, M) = \frac{11}{3}$$

$$u_2(\sigma_1, L) = u_2(\sigma_1, R) = \frac{16}{3}$$

4 Algorithms for Computing Nash Equilibrium

For the past five decades, game theorists and more recently theoretical computer scientists have sought to develop efficient algorithms for computing Nash equilibria of finite games. One of the early breakthroughs was the *complementary pivot algorithm* developed by Lemke and Howson [3] in 1964 for bimatrix games (that is, two player non-zero sum games). This was immediately followed by Mangasarian's algorithm [4] for bimatrix games. Scarf [5], in 1967, developed an algorithm for the case of three or more players. Rosenmuller [6] generalized the Lemke-Howson algorithm in 1971 to the case of games with three or more players. In the same year, Wilson [7] proposed a new algorithm for computing equilibria of games with three or more players. All of these algorithms have a worst case running time that is exponential in the size of the strategy sets and number of players.

McKelvey and McLennan wrote in 1996 an excellent survey paper on equilibrium computation algorithms [8]. Katta Murty treats the complementarity problems in a comprehensive way in his book [1]. During the decade of 2000 - 2009, there was intense renewed activity on developing more efficient algorithms. Notable efforts include the works of Govindan and Wilson [9] who used a global Newton method; Porter, Nudelman, and Shoham [10]; and Sandholm, Gilpin, and Conitzer [11]. The well known journal *Economic Theory* published a special issue on computation of Nash equilibria in finite games edited by von Stengel in 20120 [12]. This special issue summarizes the current state-of-the-art on this problem by leading researchers. The edited volume by Nisan, Roughgarden, Tardos, and Vazirani [13] also has survey articles on this problem.

4.1 Software Tools

Surprisingly, there are not many software tools available for computational game theory. The most notable is the tool GAMBIT [14] which is powerful, user-friendly, and freely downloadable. This tool is useful for finite non-cooperative games (both extensive form and strategic form). The tool GAMUT [15] is also quite useful and freely downloadable. At the Indian Institute of Science, the tool NECTAR (Nash Equilibrium CompuTation Algorithms and Resources) [16] has been developed over the years and is available on request.

5 Problems

1. Find the mixed strategy Nash equilibria for the following game.

	H	T
H	1, 1	0, 1
T	1, 0	0, 0

2. Find the mixed strategy Nash equilibria for the following game.

	A	B
A	6, 2	0, 0
B	0, 0	2, 6

   If all these numbers are multiplied by 2, will the equilibria change?

3. Find the set of all mixed strategy Nash equilibria for the following two player game.

	A	B
A	20, 0	0, 10
B	0, 90	20, 0

4. Find all mixed strategy equilibria for the following game

	A	B	C
A	-3, -3	-1, 0	4, 0
B	0, 0	2, 2	3, 1
C	0,0	2, 4	3, 3

5. (*Myerson*). Find all mixed strategy equilibria for the following two player game

	x_2	y_2	z_2
x_1	0, 0	5, 4	4, 5
y_1	4, 5	0, 0	5, 4
z_1	5, 4	4, 5	0, 0

6. (*Fudenberg and Tirole*). Show that the following two player game has a unique mixed strategy Nash equilibrium.

	L	M	R
U	1, -2	-2, 1	0, 0
M	-2, 1	1, -2	0, 0
D	0,0	0, 0	1, 1

7. Show that the pure strategy profile (a_2, b_2) is the unique mixed strategy Nash equilibrium of the following game.

	2			
1	b_1	b_2	b_3	b_4
a_1	0,7	2,5	7,0	0,1
a_2	5,2	3,3	5,2	0,1
a_3	7,0	2,5	0,7	0,1
a_4	0,0	$0,-2$	0,0	$10,-1$

8. Let $G_1 = (N, S_1, S_2, u_1, u_2)$ and $G_2 = (N_1, S_1, S_2, v_1, v_2)$ be two strategic form games with $N = \{1, 2\}$. Let

$$v_1 = a_1 u_1 + b_1 J v_2 = a_2 u_2 + b_2 J$$

where $a_1, a_2 \in (0, \infty)$, $b_1, b_2 \in R$, and $J : X_1 \times S_2 \to R$ is the constant function on $S_1 \times S_2$ with value 1. Show that G_1 and G_2 have the same set of Nash equilibria.

9. Consider the following game

	2			
1	LL	L	M	R
U	100, 2	-100, 1	0,0	-100, -100
D	-100, -100	100, 49	1,0	100,2

What are the Nash equilibria of this game.

10. Compute all Nash equilibria for the following game for each $a \in (1, \infty)$

	2	
A	a, 0	1, $2 - a$
B	1, 1	0,0

11. (*Myerson*). For the following three player game in strategic form, find all mixed strategy equilibria. Note that $A_1 = \{x_1, y_1\}, A_2 = \{x_2, y_2\}, A_3 = \{x_3, y_3\}$.

	A_2 and A_3			
A_1	x_3		y_3	
	x_2	y_2	x_2	y_2
x_1	0, 0, 0	6, 5, 4	4, 6, 5	0, 0, 0
y_1	5, 4, 6	0, 0, 0	0, 0, 0	0, 0, 0

12. Consider the following game where the numbers a, b, c, d, k_1, k_2 are strictly positive real numbers.

	H_1	H_2
P_1	$a, -k_1 a$	$b, -k_1 b$
P_2	$c, -k_2 c$	$d, -k_2 d$

For the above two player non-zero sum game, write down the necessary and sufficient conditions for mixed strategy Nash equilibrium and compute all mixed strategy Nash equilibria.

13. Consider a 3 person game with $S_1 = S_2 = S_3 = \{1, 2, 3, 4\}$. If $u_(x, y, z) = x + y + z + 4i$ for each $i = 1, 2, 3$, show that the game has a unique Nash equilibrium.

14. Suppose

$$\begin{aligned} u_1(x, y, z) &= 10 \qquad \text{if } x = y = z \\ &= 0 \qquad \text{otherwise} \end{aligned}$$

Describe all pure Nash equilibria and show that mixed Nash equilibria lead to smaller payoffs than pure Nash equilibria.

15. Consider an n player game with $S_i = \{1, 2\}\ \ \forall i$. The payoff is

$$u_i(s_1, \ldots, s_n) = s_i \prod_{j \neq i} (1 - \delta(s_i, s_j))$$

where δ is the kronecker δ given by

$$\begin{aligned} \delta(s_i, s_j) &= 1 \qquad \text{if } s_i = s_j \\ &= 0 \qquad \text{otherwise} \end{aligned}$$

If player i uses a mixed strategy in which pure strategy 1 is chosen with probability p_i, prove that $(p_1, p_2, \ldots p_n)$ defines an equilibrium point iff

$$\prod_{j \neq i} (1 - p_j) = 2 \prod_{j \neq i} p_j \qquad \forall i \in N$$

Deduce that a mixed strategy equilibrium is given by

$$p_i = \frac{1}{1 + 2^{\frac{1}{n-1}}} \qquad \forall i \in N$$

and that for $n = 2, 3$ this is the only equilibrium points.

References

[1] K.G. Murty. *Linear Complementarity, Linear and Nonlinear Programming.* Helderman-Verlag, 1988.

[2] Roger B. Myerson. *Game Theory: Analysis of Conflict.* Harvard University Press, Cambridge, Massachusetts, USA, 1997.

[3] C. E. Lemke and J. T. Howson Jr. Equilibrium points of bi-matrix games. *Journal of the Society for Industrial and Applied Mathematics*, 12:413–423, 1964.

[4] O. L. Mangasarian. Equilibrium points of bi-matrix games. *Journal of the Society for Industrial and Applied Mathematics*, 12:778–780, 1964.

[5] Herbert E. Scarf. The core of an n-person game. *Econometrica*, 35:50–69, 1967.

[6] J. Rosenmuller. On a generalization of the lemke-howson algorithm to noncooperative n-person games. *SIAM Journal on Applied Mathematics*, 21:73–79, 1971.

[7] R. Wilson. Computing equilibria of n-person games. *SIAM Journal on Applied Mathematics*, 21:80–87, 1971.

[8] R. D. McKelvey and A. McLennan. Computation of equilibria in finite games. In J. Rust, H. Amman, and D. Kendrick, editors, *Handbook of Computational Economics*, pages 87–142. Elsevier, 1996.

[9] S. Govindan and R. Wilson. A global Newton method to compute nash equilibria. *Journal of Economic Theory*, 110(1):65–86, 2003.

[10] R. Porter, E. Nudelman, and Y. Shoham. Simple search methods for finding a nash equilibrium. *Games and Economic Behavior*, 63(2):642–662, 2008.

[11] T. Sandholm, A. Gilpin, and V. Conitzer. Mixed-integer programming methods for finding nash equilibria. In *Proceedings of the American Association of Artificial Intelligence, AAAI-2005*, pages 495–501, 2005.

[12] Bernard Von Stengel. Special issue on computation of nash equilibria in finite games. *Economic theory*, 42(1), 2010.

[13] Noam Nisan, Tim Roughgarden, Eva Tardos, and Vijay Vazirani (Editors). *Algorithmic Game Theory*. Cambridge University Press, 2007.

[14] The GAMBIT Project. *GAMBIT: Software*. www.gambit-project.org, 2005.

[15] E. Nudelman, J. Wortman, Y. Shoham, and K. Leyton-Brown. Run the gamut: A comprehensive approach to evaluating game-theoretic algorithms. In *Proceedings of the Third International Joint Conference on Autonomous Agents and Multiagent Systems, AAMAS-2004*, pages 880–887, 2004.

[16] Game Theory Laboratory. *NECTAR: Software*. Department of Computer Science and Automation, Indian Institute of Science, Bangalore, India. http://lcm.csa.iisc.ernet.in/nectar, 2010.