Game Theory

Lecture Notes By

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Chapter 1: Introduction to Game Theory

Note: *This is a only a draft version, so there could be flaws. If you find any errors, please do send email to* `hari@csa.iisc.ernet.in`*. A more thorough version would be available soon in this space.*

In this chapter, we will introduce a classical and well studied form of games called *strategic form games* and expose the readers to key notions in game theory such as *utilities*, *rationality*, *intelligence*, and *common knowledge*. We also present a brief discussion on different types of games.

Game theory may be defined as the study of mathematical models of interaction between rational, intelligent decision makers [1]. The decision makers are usually referred to as players or agents. The interaction may include both *conflict* and *cooperation*. Game theory provides general mathematical techniques for analyzing situations in which two or more players make decisions that influence one another's welfare. There are many categories of games that have been proposed and discussed in game theory. We first introduce a class of games called *strategic form games* or *normal form games*, which are perhaps the most commonly studied among all types of games.

Definition 1 (Strategic Form Game): *A strategic form game Γ is a tuple $\langle N, (S_i)_{i\in N}, (u_i)_{i\in N}\rangle$, where $N = \{1, 2, \ldots, n\}$ is a finite set of players; $S_1, S_2, \ldots, S_n$ are the strategy sets of the players $1, \ldots, n$, respectively; and $u_i : S_1 \times S_2 \times \cdots \times S_n \to \mathbb{R}$ for $i = 1, 2, \ldots, n$ are mappings called the utility functions or payoff functions.*

The strategies are also called *actions* or more specifically *pure strategies*. We denote by S, the Cartesian product $S_1 \times S_2 \times \cdots \times S_n$. The set S is the collection of all strategy profiles of the players. It is important to note that the utility of an agent depends not only on its own strategy but also on the strategies of the rest of the agents. Every profile of strategies produces or induces an *outcome* in the game. A strategic form game is said to be *finite* if N and all the strategy sets $S_1, \ldots, S_n$ are finite.

A strategic form game captures each agent's decision problem of choosing a strategy that will counter the strategies adopted by the other agents. Each player is faced with this problem and therefore the players can be thought of as simultaneously choosing their strategies from the respective sets $S_1, S_2, \ldots, S_n$. We can view the play of a strategic form game as follows: each player simultaneously writes down a chosen strategy on a piece of paper and hands it over to a referee who then computes the outcome and the utilities. we will be presenting several examples of strategic form games in Chapter 3.

1 Key Notions in Game Theory

There are certain key notions which are fundamental to game theory. We discuss these notions and a few related issues.

1.1 Utilities

Utilities (also called payoffs) enable the preferences of the players to be expressed in terms of real numbers in some utility scale. Utility theory is the science of assigning numbers to outcomes in a way that captures the preferences of the players. The theory is a foundational contribution of von Neumann and Morgenstern, who stated and proved in [2] a crucial result called the *expected utility maximization theorem*. This theorem establishes for any rational decision maker that there must exist a way of assigning utility numbers to different outcomes in a way that the decision maker would always choose the option that maximizes his expected utility. This theorem holds under quite weak assumptions about how a rational decision maker should behave.

1.2 Intelligence

Another key notion in game theory is that of intelligence of the players. This notion connotes that each player in the game knows everything about the game that a game theorist knows, and the player is competent enough make any inferences about the game that a game theorist can make. In particular, an intelligent player is *strategic*, that is, would fully take into account his knowledge or expectation of behavior of other agents in determining what his best response strategy should be. Each player is assumed to have enough resources to carry out the required computations involved in determining a best response strategy.

Myerson [1] and several other authors provide the following convincing explanation to show that the two assumptions of rationality and intelligence are indeed logical and reasonable. The assumption that all individuals are rational and intelligent may not exactly be satisfied in a typical real-world situation. However, any theory that is not consistent with the assumptions of rationality and intelligence loses credibility on the following count: If a theory predicts that some individuals will be systematically deceived into making mistakes, then such a theory will lose validity when individuals learn through mistakes to understand the situations better. On the other hand, a theory based on rationality and intelligence assumptions would be sustainable.

1.3 Common Knowledge

The notion of common knowledge is an important implication of *intelligence*. Aumann [3] defines *common knowledge* as follows: A fact is common knowledge among the players if every player knows it, every player knows that every player knows it, and so on. That is, every statement of the form "every player knows that every player knows that $\cdots$ every player knows it" is true ad infinitum. If it happens that a fact is known to all the players, without the requirement of all players knowing that all players know it, etc., then such a fact is called *mutual knowledge*. In game theory, analysis often requires the assumption of common knowledge to be true; however, sometimes, the assumption of mutual knowledge suffices for the analysis. A player's *private information* is any information that the player has that is not common knowledge among all the players.

The intelligence assumption means that whatever a game theorist knows about the game must be known or understood by the players of the game. Thus the model of the game is also known to the players. Since all the players know the model and they are intelligent, they also know that they all know the model. Thus the model is common knowledge.

In a *strategic form game with complete information*, the set N, the strategy sets $S_1, \ldots, S_n$, and the utility functions $u_1, \ldots, u_n$ are common knowledge, that is every player knows them, every player knows that every player knows them, and so on. We will be studying strategic form games with complete information in this and the next few chapters. We will study games with incomplete information in Chapter 10.

1.3.1 An Example for Illustrating Common Knowledge

This example is a variant of the one presented by Myerson [1]. Assume that there are five rational and intelligent mothers A, B, C, D, and E and let a, b, c, d, and e be their daughters (or sons), respectively. The kids go the school every day, escorted by their respective mothers and the mothers get an opportunity everyday to indulge in some conservation. The conversation invariably centers around the performance and behavior of the kids. Everyday when the five mothers meet, the conversation protocol is the following. If a mother thinks her kid is *well behaved*, she will praise the virtues of her kid. On the other hand, if a mother knows that her kid is *not well behaved*, she will cry. All mothers follow this protocol.

The fact is that none of the kids is well behaved but their behaviors is unknown to their respective mothers. However, whenever a mother finds that the kid of another mother is not well behaved, she would immediately report it to all mothers except the kid's mother. For example, if A finds b badly behaved, then A would report it to C,D, and E, but not to B. This protocol is also known to all the mothers. Since none of the kids is well behaved, the fact that a kid is not well behaved is common knowledge among all the mothers except the kid's own mother.

Since each mother does not know that her kid is badly behaved, it turns out that every mother keeps praising her kid everyday. On a fine day, the class teacher meets all the mothers and makes the following statement: "one of the boys is not well behaved." Thus the fact that one of the boys is not a well behaved is now common knowledge among all the mothers. Subsequently, when the five mothers meet the next day, all of them praise their respective kids; the same happens on the 2^{nd} day, 3^{rd}, and the 4^{th} day. On the 5^{th} day, however, all the mothers cry together because all of them realize that their respective kids are not well behaved.

Note that the announcement made by the class teacher is common knowledge and that is what makes all the mothers cry on the fifth day.

1.4 Bounded Rationality

Osborne and Rubinstein [4] have the following to say regarding the important notion of *bounded rationality*.

- Game theory, in its most common form, assumes that all the players are symmetric; that is, they have identical capabilities of perception and computation. It does not model asymmetries in abilities or perceptions of situations.

- A good example of this is the game of chess. When analyzed using game theory, the game of chess can be solved using an algorithm. This was in fact shown by Zermelo in 1913. Thus chess becomes a trivial game for rational players. Zermelo's result is that the game of chess has a unique *equilibrium* outcome with the property that a player who follows the suggested strategy is guaranteed that the outcome will be at least as good as the equilibrium outcome. Only the existence of the equilibrium outcome has been shown by Zermelo. In fact, the actual equilibrium outcome is yet to be computed. A game theoretic model of chess therefore reveals an important fact about the game and suggests that it is not at all interesting for rational players. However, the game theoretic model does not capture the asymmetric abilities of the players which is what makes chess an interesting game.

Bounded rationality grapples with the problem of asymmetric abilities of the players and the fact that the players do not have infinite computational resources at their disposal.

2 Classification of Games

Any subject like game theory which has been around for more than eight decades now will abound in numerous kinds of games being defined and studied. There are innumerable classes of games and we only provide a listing of some of the well known ones here.

Non-cooperative Games and Cooperative Games

Non-cooperative games are those in which the actions of individual players are the primitives; in cooperative games, joint actions of groups of players are the primitives. John Harsanyi (1966) [5] explained that a game is cooperative if commitments (agreements, promises, threats) among players are enforceable and that a game becomes non-cooperative if the commitments are not enforceable.

Different Representational Forms

- A strategic form game (also called simultaneous move game or normal from game), which we have got introduced to in this chapter, is a model or a situation where each player chooses the plan of action once and for all and all players exercise their decisions simultaneously.

- An extensive form game specifies a possible order of events and each player can consider his plan of action whenever a decision has to be made by him.

- A coalitional form game or characteristic form game is one where every subset of players of players is represented, by associating a value with each subset of players. This form is appropriate for cooperative games.

Games with Perfect Information and Games with Imperfect Information

When the players are fully informed about the entire past history (each player, before making a move, knows the past moves of all other players as well as his own past moves), the game is said to be of perfect information. Otherwise the game is said to be with imperfect information.

Complete Information and Incomplete Information

A game with incomplete information is one in which, at the first point in time when the players can begin to plan their moves, some players have private information about the game that other players do not know. In a game with complete information, every aspect of the game is common knowledge.

Other Categorizations

There are many other categories of games, such as dynamic games, repeated games, stochastic games, games with communication, multi-level games (Stackelberg games), differential games, etc. These games are beyond the scope of our discussion but can be looked up in the references at the end of this chapter.

3 To Probe Further

The material discussed in this chapter draws upon mainly from three sources, namely the books by Myerson [1], Mascolell, Whinston, and Green [6], and Osborne and Rubinstein [4].

The classic treatise by John von Neumann and Oskar Morgenstern [2], published in 1944, provides a comprehensive foundation for game theory. To this day, even after seven decades of its first appearance, the book continues to be a valuable reference.

The first chapter of Myerson's book [1] presents a detailed treatment of decision theory and utility theory. We will be treating utility theory in detail in Chapter 6.

For an undergraduate level treatment of game theory, we recommend the books by Osborne [7], Straffin [8], and Binmore [9]. For a graduate level treatment, we recommend the books by Myerson [1] and Osborne and Rubinstein [4].

References

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[3] Robert J. Aumann. Agreeing to disagree. *The Annals of Statistics*, 4(6):1236–1239, 1976.

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