Game Theory

Lecture Notes By

Y. Narahari

Department of Computer Science and Automation

Indian Institute of Science

Bangalore, India

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The quasilinear Environment

Note: *This is a only a draft version, so there could be flaws. If you find any errors, please do send email to* `hari@csa.iisc.ernet.in`*. A more thorough version would be available soon in this space.*

1 The Quasilinear Environment

This is the most extensively studied special class of environments where the Gibbard–Satterthwaite theorem does not hold. In fact, the rest of this chapter assumes this environment most of the time. In the quasilinear environment, an alternative $x \in X$ is a vector of the form $x = (k, t_1, \ldots, t_n)$, where k is an element of a set K, which is called the set of project choices or set of allocations. The set K is usually assumed to be finite. The term $t_i \in \mathbb{R}$ represents the monetary transfer to agent i. If $t_i > 0$ then agent i will receive the money and if $t_i < 0$ then agent i will pay the money. We assume that we are dealing with a system in which the n agents have no external source of funding, i.e., $\sum_{i=1}^{n} t_i \leq 0$. This condition is known as the *weak budget balance* condition. The set of alternatives X is therefore

$$X = \left\{ (k, t_1, \ldots, t_n) : k \in K;\ t_i \in \mathbb{R}\ \ \forall\, i \in N;\ \ \sum_i t_i \leq 0 \right\}.$$

A social choice function in this quasilinear environment takes the form $f(\theta) = (k(\theta), t_1(\theta), \ldots, t_n(\theta))$ where, for every $\theta \in \Theta$, we have $k(\theta) \in K$ and $\sum_i t_i(\theta) \leq 0$. Note that here we are using the symbol k both as an element of the set K and as a function going from Θ to K. It should be clear from the context as to which of these two we are referring. For a direct revelation mechanism $\mathscr{D} = ((\Theta_i)_{i \in N}, f(\cdot))$ in this environment, the agent i's utility function takes the quasilinear form

$$u_i(x, \theta_i) = u_i((k, t_1, \ldots, t_n), \theta_i) = v_i(k, \theta_i) + m_i + t_i$$

where m_i is agent i's initial endowment of the money and the function $v_i(\cdot)$ is known as agent i's valuation function. Recall from our discussion of mechanism design environment (Section **??**) that the utility functions $u_i(\cdot)$ are common knowledge. In the context of a quasilinear environment, this implies that for any given type θ_i of any agent i, the social planner and every other agent j have a way to know the function $v_i(., \theta_i)$. In many cases, the set Θ_i of the direct revelation mechanism

$\mathscr{D} = ((\Theta_i)_{i\in N}, f(\cdot))$ is actually the set of all feasible valuation functions v_i of agent i. That is, each possible function represents the possible types of agent i. Therefore, in such settings, reporting a type is the same as reporting a valuation function.

Immediate examples of quasilinear environment include many of the previously discussed examples, such as the first price and second price auctions (Example **??**), the public project problem (Example **??**), the network formation problem (Example **??**), bilateral trade (Example **??**), etc. In the quasilinear environment, we can define two important properties of a social choice function, namely, allocative efficiency and budget balance.

Definition 1.1 (Allocative Efficiency (AE)) *We say that a social choice function $f(\cdot) = (k(\cdot), t_1(\cdot), \ldots, t_n(\cdot))$ is allocatively efficient if for each $\theta \in \Theta$, $k(\theta)$ satisfies the following condition*[1]

$$k(\theta) \in \underset{k \in K}{\arg\max} \sum_{i=1}^{n} v_i(k, \theta_i). \tag{1}$$

Equivalently,

$$\sum_{i=1}^{n} v_i(k(\theta), \theta_i) = \max_{k \in K} \sum_{i=1}^{n} v_i(k, \theta_i).$$

The above definition implies that for every $\theta \in \Theta$, the allocation $k(\theta)$ will maximize the sum of the values of the players. In other words, every allocation is a value maximizing allocation, or the objects are allocated to the players who value the objects most. This is an extremely desirable property to have for any social choice function. The above definition implicitly assumes that for any given θ, the function $\sum_{i=1}^{n} v_i(., \theta_i) : K \to \mathbb{R}$ attains a maximum over the set K.

Example 1 (Public Project Problem) Consider the public project problem with two agents $N = \{1, 2\}$. Let the cost of the public project be 50 units of money. Let the type sets of the two players be given by

$$\Theta_1 = \Theta_2 = \{20, 60\}.$$

Each agent either has a low willingness to pay, 20, or a high willingness to pay, 60. Let the set of project choices be

$$K = \{0, 1\}$$

with 1 indicating that the project is taken up and 0 indicating that the project is dropped.

Assume that if $k = 1$, then the two agents will equally share the cost of the project by paying 25 each. If $k = 0$, the agents do not pay anything. A reasonable way of defining the valuation function would be

$$v_i(k, \theta_i) = k(\theta_i - 25).$$

This means, if $k = 0$, the agents derive zero value while if $k = 1$, the value derived is willingness to pay minus 25.

Define the following allocation function:

$$\begin{aligned} k(\theta_1, \theta_2) &= 0 \text{ if } \theta_1 = \theta_2 = 20 \\ &= 1 \text{ otherwise.} \end{aligned}$$

[1] *We will be using the symbol $k^*(\cdot)$ for a function $k(\cdot)$ that satisfies Equation (1).*

This means, the project is taken up only when at least one of the agents has a high willingness to pay. We can see that this function is allocatively efficient. This may be easily inferred from Table 1, which shows the values derived by the agents for different type profiles. The second column gives the actual value of k.

(θ_1, θ_2)	k	$v_1(k, \theta_1)$ when $k = 0$	$v_2(k, \theta_2)$ when $k = 0$	$v_1(k, \theta_1)$ when $k = 1$	$v_2(k, \theta_2)$ when $k = 1$
$(20, 20)$	0	0	0	-5	-5
$(20, 60)$	1	0	0	-5	35
$(60, 20)$	1	0	0	35	-5
$(60, 60)$	1	0	0	35	35

Table 1: Values for different type profiles when $v_i(k, \theta_i) = k(\theta_i - 25)$

Example 2 (A Non-Allocatively Efficient SCF) Let the v function be defined as under:

$$v_i(k, \theta_i) = k\theta_i \qquad i = 1, 2.$$

With respect to the above function, the allocation function k defined in the previous example can be seen to be not allocatively efficient. The values for different type profiles are shown in Table 2. If the type profile is $(20, 20)$, the allocation is $k = 0$ and the total value of allocation is 0. However, the total value is 40 if the allocation were $k = 1$.

(θ_1, θ_2)	k	$v_1(k, \theta_1)$ when $k = 0$	$v_2(k, \theta_2)$ when $k = 0$	$v_1(k, \theta_1)$ when $k = 1$	$v_2(k, \theta_2)$ when $k = 1$
$(20, 20)$	0	0	0	20	20
$(20, 60)$	1	0	0	20	60
$(60, 20)$	1	0	0	60	20
$(60, 60)$	1	0	0	60	60

Table 2: Values for different type profiles when $v_i(k, \theta_i) = k\theta_i$

Definition 1.2 (Budget Balance (BB)) *We say that a social choice function $f(\cdot) = (k(\cdot), t_1(\cdot), \ldots, t_n(\cdot))$ is budget balanced if for each $\theta \in \Theta$, $t_1(\theta), \ldots, t_n(\theta)$ satisfy the following condition:*

$$\sum_{i=1}^{n} t_i(\theta) = 0. \tag{2}$$

Many authors prefer to call this property *strong budget balance*, and they refer to the property of having $\sum_{i=1}^{n} t_i(\theta) \leq 0$ as *weak budget balance*. In this monograph, we will use the term budget balance to refer to strong budget balance.

Budget balance ensures that the total receipts are equal to total payments. This means that the system is a closed one, with no surplus and no deficit. The weak budget balance property means that the total payments are greater than or equal to total receipts.

The following lemma establishes an important relationship of these two properties of an SCF with the ex-post efficiency of the SCF.

Lemma 1.1 *A social choice function $f(\cdot) = (k(\cdot), t_1(\cdot), \ldots, t_n(\cdot))$ is ex-post efficient in quasilinear environment if and only if it is allocatively efficient and budget balanced.*

Proof: Let us assume that $f(\cdot) = (k(\cdot), t_1(\cdot), \ldots, t_n(\cdot))$ is allocatively efficient and budget balanced. This implies that for any $\theta \in \Theta$, we have

$$\begin{aligned}
\sum_{i=1}^{n} u_i(f(\theta), \theta_i) &= \sum_{i=1}^{n} v_i(k(\theta), \theta_i) + \sum_{i=1}^{n} t_i(\theta) \\
&= \sum_{i=1}^{n} v_i(k(\theta), \theta_i) + 0 \\
&\geq \sum_{i=1}^{n} v_i(k, \theta_i) + \sum_{i=1}^{n} t_i; \;\; \forall \, x = (k, t_1, \ldots, t_n) \\
&= \sum_{i=1}^{n} u_i(x, \theta_i); \;\; \forall \, (k, t_1, \ldots, t_n) \in X.
\end{aligned}$$

That is if the SCF is allocatively efficient and budget balanced then for any type profile θ of the agent, the outcome chosen by the social choice function will be such that it maximizes the total utility derived by all the agents. This will automatically imply that the SCF is ex-post efficient.

To prove the other part, we will first show that if $f(\cdot)$ is not allocatively efficient, then, it cannot be ex-post efficient and next we will show that if $f(\cdot)$ is not budget balanced then it cannot be ex-post efficient. These two facts together will imply that if $f(\cdot)$ is ex-post efficient then it will have to be allocatively efficient and budget balanced, thus completing the proof of the lemma.

To start with, let us assume that $f(\cdot)$ is not allocatively efficient. This means that $\exists \, \theta \in \Theta$, and $k \in K$ such that

$$\sum_{i=1}^{n} v_i(k, \theta_i) > \sum_{i=1}^{n} v_i(k(\theta), \theta_i).$$

This implies that there exists at least one agent j for whom $v_j(k, \theta_i) > v_j(k(\theta), \theta_i)$. Now consider the following alternative x

$$x = \Big(k, (t_i = t_i(\theta) + v_i(k(\theta), \theta_i) - v_i(k, \theta_i))_{i \neq j}, t_j = t_j(\theta)\Big).$$

It is easy to verify that $u_i(x, \theta_i) = u_i(f(\theta), \theta_i) \; \forall \; i \neq j$ and $u_j(x, \theta_i) > u_j(f(\theta), \theta_i)$, implying that $f(\cdot)$ is not ex-post efficient.

Next, we assume that $f(\cdot)$ is not budget balanced. This means that there exists at least one agent j for whom $t_j(\theta) < 0$. Let us consider the following alternative x

$$x = \Big(k, (t_i = t_i(\theta))_{i \neq j}, t_j = 0\Big).$$

It is easy to verify that for the above alternative x, we have $u_i(x, \theta_i) = u_i(f(\theta), \theta_i) \; \forall \; i \neq j$ and $u_j(x, \theta_i) > u_j(f(\theta), \theta_i)$ implying that $f(\cdot)$ is not ex-post efficient.

Q.E.D.

The next lemma summarizes another fact about social choice functions in quasilinear environment.

Lemma 1.2 *All social choice functions in quasilinear environments are nondictatorial.*

Proof: If possible, assume that a social choice function, $f(\cdot)$, is dictatorial in the quasilinear environment. This means that there exists an agent called the dictator, say $d \in N$, such that for each $\theta \in \Theta$, we have

$$u_d(f(\theta), \theta_d) \geq u_d(x, \theta_d) \; \forall \, x \in X.$$

However, because of the environment being quasilinear, we have $u_d(f(\theta), \theta_d) = v_d(k(\theta), \theta_d) + t_d(\theta)$. Now consider the following alternative $x \in X$:

$$x = \begin{cases} (k(\theta), (t_i = t_i(\theta))_{i \neq d}, t_d = t_d(\theta) - \sum_{i=1}^{n} t_i(\theta)) & : \quad \sum_{i=1}^{n} t_i(\theta) < 0 \\ (k(\theta), (t_i = t_i(\theta))_{i \neq d,j}, t_d = t_d(\theta) + \epsilon, t_j = t_j(\theta) - \epsilon) & : \quad \sum_{i=1}^{n} t_i(\theta) = 0 \end{cases}$$

where $\epsilon > 0$ is any arbitrary number, and j is any agent other than d. It is easy to verify, for the above outcome x, that we have $u_d(x, \theta_d) > u_d(f(\theta), \theta_d)$, which contradicts the fact that d is a dictator.

Q.E.D.

In view of Lemma 1.2, the social planner need not have to worry about the nondictatorial property of the social choice function in quasilinear environments and he can simply look for whether there exists any SCF that is both ex-post efficient and dominant strategy incentive compatible. Furthermore, in the light of Lemma 1.1, we can say that the social planner can look for an SCF that is allocatively efficient, budget balanced, and dominant strategy incentive compatible. Once again the question arises whether there could exist social choice functions which satisfy all these three properties — AE, BB, and DSIC. We explore this and other questions in the forthcoming sections.

2 Problems

1. Let $f : \Theta_1 \times \ldots \times \Theta_n \to X$ be a social choice function. If the utility functions are quasi-linear of the form
$$u_i(k^*(\theta), t_1(\theta), \ldots, t_n(\theta), \theta_i) = v_i(k^*(\theta), \theta_i) + t_i(\theta) + m_i,$$
then show that f is ex-post efficient iff f is allocatively efficient and strictly budget balanced.

2. Show that no social choice function in the quasi-linear environment can be dictatorial.