# Game Theory

**Lecture Notes By**

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**July 2012**

## The Vickrey-Clarke-Groves Mechanisms

**Note**: *This is a only a draft version, so there could be flaws. If you find any errors, please do send email to* `hari@csa.iisc.ernet.in`*. A more thorough version would be available soon in this space.*

# 1 Groves Mechanisms

The main result in this section is that in the quasilinear environment, there exist social choice functions that are both allocatively efficient and dominant strategy incentive compatible. These are in general called the VCG (Vickrey–Clarke–Groves) mechanisms.

## 1.1 VCG Mechanisms

The VCG mechanisms are named after their famous inventors William Vickrey, Edward Clarke, and Theodore Groves. It was Vickrey who introduced the famous Vickrey auction (second price sealed bid auction) in 1961 [1]. To this day, the Vickrey auction continues to enjoy a special place in the annals of mechanism design. Clarke [2] and Groves [3] came up with a generalization of the Vickrey mechanisms and helped define a broad class of dominant strategy incentive compatible mechanisms in the quasilinear environment. VCG mechanisms are by far the most extensively used among quasilinear mechanisms. They derive their popularity from their mathematical elegance and the strong properties they satisfy.

## 1.2 The Groves' Theorem

The following theorem provides a sufficient condition for an allocatively efficient social function in quasilinear environment to be dominant strategy incentive compatible. We will refer to this theorem in the sequel as Groves theorem, rather than Groves' theorem.

**Theorem 1 (Groves Theorem)** *Let the SCF* $f(\cdot) = (k^*(\cdot), t_1(\cdot), \ldots, t_n(\cdot))$ *be allocatively efficient. Then* $f(\cdot)$ *is dominant strategy incentive compatible if it satisfies the following payment structure* *(popularly known as the Groves payment (incentive) scheme):*

$$t_i(\theta) = \left[\sum_{j \neq i} v_j(k^*(\theta), \theta_j)\right] + h_i(\theta_{-i}) \;\; \forall\, i = 1, \ldots, n \tag{1}$$

*where* $h_i : \Theta_{-i} \to \mathbb{R}$ *is any arbitrary function that honors the feasibility condition* $\sum_i t_i(\theta) \leq 0 \;\forall\, \theta \in \Theta$.

**Proof**: The proof is by contradiction. Suppose $f(\cdot)$ satisfies both allocative efficiency and the Groves payment structure but is not DSIC. This implies that $f(\cdot)$ does not satisfy the following necessary and sufficient condition for DSIC: $\forall i \in N \;\; \forall \theta \in \Theta$,

$$u_i(f(\theta_i, \theta_{-i}), \theta_i) \geq u_i(f(\theta_i^{'}, \theta_{-i}), \theta_i) \;\; \forall \theta_i^{'} \in \Theta_i \;\; \forall \theta_{-i} \in \Theta_{-i}.$$

This implies that there exists at least one agent $i$ for which the above is false. Let $i$ be one such agent. That is, for agent $i$,

$$u_i(f(\theta_i^{'}, \theta_{-i}), \theta_i) > u_i(f(\theta_i, \theta_{-i}), \theta_i)$$

for some $\theta_i \in \Theta_i$, for some $\theta_{-i} \in \Theta_{-i}$, and for some $\theta_i^{'} \in \Theta_i$. Thus, for agent $i$, there would exist $\theta_i \in \Theta_i, \theta_i^{'} \in \Theta_i, \theta_{-i} \in \Theta_{-i}$ such that

$$v_i(k^*(\theta_i^{'}, \theta_{-i}), \theta_i) + t_i(\theta_i^{'}, \theta_{-i}) + m_i > v_i(k^*(\theta_i, \theta_{-i}), \theta_i) + t_i(\theta_i, \theta_{-i}) + m_i.$$

Recall that

$$t_i(\theta_i, \theta_{-i}) = h_i(\theta_{-i}) + \sum_{j \neq i}(k^*(\theta_i, \theta_{-i}), \theta_j)$$

$$t_i(\theta_i^{'}, \theta_{-i}) = h_i(\theta_{-i}) + \sum_{j \neq i}(k^*(\theta_i^{'}, \theta_{-i}), \theta_j).$$

Substituting these, we get

$$v_i(k^*(\theta_i^{'}, \theta_{-i}), \theta_i) + \sum_{j \neq i} v_i(k^*(\theta_i^{'}, \theta_{-i}), \theta_j) > v_i(k^*(\theta_i, \theta_{-i}), \theta_i) + \sum_{j \neq i} v_i(k^*(\theta_i, \theta_{-i}), \theta_j),$$

which implies

$$\sum_{i=1}^{n} v_i(k^*(\theta_i^{'}, \theta_{-i}), \theta_i) > \sum_{i=1}^{n} v_i(k^*(\theta_i, \theta_{-i}), \theta_i).$$

The above contradicts the fact that $f(\cdot)$ is allocatively efficient. This completes the proof.

*Q.E.D.*

The following are a few interesting implications of the above theorem.

1. Given the announcements $\theta_{-i}$ of agents $j \neq i$, the monetary transfer to agent $i$ depends on his announced type only through effect of the announcement of agent $i$ on the project choice $k^*(\theta)$.

2. The change in the monetary transfer of agent $i$ when his type changes from $\theta_i$ to $\hat{\theta}_i$ is equal to the effect that the corresponding change in project choice has on total value of the rest of the agents. That is,

$$t_i(\theta_i, \theta_{-i}) - t_i(\hat{\theta}_i, \theta_{-i}) = \sum_{j \neq i} \left[v_j(k^*(\theta_i, \theta_{-i}), \theta_j) - v_j(k^*(\hat{\theta}_i, \theta_{-i}), \theta_j)\right].$$

   Another way of describing this is to say that the change in monetary transfer to agent $i$ reflects exactly the externality he is imposing on the other agents.

After the famous result of Groves, a direct revelation mechanism in which the implemented SCF is allocatively efficient and satisfies the Groves payment scheme is called a *Groves Mechanism.*

**Definition 1 (Groves Mechanisms)** *A direct mechanism,* $\mathscr{D} = ((\Theta_i)_{i\in N}, f(\cdot))$ *in which* $f(\cdot) = (k(\cdot), t_1(\cdot), \ldots, t_n(\cdot))$ *satisfies allocative efficiency (**??**) and Groves payment rule (1) is known as a Groves mechanism.*

In mechanism design parlance, Groves mechanisms are popularly known as Vickrey–Clarke–Groves (VCG) mechanisms because the Clarke mechanism is a special case of Groves mechanism, and the Vickrey mechanism is a special case of Clarke mechanism. We will discuss this relationship later in this monograph.

The Groves theorem provides a sufficiency condition under which an allocatively efficient (AE) SCF will be DSIC. The following theorem due to Green and Laffont [4] provides a set of conditions under which the condition of Groves Theorem also becomes a necessary condition for an AE SCF to be DSIC. In this theorem, we let $\mathscr{F}$ denote the set of all possible functions $f : K \to \mathbb{R}$.

**Theorem 2 (First Characterization Theorem of Green–Laffont)** *Suppose for each agent* $i \in N$ *that* $\{v_i(., \theta_i) : \theta_i \in \Theta_i\} = \mathscr{F}$, *that is, every possible valuation function from* $K$ *to* $\mathbb{R}$ *arises for some* $\theta_i \in \Theta_i$. *Then any allocatively efficient social choice function* $f(\cdot)$ *will be dominant strategy incentive compatible if and only if it satisfies the Groves payment scheme given by Equation (1).*

Note that in the above theorem, every possible valuation function from $K$ to $\mathbb{R}$ arises for any $\theta_i \in \Theta_i$. In the following characterization theorem, again due to Green and Laffont [4], $\mathscr{F}$ is replaced with with $\mathscr{F}_c$ where $\mathscr{F}_c$ denotes the set of all possible continuous functions $f : K \to \mathbb{R}$.

**Theorem 3 (Second Characterization Theorem of Green–Laffont)** *Suppose for each agent* $i \in N$ *that* $\{v_i(., \theta_i) : \theta_i \in \Theta_i\} = \mathscr{F}_c$, *that is, every possible continuous valuation function from* $K$ *to* $\mathbb{R}$ *arises for some* $\theta_i \in \Theta_i$. *Then any allocatively efficient social choice function* $f(\cdot)$ *will be dominant strategy incentive compatible if and only if it satisfies the Groves payment scheme given by Equation (1).*

## 1.3 Groves Mechanisms and Budget Balance

Note that a Groves mechanism always satisfies the properties of AE and DSIC. Therefore, if a Groves mechanism is budget balanced, then it will solve the problem of the social planner because it will then be ex-post efficient and dominant strategy incentive compatible. By looking at the definition of the Groves mechanism, one can conclude that it is the functions $h_i(\cdot)$ that decide whether or not the Groves mechanism is budget balanced. The natural question that arises now is whether there exists a way of defining functions $h_i(\cdot)$ such that the Groves mechanism is budget balanced. In what follows, we present one possibility result and one impossibility result in this regard.

### 1.3.1 Possibility and Impossibility Results for Quasilinear Environments

Green and Laffont [4] showed that in a quasilinear environment, if the set of possible types for each agent is sufficiently rich then ex-post efficiency and DSIC cannot be achieved together. The precise statement is given in the form of the following theorem.

**Theorem 4 (Green–Laffont Impossibility Theorem)** *Suppose for each agent* $i \in N$ *that* $\mathscr{F} = \{v_i(., \theta_i) : \theta_i \in \Theta_i\}$, *that is, every possible valuation function from* $K$ *to* $\mathbb{R}$ *arises for some* $\theta_i \in \Theta_i$. *Then there is no social choice function that is ex-post efficient and DSIC.*

Thus, the above theorem says that if the set of possible types for each agent is sufficiently rich then there is no hope of finding a way to define the functions $h_i(\cdot)$ in Groves payment scheme so that we have $\sum_{i=1}^{n} t_i(\theta) = 0$. However, one special case in which a positive result arises is summarized in the form of following possibility result.

**Theorem 5 (A Possibility Result for Budget Balance of Groves Mechanisms)** *If there is at least one agent whose preferences are known (that is, the type set is a singleton set) then it is possible to choose the functions $h_i(\cdot)$ so that $\sum_{i=1}^{n} t_i(\theta) = 0$.*

**Proof**: Let agent $i$ be such that his preferences are known, that is $\Theta_i = \{\theta_i\}$. In view of this condition, it is easy to see that for an allocatively efficient social choice function $f(\cdot) = (k^*(\cdot), t_1(\cdot), \ldots, t_n(\cdot))$, the allocation $k^*(\cdot)$ depends only on the types of the agents other than $i$. That is, the allocation $k^*(\cdot)$ is a mapping from $\Theta_{-i}$ to $K$. Let us define the functions $h_j(\cdot)$ in the following manner:

$$h_j(\theta_{-j}) = \begin{cases} h_j(\theta_{-j}) & : \quad j \neq i \\ -\sum_{r \neq i} h_r(\theta_{-r}) - (n-1)\sum_{r=1}^{n} v_r(k^*(\theta), \theta_r) & : \quad j = i. \end{cases}$$

It is easy to see that under the above definition of the functions $h_i(\cdot)$, we will have $t_i(\theta) = -\sum_{j \neq i} t_j(\theta)$. *Q.E.D.*

Figure 1 summarizes the main results of this section by showing what the space of social choice functions looks like in the quasilinear environment. The exhibit brings out various possibilities and impossibilities in the quasilinear environment, based on the results that we have discussed so far.

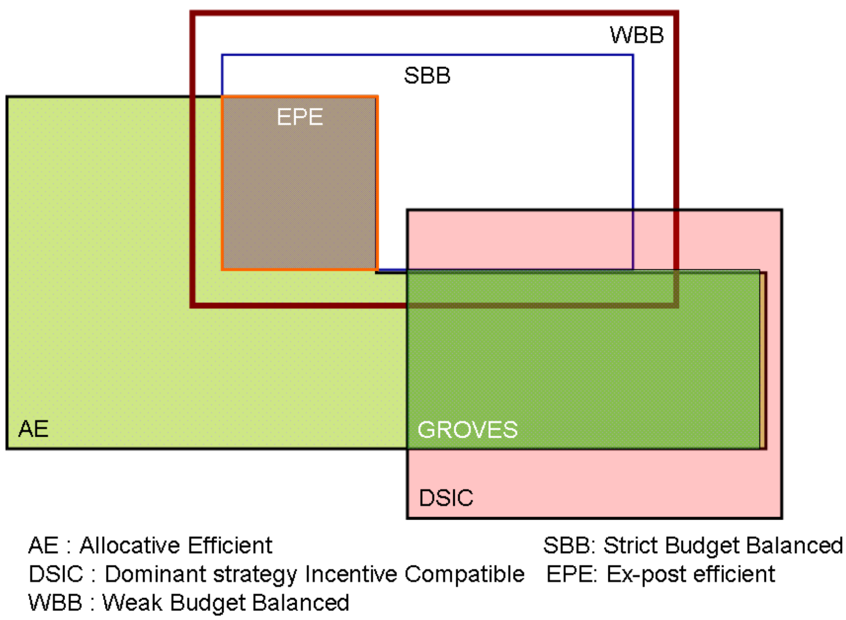


Figure 1: Space of social choice functions in quasilinear environment

## 2 Clarke (Pivotal) Mechanisms

A special case of Groves mechanism was developed independently by Clarke in 1971 [2] and is known as the *Clarke*, or the *pivotal* mechanism. It is a special case of Groves mechanisms in the sense of using a natural special form for the function $h_i(\cdot)$. In the Clarke mechanism, the function $h_i(\cdot)$ is given by the following relation:

$$h_i(\theta_{-i}) = -\sum_{j \neq i} v_j(k^*_{-i}(\theta_{-i}), \theta_j) \;\; \forall\, \theta_{-i} \in \Theta_{-i}, \forall\, i = 1, \ldots, n \tag{2}$$

where $k^*_{-i}(\theta_{-i}) \in K_{-i}$ is the choice of a project that is allocatively efficient if there were only the $n-1$ agents $j \neq i$. Formally, $k^*_{-i}(\theta_{-i})$ must satisfy the following condition.

$$\sum_{j \neq i} v_j(k^*_{-i}(\theta_{-i}), \theta_j) \geq \sum_{j \neq i} v_j(k, \theta_j) \;\forall\, k \in K_{-i} \tag{3}$$

where the set $K_{-i}$ is the set of project choices available when agent $i$ is absent. Substituting the value of $h_i(\cdot)$ from Equation (2) in Equation (1), we get the following expression for agent $i$'s transfer in the Clarke mechanism:

$$t_i(\theta) = \left[\sum_{j \neq i} v_j(k^*(\theta), \theta_j)\right] - \left[\sum_{j \neq i} v_j(k^*_{-i}(\theta_{-i}), \theta_j)\right]. \tag{4}$$

The above payment rule has an appealing interpretation: Given a type profile $\theta = (\theta_1, \ldots, \theta_n)$, the monetary transfer to agent $i$ is given by the total value of all agents other than $i$ under an efficient allocation when agent $i$ is present in the system minus the total value of all agents other than $i$ under an efficient allocation when agent $i$ is absent in the system.

### 2.1 Clarke Mechanisms and Weak Budget Balance

Recall from the definition of Groves mechanisms that, for weak budget balance, we should choose the functions $h_i(\theta_{-i})$ in a way that the weak budget balance condition $\sum_{i=1}^{n} t_i(\theta) \leq 0$ is satisfied. In this sense, the Clarke mechanism is a useful special case because it achieves weak budget balance under fairly general settings. In order to understand these general sufficiency conditions, we define following quantities

$$\begin{aligned}
B^*(\theta) &= \left\{ k \in K : k \in \begin{array}{c}\arg\max \\ k \in K\end{array} \sum_{j=1}^{n} v_j(k, \theta_j) \right\} \\
B^*(\theta_{-i}) &= \left\{ k \in K_{-i} : k \in \begin{array}{c}\arg\max \\ k \in K_{-i}\end{array} \sum_{j \neq i} v_j(k, \theta_j) \right\}
\end{aligned}$$

where $B^*(\theta)$ is the set of project choices that are allocatively efficient when all the agents are present in the system. Similarly, $B^*(\theta_{-i})$ is the set of project choices that are allocatively efficient if all agents except agent $i$ were present in the system. It is obvious that $k^*(\theta) \in B^*(\theta)$ and $k^*_{-i}(\theta_{-i}) \in B^*(\theta_{-i})$.

Using the above quantities, we define the following properties of a direct revelation mechanism in quasilinear environment.

**Definition 2 (No Single Agent Effect)** *We say that mechanism $\mathscr{M}$ has no single agent effect if for each agent $i$, for each $\theta \in \Theta$, and for each $k^*(\theta) \in B^*(\theta)$, we have a $k \in K_{-i}$ such that*

$$\sum_{j \neq i} v_j(k, \theta_j) \geq \sum_{j \neq i} v_j(k^*(\theta), \theta_j).$$

In view of the above properties, we have the following proposition that gives a sufficiency condition for Clarke mechanism to be weak budget balanced.

**Proposition 1** *If the Clarke mechanism has no single agent effect, then the monetary transfer to each agent would be non-positive, that is,* $t_i(\theta_i) \leq 0 \; \forall \; \theta \in \Theta$*;* $\forall \; i = 1, \ldots, n$*. In such a situation, the Clarke mechanism would satisfy the weak budget balance property.*

**Proof:** Note that by virtue of no single agent effect, for each agent $i$, each $\theta \in \Theta$, and each $k^*(\theta) \in B^*(\theta)$, there exists a $k \in K_{-i}$ such that

$$\sum_{j \neq i} v_j(k, \theta_j) \geq \sum_{j \neq i} v_j(k^*(\theta), \theta_j).$$

However, by definition of $k^*_{-i}(\theta_{-i})$, given by Equation (3), we have

$$\sum_{j \neq i} v_j(k^*_{-i}(\theta_{-i}), \theta_j) \quad \geq \quad \sum_{j \neq i} v_j(k, \theta_j) \; \forall \; k \in K_{-i}.$$

Combining the above two facts, we get

$$\begin{aligned}
\sum_{j \neq i} v_j(k^*_{-i}(\theta_{-i}), \theta_j) \quad &\geq \quad \sum_{j \neq i} v_j(k^*(\theta), \theta_j) \\
\Rightarrow 0 \quad &\geq \quad t_i(\theta) \\
\Rightarrow 0 \quad &\geq \quad \sum_{i=1}^{n} t_i(\theta).
\end{aligned}$$

*Q.E.D.*

In what follows, we present an interesting corollary of the above proposition.

**Corollary 1**

1. $t_i(\theta) = 0$ *iff* $k^*(\theta) \in B^*(\theta_{-i})$*. That is, agent i's monetary transfer is zero iff his announcement does not change the project decision relative to what would be allocatively efficient for agents* $j \neq i$ *in isolation.*

2. $t_i(\theta) < 0$ *iff* $k^*(\theta) \notin B^*(\theta_{-i})$*. That is, agent i's monetary transfer is negative iff his announcement changes the project decision relative to what would be allocatively efficient for agents* $j \neq i$ *in isolation. In such a situation, the agent i is known to be "pivotal" to the efficient project choice, and he pays a tax equal to his effect on the other agents.*

## 2.2 Clarke Mechanisms and Individual Rationality

We have studied individual rationality (also called voluntary participation) property in Section **??**. The following proposition investigates the individual rationality of the Clarke mechanism. First, we provide two definitions.

**Definition 3 (Choice Set Monotonicity)** *We say that a mechanism* $\mathscr{M}$ *is choice set monotone if the set of feasible outcomes X (weakly) increases as additional agents are introduced into the system. An implication of this property is* $K_{-i} \subset K \; \forall \; i = 1, \ldots, n$*.*

**Definition 4 (No Negative Externality)** *Consider a choice set monotone mechanism $\mathscr{M}$. We say that the mechanism $\mathscr{M}$ has no negative externality if for each agent $i$, each $\theta \in \Theta$, and each $k^*_{-i}(\theta_{-i}) \in B^*(\theta_{-i})$, we have*

$$v_i(k^*_{-i}(\theta_{-i}), \theta_i) \geq 0.$$

We now state and prove a proposition which provides a sufficient condition for the ex-post individual rationality of the Clarke mechanism. Recall from Section **??** the notation $\overline{u_i}(\theta_i)$, which represents the utility that agent $i$ receives by withdrawing from the mechanism.

**Proposition 2 (Ex-Post Individual Rationality of Clarke Mechanism )** *Let us consider a Clarke mechanism in which*

1. $\overline{u_i}(\theta_i) = 0 \; \forall \theta_i \in \Theta_i; \; \forall \; i = 1, \ldots, n,$
2. *The mechanism satisfies choice set monotonicity property,*
3. *The mechanism satisfies no negative externality property.*

*Then the Clarke mechanism is ex-post individual rational.*

**Proof:** Recall that utility $u_i(f(\theta), \theta_i)$ of an agent $i$ in Clarke mechanism is given by

$$\begin{aligned} u_i(f(\theta), \theta_i) &= v_i(k^*(\theta), \theta_i) + \left[ \sum_{j \neq i} v_j(k^*(\theta), \theta_j) \right] - \left[ \sum_{j \neq i} v_j(k^*_{-i}(\theta_{-i}), \theta_j) \right] \\ &= \left[ \sum_{j} v_j(k^*(\theta), \theta_j) \right] - \left[ \sum_{j \neq i} v_j(k^*_{-i}(\theta_{-i}), \theta_j) \right]. \end{aligned}$$

By virtue of choice set monotonicity, we know that $k^*_{-i}(\theta_{-i}) \in K$. Therefore, we have

$$\begin{aligned} u_i(f(\theta), \theta_i) &\geq \left[ \sum_{j} v_j(k^*_{-i}(\theta_{-i}), \theta_j) \right] - \left[ \sum_{j \neq i} v_j(k^*_{-i}(\theta_{-i}), \theta_j) \right] \\ &= v_i(k^*_{-i}(\theta_{-i}), \theta_i) \\ &\geq 0 = \overline{u_i}(\theta_i). \end{aligned}$$

The last step follows due to the fact that the mechanism has no negative externality. *Q.E.D.*

**Example 1 (Individual Rationality in Sealed Bid Auction)** Let us consider the example of first-price sealed bid auction. If for each possible type $\theta_i$, the utility $\overline{u_i}(\theta_i)$ derived by the agents $i$ from not participating in the auction is 0, then it is easy to see that the SCF used in this example would be ex-post IR.

Let us next consider the example of a second-price sealed bid auction. If for each possible type $\theta_i$, the utility $\overline{u_i}(\theta_i)$ derived by the agents $i$ from not participating in the auction is 0, then it is easy to see that the SCF used in this example would be ex-post IR. Moreover, the ex post IR of this example also follows directly from Proposition 2 because this is a special case of the Clarke mechanism satisfying all the required conditions in the proposition.

## 3 Examples of VCG Mechanisms

VCG mechanisms derive their popularity on account of the elegant mathematical and economic properties that they have and the revealing first level insights they provide during the process of designing mechanisms for a game theoretic problem. For this reason, invariably, mechanism design researchers try out VCG mechanisms first. However, VCG mechanisms do have many limitations. The virtues and limitations of VCG mechanisms are captured by Ausubel and Milgrom [5], whereas a recent paper by Rothkopf [6] summarizes the practical limitations of applying VCG mechanisms. In this section, we provide a number of examples to illustrate some interesting nuances of VCG mechanisms.

**Example 2 (Vickrey Auction for a Single Indivisible Item)** Consider 5 bidders $\{1, 2, 3, 4, 5\}$, with valuations $v_1 = 20; v_2 = 15; v_3 = 12; v_4 = 10; v_5 = 6$, participating in a sealed bid auction for a single indivisible item. If Vickrey auction is the mechanism used, then it is a dominant strategy for the agents to bid their valuations. Agent 1 with valuation 20 will be the winner, and the monetary transfer to agent 1

$$\begin{aligned} &= \sum_{j \neq 1} v_j(k^*(\theta), \theta_j) - \sum_{j \neq 1} v_j(k^*(\theta_{-1}), \theta_j) \\ &= 0 - 15 = -15. \end{aligned}$$

This means agent 1 would pay an amount equal to 15, which happens to be the second highest bid (in this case the second highest valuation). Note that 15 is the change in the total value of agents other than agent 1 when agent 1 drops out of the system. This is the externality that agent 1 imposes on the rest of the agents. This externality becomes the payment of agent 1 when he wins the auction.

Another way of determining the payment by agent 1 is to compute his marginal contribution to the system. The total value in the presence of agent 1 is 20, while the total value in the absence of agent 1 is 15. Thus the marginal contribution of agent 1 is 5. The above marginal contribution is given as a discount to agent 1 by the Vickrey payment mechanism, and agent 1 pays $20 - 5 = 15$. Such a discount is known as the *Vickrey discount*.

If we use the Clarke mechanism, we have

$$\begin{aligned} t_i(\theta_i, \theta_{-i}) &= \sum_{j \neq i} v_j(k^*(\theta), \theta_j) - \sum_{j \neq i} v_j(k^*_{-i}(\theta_{-i}), \theta_j) \\ &= \sum_{j \in N} v_j(k^*(\theta), \theta_j) - v_i(k^*(\theta), \theta_i) - \sum_{j \neq i} v_j(k^*_{-i}(\theta_{-i}), \theta_j) \\ &= \sum_{j \in N} v_j(k^*(\theta), \theta_j) - \sum_{j \neq i} v_j(k^*_{-i}(\theta_{-i}), \theta_j) - v_i(k^*(\theta), \theta_i). \end{aligned}$$

The difference in the first two terms represents the marginal contribution of agent $i$ to the system while the term $v_i(k^*(\theta), \theta_i)$ is the value received by agent $i$.

**Example 3 (Vickrey Auction for Multiple Identical Items)** Consider the same set of bidders as above but with the difference that there are 3 identical items available for auction. Each bidder wants only one item. If we apply the Clarke mechanism for this situation, bidders 1, 2, and 3 become the winners. The payment by bidder 1

$$
\begin{aligned}
&= \sum_{j \neq 1} v_j(k^*(\theta), \theta_j) - \sum_{j \neq 1} v_j(k^*_{-1}(\theta_{-1}), \theta_j) \\
&= (15 + 12) - (15 + 12 + 10) \\
&= -10.
\end{aligned}
$$

Thus bidder 1 pays an amount equal to the highest nonwinning bid. Similarly, one can verify that the payment to be made by the other two winners (namely agent 2 and agent 3) is also equal to 10. This payment is consistent with their respective marginal contributions.

$$\text{Marginal contribution of agent } 1 = (20 + 15 + 12) - (15 + 12 + 10) = 10$$

$$\text{Marginal contribution of agent } 2 = (20 + 15 + 12) - (20 + 12 + 10) = 5$$

$$\text{Marginal contribution of agent } 3 = (20 + 15 + 12) - (20 + 15 + 10) = 2.$$

In the above example, let the demand by agent 1 be 2 units with the rest of agents continuing to have unit demand. Now the allocation will allocate 2 units to agent 1 and 1 unit to agent 2.

$$\text{Payment by agent } 1 = 15 - (15 + 12 + 10) = -22$$

$$\text{Payment by agent } 2 = 40 - (40 + 12) = -12.$$

This is because the marginal contribution of agent 1 and agent 2 are given by: agent 1: $55 - 37 = 18$; agent 2: $55 - 52 = 3$.

**Example 4 (Generalized Vickrey Auction)** Generalized Vickrey auction (GVA) refers to an auction that results when the Clarke mechanism is applied to a combinatorial auction. A combinatorial auction is one where the bids correspond to bundles or combinations of different items. In a *forward combinatorial auction*, a bundle of different types of goods is available with the seller; the buyers are interested in purchasing certain subsets of the items. In a *reverse combinatorial auction*, a bundle of different types of goods is required by the buyer; several sellers are interested in selling subsets of the goods to the buyer. There is a rich body of literature on combinatorial auctions, for example see the edited volume [7]. We discuss a simple example here. Let a seller be interested in auctioning two items A and B. Let there be three buying agents $\{1, 2, 3\}$. Let us abuse the notation slightly and denote the subsets $\{A\}$, $\{B\}$, $\{A, B\}$ by A, B, and AB, respectively. These are called combinations or bundles. Assume that the agents have *valuations* for the bundles as shown in Table 1. In the above table, a "*" indicates that the agent is not interested in that bundle. Note from Table 1 that agent 1 values bundle AB at 10 and does not have any valuation for bundle A and bundle B. Agent 2 is only interested in bundle A and has a valuation of 5 for this bundle. Agent 3 is only interested in bundle B and has a valuation of 5 for this bundle. If we apply the Clarke mechanism to this situation, the bids from the agents will be identical to the valuations because of the DSIC property of the Clarke mechanism. There are two allocatively efficient allocations, namely: (1) Allocate bundle AB to agent 1; (2) Allocate bundle A to agent 2 and bundle B to agent 3. Each of these allocations has a total

| | A | B | AB |
|---|---|---|---|
| Agent 1 | * | * | 10 |
| Agent 2 | 5 | * | * |
| Agent 3 | * | 5 | * |

Table 1: Valuations of agents for bundles in scenario 1

| | A | B | AB |
|---|---|---|---|
| Agent 1 | * | * | 10 |
| Agent 2 | 10 | * | * |
| Agent 3 | * | 10 | * |

Table 2: Valuations of agents for bundles in scenario 2

value of 10. Suppose we choose allocation (2), which awards bundle A to agent 2 and bundle B to agent 3. To compute the payments to be made by agents 2 and 3, we have to use the Clarke payment rule. For this, we analyze what would happen in the absence of agent 2 and agent 3 separately. If agent 2 is absent, the allocation will award the bundle AB to agent 1 resulting in a total value of 10. Therefore, the Vickrey discount to agent 2 is $10 - 10 = 0$, which means payment to be made by agent 2 is $5 + 0 = 5$. Similarly the Vickrey discount to agent 3 is also 0 and the payment to be made by agent 3 is also equal to 5. The total revenue to the seller is $5 + 5 = 10$. Even if allocation (1) is chosen (that is, award bundle AB to agent 1), the total revenue to the seller remains as 10. This is a situation where the seller is able to capture the entire consumer surplus.

A contrasting situation will result if the valuations are as shown in Table 2. In this case, the winning allocation is: award bundle A to agent 2 and bundle B to agent 3, resulting in a total value of 20. If agent 2 is not present, the allocation will be to award bundle AB to agent 1, thus resulting in a total value of 10. Similarly, if agent 3 were not present, the allocation would be to award bundle AB to agent 1, thus resulting in a total value of 10. This would mean a Vickrey discount of 10 each to agent 2 and agent 3, which in turn means that the the payment to be made by agent 2 and agent 3 is 0 each! This represents a situation where the seller will end up with a zero revenue in the process of guaranteeing allocative efficiency and dominant strategy incentive compatibility. Worse still, if agent 2 and agent 3 are both the false names of a single agent, then the auction itself is seriously manipulated!

We now study a third scenario where the valuations are as described in Table 3. Here, the allocation is to award bundle AB to agent 1, resulting in a total value of 10. If agent 1 were absent, the allocation would be to award bundle A to agent 2 and bundle B to agent 3, which leads to a total value of 4.

| | A | B | AB |
|---|---|---|---|
| Agent 1 | * | * | 10 |
| Agent 2 | 2 | * | * |
| Agent 3 | * | 2 | * |

Table 3: Valuations of agents for bundles in scenario 3

The Vickrey discount to agent 1 is therefore $10 - 4 = 6$, and the payment to be made by agent 1 is 4. The revenue to the seller is also 4. Contrast this scenario with scenario 2, where the valuations of bidders 2 and 3 were higher, but they were able to win the bundles by paying nothing. This shows that the GVA mechanism is not foolproof against bidder collusion (in this case, bidders 2 and 3 can collude and deny the bundle to agent 1 and also seriously reduce the revenue to the seller.

**Example 5 (Strategy Proof Mechanism for the Public Project Problem)** Consider the public project problem discussed in Example **??**. We shall compute the Clarke payments by each agent for each type of profile. We will also compute the utilities. First consider the type profile (20,20). Since $k = 0$, the values derived by either agent is zero. Hence the Clarke payment by each agent is zero, and the utilities are also zero.

Next consider the type profile (60, 20). Note that $k(60, 20) = 1$. Agent 1 derives a value 35 and agent 2 derives a value $-5$. If agent 1 is not present, then agent 2 is left alone and the allocation will be 0 since its willingness to pay is 20. Thus the value to agent 2 when agent 1 is not present is 0. This means

$$t_1(60, 20) = -5 - 0 = -5.$$

This implies agent 1 would pay an amount of 5 units in addition to 25 units, which is its contribution to the cost of the project. The above payment is consistent with the marginal contribution of agent 1, which is equal to $(60 - 25) + (20 - 25) - 0 = 35 - 5 = 30$.

We can now determine the utility of agent 1, which will be

$$\begin{aligned} u_1(60, 20) &= v_1(60, 20) + t_1(60, 20) \\ &= 35 - 5 = 30. \end{aligned}$$

To compute $t_2(60, 20)$, we first note that the value to the agent 1 when agent 2 is not present is $(60 - 50)$. Therefore

$$t_2(60, 20) = 35 - 10 = 25.$$

This means agent 2 receives 25 units of money; of course, this is besides the 25 units of money it pays towards the cost of the project. Now

$$\begin{aligned} u_2(60, 20) &= v_2(60, 20) + t_2(60, 20) \\ &= -5 + 25 \\ &= 20. \end{aligned}$$

Likewise, we can compute the payments and utilities of the agents for all the type profiles. Table 4 provides these values. Note that this mechanism is ex-post individually rational assuming that the utility for not participating in the mechanism is zero.

| $(\theta_1, \theta_2)$ | $t_1(\theta_1, \theta_2)$ | $t_2(\theta_1, \theta_2)$ | $u_1(\theta_1, \theta_2)$ | $u_2(\theta_1, \theta_2)$ |
|---|---|---|---|---|
| (20, 20) | 0 | 0 | 0 | 0 |
| (60, 20) | -5 | 25 | 30 | 20 |
| (20, 60) | 25 | -5 | 20 | 30 |
| (60, 60) | 25 | 25 | 60 | 60 |

Table 4: Payments and utilities for different type profiles

**Example 6 (Strategy Proof Network Formation)** Consider the problem of forming a supply chain as depicted in Figure 2. The node S represents a starting state and T represents a target state; A and B are two intermediate states. $SP_1, SP_2, SP_3, SP_4$ are four different service providers. In the figure, the service providers are represented as owners of the respective edges. The cost of

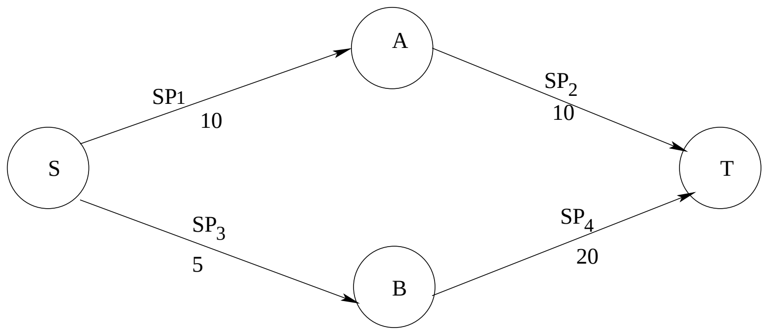


Figure 2: A network formation problem - case 1

providing service (willingness to sell) is indicated on each edge. The problem is to procure a path from S to T having minimum cost. Let $(y_1, y_2, y_3, y_4)$ represent the allocation vector. The feasible allocation vectors are

$$K = \{(1,1,0,0),(0,0,1,1)\}.$$

Among these, the allocation $(1,1,0,0)$ is allocatively efficient since it minimizes the cost of allocation. We shall define the value as follows:

$$v_i((y_1, y_2, y_3, y_4); \theta_i) = -y_i\theta_i.$$

The above manner of defining the values reflects the fact that cost minimization is the same as value maximization. Applying Clarke's payment rule, we obtain

$$\begin{aligned} t_1(\theta) &= -10 - (-25) = 15 \\ t_2(\theta) &= -10 - (-25) = 15. \end{aligned}$$

Note that each agent gets a surplus of 5, being its marginal contribution. The utilities for these two agents are

$$\begin{aligned} u_1(\theta) &= -10 + 15 = 5 \\ u_2(\theta) &= -10 + 15 = 5. \end{aligned}$$

The payments and utilities for $SP_3$ and $SP_4$ are zero. Let us study the effect of changing the willingness to sell of $SP_4$. Let us make it as 15. Then, we find that both the allocations $(1,1,0,0)$ and $(0,0,1,1)$ are allocatively efficient. If we choose the allocation $(1,1,0,0)$, we get the payments as

$$\begin{aligned} t_1(\theta) &= 10 \\ t_2(\theta) &= 10 \\ u_1(\theta) &= 0 \\ u_2(\theta) &= 0. \end{aligned}$$

This means that the payments to the service providers are equal to the costs. There is no surplus payment to the winning agents. In this case, the mechanism is friendly to the buyer and unfriendly to the sellers.

If we make the willingness to sell of $SP_4$ as 95, the allocation $(1, 1, 0, 0)$ is efficient and we get the payments as

$$\begin{aligned} t_1(\theta) &= 90 \\ t_2(\theta) &= 90 \\ u_1(\theta) &= 80 \\ u_2(\theta) &= 80. \end{aligned}$$

In this case, the mechanism is extremely unfriendly to the buyer but is very attractive to the sellers.

Let us introduce one more edge from B to A and see the effect. See Figure 3.

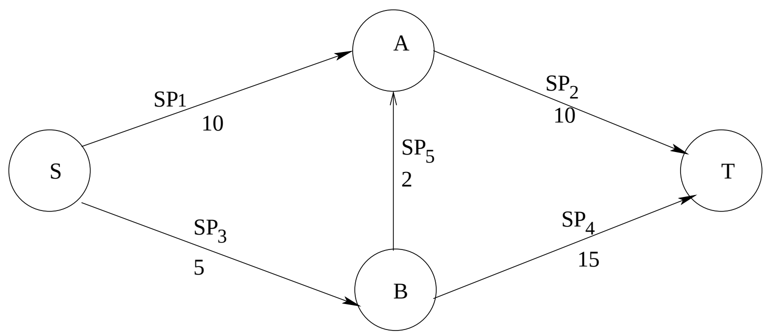


Figure 3: A network formation problem - case 2

The efficient allocation here is (0, 1, 1, 0, 1). The payments are

$$\begin{aligned} t_2(\theta) &= 13 \\ t_3(\theta) &= 8 \\ t_5(\theta) &= 5 \\ u_2(\theta) &= 3 \\ u_3(\theta) &= 3 \\ u_2(\theta) &= 3. \end{aligned}$$

This shows that the total payments to be made by the buyer is 26 whereas the total payment if the service provider $SP_5$ were absent is 20. Thus in spite of an additional agent being available, the payment to the buyer is higher. This shows a kind of non-monotonicity exhibited by the Clarke payment rule.

**Example 7 (A Groves (but not Clarke) Mechanism)** Consider a sealed bid auction for a single indivisible item where the bidder with the highest bid is declared as the winner and the winner pays an amount equal to equal to twice the bid of the bidder with the lowest valuation among the rest of the agents. Such a payment rule is not a Clarke payment rule but does belong to the Groves payment scheme.

As an example consider 5 bidders with valuations 20, 15, 12, 10, 8. The bidder with valuation 20 is declared the winner and will pay an amount = 16. On the other hand, if there are only three bidders with values 20, 15, 12, the first bidder wins but has to pay 24. It is clear that this is not individually rational.

## 4 Problems

1. Apply the generalized Vickrey auction to the following combinatorial auction scenario. There are three bidders and two objects. The valuation matrix is as follows.

```
            {1}    {2}   {1,2}

Bidder 1    6      10    10
Bidder 2    8      5     8
Bidder 3    8      0     9
```

2. The Clarke mechanism is used by a buyer for procuring a bundle $\{A, B, C, D, E\}$. The following are the bids received from 5 sellers.

   - Seller 1: (A, 20), (B, 30), (AB, 45)
   - Seller 2: (B, 25), (C, 35), (BC, 50)
   - Seller 3: (C, 30), (D, 40), (CD, 60)
   - Seller 4: (D, 35), (E, 45), (DE, 70)
   - Seller 5: (E, 40), (A, 15), (EA, 50)

   Assume XOR bids (that is, at most one bid will be selected from any seller). Compute the allocation and the payments that the winning bidders will receive. Is this mechanism individually rational. Why?

3. Consider a single-item, multi-unit exchange with two selling agents and two buying agents. The selling agents specify marginally decreasing, piecewise constant *asks* and buying agents specify marginally decreasing, piecewise constant *bids*. It is required to maximize the total surplus (total receipts minus total payments by the exchange). The asks received from the sellers are:

```
Ask-1: ((1-50, 10), (51-150, 8), (151-200, 6)) /* unit price of Rs. 10
       for the 50 items; unit price of Rs. 8 for the next 100 items;
       unit price of Rs. 6 for the next 50 items) */

Ask-2: ((1-50, 12), (51-100, 10), (101-200, 6))
```

   The bids received from the buying agents are:

```
Bid-1: ((1-50, 12), (51-100, 11), (101-150, 10))

Bid-2: ((1-75, 12), (76-150, 10))
```

Assume that Clarke mechanism is used. For this problem:

- Compute a surplus maximizing allocation for the given bids
- What is the payment required to be made by the two buyers.
- What is the payment received by the two sellers.

4. Consider the following situation. It is required to lease a certain quantity of telecom bandwidth from Bangalore to Delhi. There are two service providers in the game $\{1, 2\}$. Provider 1 offers a direct service from Bangalore to New Delhi with a bid of 100 million rupees. Provider 1 also offers a service from Mumbai to Delhi with a bid of 30 million rupees. On the other hand, Provider 2 can only offer a service from Bangalore to Mumbai and the bid is 50 million rupees. Design a dominant strategy incentive compatible mechanism (allocation rule and payment rule) for the above situation assuming standard quasi-linear utilities.

5. Consider a forward auction for sale of $m$ identical objects. Let there be $n$ bidders where $n > m$. The valuations of the bidders for the object are $v_1$, ..., $v_n$, respectively. Each bidder is interested in at most one unit. For this auction scenario, write down an allocation rule that is allocatively efficient. What will be the Clarke payment in this case? Do you see any difficulty. How can you overcome the difficulty, if any?

6. Consider a second price auction. Suppose each bidder $i$ has a value $v_i$ and a budget $c_i$. If a bidder wins the object and has to pay more than the budget, the bidder will renege and is charged with a small penalty $\epsilon > 0$. Show that it is a weakly dominant strategy to bid $min(v_i, c_i)$ in the auction.

7. (Second price auction with budget). Consider a second price auction for a single indivisible item. Suppose each bidder $i$ has a value $v_i > 0$ and a budget $c_i > 0$. If a bidder wins the object and has to pay higher than the budget, the bidder will simply drop out from the auction but is charged with a small penalty $\epsilon > 0$. Compute a bid in the auction for each player $i$ which will be a weakly dominant strategy for the player.

8. Provide simple examples for the following:

   - A Groves mechanism that is not a Clarke mechanism.
   - A Clarke mechanism that is not individually rational.
   - A mechanism that is DSIC and SBB but not allocatively efficient
   - A mechanism that is BIC, AE, IR but not SBB
   - A Clarke mechanism that is not even weakly budget balanced

9. Consider a sealed bid auction for a single indivisible item where the bidders with the highest bidder is declared as the winner pays an amount equal to equal to twice the valuation of the bidder with the lowest valuation among the rest of the agents. Is this a Clarke mechanism. Is this individually rational.

10. Consider the following reverse auction. A buyer wants to buy $m$ identical objects. There are $n$ selling agents ($n > m$) with values $v_1, \ldots, v_n$, each of whom can supply exactly one unit. The auction uses Clarke mechanism. What would each seller get in this auction as payment. Do you see any difficulty. Can you design a better mechanism from the sellers' viewpoint. The new mechanism also should be DSIC and AE.

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