Game Theory

Lecture Notes By

Y. Narahari

Department of Computer Science and Automation

Indian Institute of Science

Bangalore, India

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Social Choice Functions and Mechanisms

Note: *This is a only a draft version, so there could be flaws. If you find any errors, please do send email to* `hari@csa.iisc.ernet.in`*. A more thorough version would be available soon in this space.*

1 Examples of Social Choice Functions

Example 1 (Technology Driven Supplier Selection) Suppose there is a buyer who wishes to procure a certain volume of an item that is produced by two suppliers, call them 1 and 2. We have $N = \{1, 2\}$. Supplier 1 is known to use technology a_1 to produce these items, while supplier 2 uses one of two possible technologies, a high end technology a_2 and a low end technology b_2. The technology a_2 is known to be superior to a_1 also. The technology elements could be taken as the types of the suppliers, so we have $\Theta_1 = \{a_1\}$; $\Theta_2 = \{a_2, b_2\}$. See Figure 1.

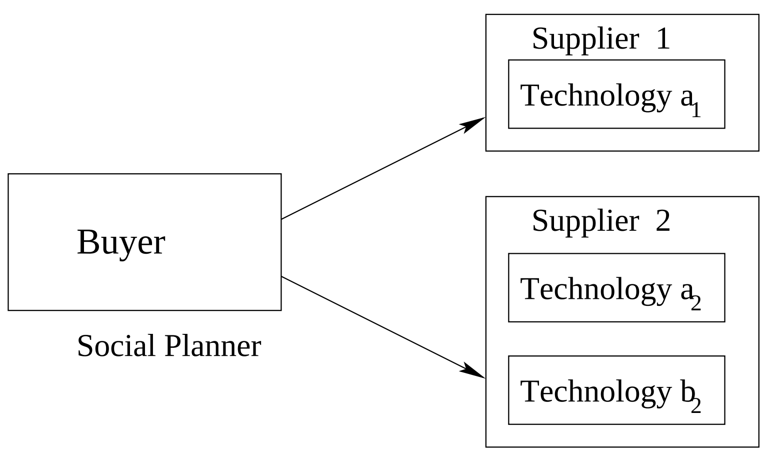

Figure 1: A sourcing scenario with one buyer and two suppliers

Let us define three outcomes (alternatives) x, y, z for this situation. The alternative x means that the entire volume required is sourced from supplier 1 while the alternative z means that the entire volume required is sourced from supplier 2. The alternative y indicates that 50% of the requirement is sourced from supplier 1 and the rest is sourced from supplier 2.

Since the buyer already has a long-standing relationship with supplier 1, this supplier is the preferred one. However, because of the superiority of technology a_2 over a_1 and b_2, the buyer would like to certainly source some quantity from supplier 2 if it is known that supplier 2 is guaranteed to use technology a_2. To reflect these facts, we assume the payoff functions to be given by:

$$u_1(x, a_1) = 100; \;\; u_1(y, a_1) = 50; \;\; u_1(z, a_1) = 0$$

$$u_2(x, a_2) = 0; \;\; u_2(y, a_2) = 50; \;\; u_2(z, a_2) = 100$$

$$u_2(x, b_2) = 0; \;\; u_2(y, b_2) = 50; \;\; u_2(z, b_2) = 25.$$

Note that $\Theta = \{(a_1, a_2), (a_1, b_2)\}$. Consider the social choice function $f(a_1, a_2) = y$ and $f(a_1, b_2) = x$. This means that when it is guaranteed that supplier 2 will use technology a_2, the buyer would like to procure from both the suppliers, whereas if it is known that supplier 2 uses technology b_2, the buyer would rather source the entire requirement from supplier 1.

Likewise, there are eight other social choice functions that one can define here. These are:

$$\begin{aligned}
f(a_1, a_2) = x; &\quad f(a_1, b_2) = x \\
f(a_1, a_2) = x; &\quad f(a_1, b_2) = y \\
f(a_1, a_2) = x; &\quad f(a_1, b_2) = z \\
f(a_1, a_2) = y; &\quad f(a_1, b_2) = y \\
f(a_1, a_2) = y; &\quad f(a_1, b_2) = z \\
f(a_1, a_2) = z; &\quad f(a_1, b_2) = x \\
f(a_1, a_2) = z; &\quad f(a_1, b_2) = y \\
f(a_1, a_2) = z; &\quad f(a_1, b_2) = z.
\end{aligned}$$

It would be interesting to look at the implications of these social choice functions. We will return to this example later on, in many different contexts.

Example 2 (Selling a Single Indivisible item) . Consider a selling agent and two buying agents, so $N = \{0, 1, 2\}$. 0 is the seller and 1 and 2 are the buying agents. The *good* is to be allocated to one of the buyers in return for a monetary consideration. An outcome here can be represented by

$$x = (y_0, y_1, y_2, t_0, t_1, t_2)$$

where, for $i = 0, 1, 2$,

$$\begin{aligned}
y &= 1 \quad \text{for agent } i \text{ gets the good} \\
&= 0 \quad \text{if agent } i \text{ does not get the good} \\
t_i &= \text{monetary transfer received by the agent } i
\end{aligned}$$

The set X of all feasible outcomes is given by

$$X = \{(y_0, y_1, y_2, t_0, t_1, t_2) : y_i \in \{0, 1\}, \sum_i y_i = 1, t_i \in R, \sum_i t_i \leq 0\}$$

The constraint $\sum_i t_i \leq 0$ implies that the total money received by all the agents is less than or equal to zero. That is, total money paid by all the agents is greater than or equal to zero (that is, buyers pay at least as much as sellers receive. The excess between the payment and receipts is the surplus).

For $x = (y_0, y_1, y_2, t_0, t_1, t_2)$, define the utilities as

$$u_i(x, \theta_i) = \theta_i y_i + (\overline{m_i} + t_2)$$

where $\theta_i \in R$ can be viewed as agent i's valuation of the good. Let Θ_i be the real interval $[\underline{\theta_i}, \overline{\theta_i}. \; \overline{m_i}]$ is agent i's initial endowment of money.

Assume that u_i is a Bernoulli-utility (a special case of von Neumann Morgenstern utility defined on deterministic values of money). Utility functions of the form

$$u_i(y_0, y_1, y_2, t_0, t_1, t_2) = \theta_i y_i + (\overline{m_i} + t_i)$$

are said to be of *quasi-linear* form (A function is said to be quasi-linear if it is linear in some of the variables and possibly non-linear in the other variables).

We make the following assumptions regarding valuations.

- The seller (agent 0) derives no value from the good. That is, $\theta_0 = 0$. More generally, the seller derives a *known* value $\overline{\theta_0}$ from the good.
- The valuations θ_1 and θ_2 of the buyers are drawn independently from the uniform distribution on $[0, 1]$ and this fact is *common knowledge* among all the players.

Consider the following social choice function.

- The seller gives the good to the buyer with the highest valuation and to buyer 1 if there is a tie
- The winning buyer gives the seller a payment equal to his valuation (the losing buyer does not make any payment)

The above social choice function $f(\theta) = (y_0(\theta), y_1(\theta), y_2(\theta), t_0(\theta), t_1(\theta), t_2(\theta))$ can be precisely written as

$$\begin{aligned}
y_0(\theta) &= 0 \quad \forall \theta \\
y_1(\theta) = 1 \quad \text{if } \theta_1 \geq \theta_2 & \\
&= 0 \quad \text{if } \theta_1 < \theta_2 \\
y_2(\theta) = 1 \quad \text{if } \theta_1 < \theta_2 & \\
&= 0 \quad \text{if } \theta_1 \geq \theta_2
\end{aligned}$$

$$\begin{aligned}
t_1(\theta) &= -\theta_1 y_1(\theta) \\
t_2(\theta) &= -\theta_2 y_2(\theta) \\
t_0(\theta) &= -(t_1(\theta) + t_2(\theta))
\end{aligned}$$

The social choice function is very attractive to the seller since the seller will capture all of the consumption benefits that are generated by the good. Now we ask the question: can we implement this social choice function?

Suppose we consider the following social choice function, which has the same allocation rule as the one we have just studied but has a different payment rule:

1. The seller gives the good to the buyer with the highest valuation and to buyer 1 if there is a tie
2. The winning buyer gives the seller a payment equal to the second highest valuation (the losing buyer does not make any payment).

The social choice function is now the following.

$$
\begin{aligned}
y_0(\theta) &= 0 \qquad \forall\, \theta \\
y_1(\theta) &= 1 \qquad \text{if } \theta_1 \geq \theta_2 \\
&= 0 \qquad \text{otherwise} \\
y_2(\theta) &= 1 \qquad \text{if } \theta_1 < \theta_2 \\
&= 0 \qquad \text{otherwise} \\
t_1(\theta) &= -\theta_2 y_1(\theta) \\
t_2(\theta) &= -\theta_1 y_2(\theta) \\
t_0(\theta) &= -(t_1(\theta) + t_{@}(\theta))
\end{aligned}
$$

Example 3 (Procurement of a Single Indivisible Resource) Procurement is a ubiquitous activity in any organization. Every organization procures a variety of direct and indirect materials. For example, a factory procures raw material or subassemblies from a pool of suppliers. In a computational grid, a grid user procures computational or storage resources from the grid. A network user procures network resources. A dynamic supply chain planner procures supply chain service providers. Every organization procures indirect materials such as office supplies and services.

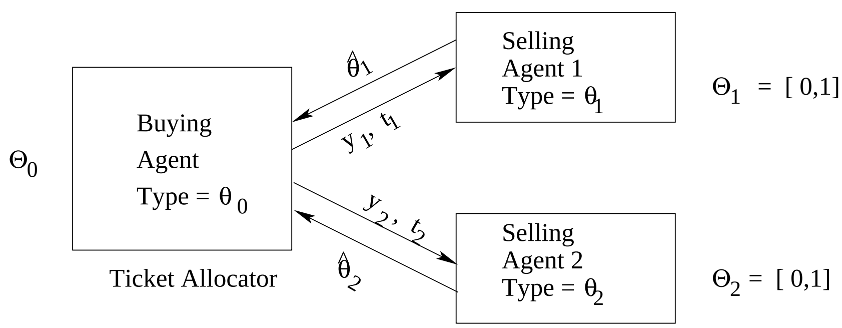

Figure 2: Procurement with two suppliers

The generic procurement situation involves a buyer, a pool of suppliers, and items to be procured. We consider a simple abstraction of the problem by considering a buying agent (call the agent 0) and two selling agents (call them 1 and 2), so we have $N = \{0, 1, 2\}$. See Figure 2. An indivisible item or resource is to be procured from one of the sellers in return for a monetary consideration. An outcome here can be represented by $x = (y_0, y_1, y_2, t_0, t_1, t_2)$. For $i = 0$, we have

$$
\begin{aligned}
y_0 &= 0 \quad \text{if the buyer buys the good} \\
&= 1 \quad \text{otherwise} \\
t_0 &= \text{monetary transfer received by the buyer.}
\end{aligned}
$$

For $i = 1, 2$, we have

$$
\begin{aligned}
y_i &= 1 \quad \text{if agent } i \text{ supplies the good to the buyer} \\
&= 0 \quad \text{if agent } i \text{ does not supply the good} \\
t_i &= \text{monetary transfer received by the agent } i.
\end{aligned}
$$

The set X of all feasible outcomes is given by

$$X = \{(y_0, y_1, y_2, t_0, t_1, t_2) : y_i \in \{0, 1\}, \sum_{i=0}^{2} y_i = 1, t_i \in \mathbb{R}, \sum_{i=0}^{2} t_i \leq 0\}.$$

The constraint $\sum_i t_i \leq 0$ implies that the total money received by all the agents is less than or equal to zero. That is, total money paid by all the agents is greater than or equal to zero (that is, buyer pays at least as much as the sellers receive. The excess between the payment and receipts is the surplus). For $x = (y_0, y_1, y_2, t_0, t_1, t_2)$, define the utilities to be of the form:

$$u_i(x, \theta_i) = u_i((y_0, y_1, y_2, t_0, t_1, t_2), \theta_i) = -y_i\theta_i + t_i \ ; \quad i = 1, 2$$

where $\theta_i \in \mathbb{R}$ can be viewed as seller i's valuation of the good. Such utility functions are said to be of *quasilinear* form (because it is linear in some of the variables and possibly non-linear in the other variables). We will be studying such utility forms quite extensively in this chapter.

We make the following assumptions regarding valuations.

- The buyer has a *known* value $\underline{\theta_0}$ for the good. This valuation does not depend on the choice of the seller from whom the item is purchased.
- Let Θ_i be the real interval $[\underline{\theta_i}, \overline{\theta_i}]$. The types θ_1 and θ_2 of the sellers are drawn independently from the interval $[\theta_i, \bar{\theta}_i]$ and this fact is *common knowledge* among all the players. The type of a seller is to be viewed as the *willingness to sell* (minimum price below which the seller is not interested in selling the item).

Consider the following social choice function.

- The buyer buys the good from the seller with the lowest willingness to sell. If both the sellers have the same type, the buyer will buy the object from seller 1.
- The buyer pays the selected selling agent his willingness to sell.

The above social choice function $f(\theta) = (y_0(\theta), y_1(\theta), y_2(\theta), t_0(\theta), t_1(\theta), t_2(\theta))$ can be precisely written as

$$
\begin{aligned}
y_0(\theta) &= 0 \quad \forall \theta \\
y_1(\theta) &= 1 \quad \text{if } \theta_1 \leq \theta_2 \\
&= 0 \quad \text{if } \theta_1 > \theta_2 \\
y_2(\theta) &= 1 \quad \text{if } \theta_1 > \theta_2 \\
&= 0 \quad \text{if } \theta_1 \leq \theta_2
\end{aligned}
$$

$$
\begin{aligned}
t_1(\theta) &= \theta_1 \, y_1(\theta) \\
t_2(\theta) &= \theta_2 \, y_2(\theta)
\end{aligned}
$$

$$t_0(\theta) = -(t_1(\theta) + t_2(\theta)).$$

We will refer to the above SCF as SCF-PROC1 in the sequel.

Suppose we consider another social choice function, which has the same allocation rule as the one we have just studied but has a different payment rule: The buyer now pays the winning seller a payment equal to the second lowest willingness to sell (as usual, the losing seller does not receive any payment). The new social choice function, which we will call SCF-PROC2, will be the following.

$$\begin{aligned}
y_0(\theta) &= 0 \qquad \forall\, \theta \\
y_1(\theta) &= 1 \qquad \text{if } \theta_1 \leq \theta_2 \\
&= 0 \qquad \text{otherwise} \\
y_2(\theta) &= 1 \qquad \text{if } \theta_1 > \theta_2 \\
&= 0 \qquad \text{otherwise} \\
t_1(\theta) &= \theta_2\, y_1(\theta) \\
t_2(\theta) &= \theta_1\, y_2(\theta) \\
t_0(\theta) &= -(t_1(\theta) + t_2(\theta)).
\end{aligned}$$

Let us define one more SCF, which we call SCF-PROC3, in the following way. SCF-PROC3 has the allocation rule as SCF-PROC1 and SCF-PROC2, but the payments are defined as:

$$\begin{aligned}
t_1(\theta) &= \frac{(1+\theta_1)}{2}\; y_1(\theta) \\
t_2(\theta) &= \frac{(1+\theta_2)}{2}\; y_2(\theta) \\
t_0(\theta) &= -(t_1(\theta) + t_2(\theta)).
\end{aligned}$$

The reason for defining the payment rule in the above way will become clear in the next section, where we will discuss the implementability of SCF-PROC1, SCF-PROC2, and SCF-PROC3.

Example 4 (Funding a Public Project) There is a set of agents $N = \{1, 2, \ldots, n\}$ who have a stake in a common infrastructure, for example, a bridge, community building, Internet infrastructure, etc. For example, the agents could be firms forming a business cluster and interested in creating a shared infrastructure. The cost of the project is to be shared by the agents themselves since there is no source of external funding. Let $k = 1$ indicate that the project is taken up, with $k = 0$ indicating that the project is dropped. Let $t_i \in \mathbb{R}$ denote the payment received by agent i (which means $-t_i$ is the payment made by agent i) for each $i \in N$. Let the cost of the project be C. Since the agents have to fund the project themselves,

$$C \leq -\sum_{i \in N} t_i \quad \text{if } k = 1$$

If $k = 0$, we have

$$0 \leq -\sum_{i \in N} t_i$$

Combining the above two possibilities, we get the condition

$$\sum_{i \in N} t_i \leq -kC\ ; \quad k \in \{0, 1\}.$$

Thus, a natural set of outcomes for this problem is:

$$X = \{(k, t_1, \ldots, t_n) : k \in \{0, 1\},\ t_i \in \mathbb{R}\ \forall i \in N,\ \sum_{i \in N} t_i \leq -kC.\}$$

We assume the utility of agent i, when its type is θ_i corresponding to an outcome $(k, t_1, t_2, \ldots, t_n)$ to be given by

$$u_i((k, t_1, \ldots, t_n), \theta_i) = k\theta_i + t_i.$$

The type θ_i of agent i has the natural interpretation of being the willingness to pay of agent i (maximum amount that agent i is prepared to pay) towards the project. A social choice function in this context is $f(\theta) = (k(\theta), t_1(\theta), \ldots, t_n(\theta))$ given by

$$k(\theta) = \begin{cases} 1 & \text{if } \sum\limits_{i \in N} \theta_i \geq C \\ 0 & \text{otherwise} \end{cases}$$

$$t_i(\theta) = -\left(\frac{k(\theta)C}{n}\right).$$

The way $k(\theta)$ is defined ensures that the project is taken up only if the combined willingness to pay of all the agents is at least the cost of the project. The definition of $t_i(\theta)$ above follows the egalitarian principle, namely that the agents share the cost of the project equally among themselves.

Example 5 (Bilateral Trade) Consider two agents 1 and 2 where agent 1 is the seller of an indivisible private good and agent 2 is a prospective buyer of the good. See Figure 3. An outcome here

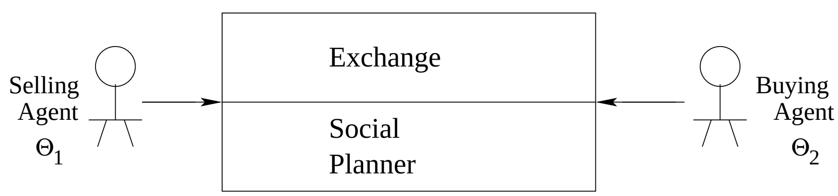

Figure 3: A bilateral trade environment

is of the form $x = (y_1, y_2, t_1, t_2)$ where $y_i = 1$ if agent i gets the good and t_i denotes the payment received by agent i $(i = 1, 2)$. A natural set of outcomes here is

$$X = \{(y_1, y_2, t_1, t_2) :\ y_1 + y_2 = 1;\ y_1, y_2 \in \{0, 1\},\ t_1 + t_2 \leq 0\}.$$

The condition $t_1 + t_2 \leq 0$ indicates that the the amount paid by the buyer should be at least equal to the amount received by the seller (the surplus could perhaps be retained by a market maker or mediator). The utility of the agent i $(i = 1, 2)$ would be of the form

$$u_i((y_1, y_2, t_1, t_2), \theta_i) = y_i\theta_i + t_i.$$

The type θ_1 of agent 1 (seller) can be interpreted as the willingness to sell of the agent (minimum price at which agent 1 is willing to sell). The type θ_2 of agent 2 (buyer) has the natural interpretation of willingness to pay (maximum price the buyer is willing to pay). A social choice function here would be $f(\theta) = (y_1(\theta), y_2(\theta), t_1(\theta), t_2(\theta))$ defined as

$$
\begin{aligned}
y_1(\theta_1, \theta_2) &= 1 \quad \theta_1 > \theta_2 \\
&= 0 \quad \theta_1 \leq \theta_2 \\
y_2(\theta_1, \theta_2) &= 1 \quad \theta_1 \leq \theta_2 \\
&= 0 \quad \theta_1 > \theta_2 \\
t_1(\theta_1, \theta_2) &= y_2(\theta_1, \theta_2)\frac{\theta_1 + \theta_2}{2} \\
t_2(\theta_1, \theta_2) &= -y_2(\theta_1, \theta_2)\frac{\theta_1 + \theta_2}{2}.
\end{aligned}
$$

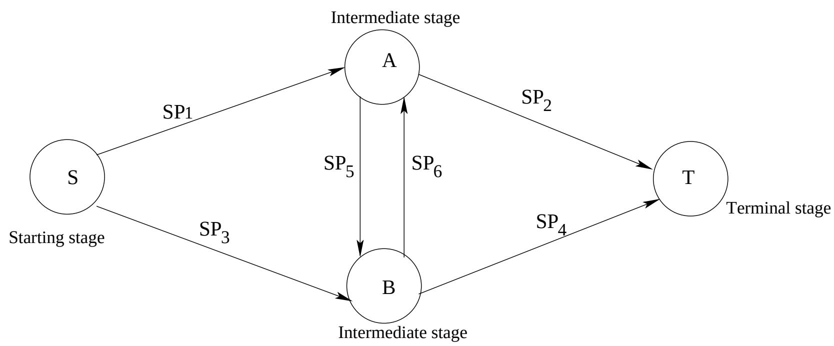

Figure 4: A graph representation of network formation

Example 6 (Network Formation Problem) Consider a supply chain network scenario where a supply chain planner (SCP) is interested in forming an optimal network for delivering products/services. Multiple service providers or supply chain partners are needed for executing the end-to-end process. The directed graph shown in Figure 4 describes different possible ways in which the supply chain can be formed. The node S denotes the starting stage, and the node T denotes the terminal stage in the supply chain. $SP_1, SP_2, \ldots, SP_6$ denote the service providers. They own the respective edges of the directed graph. The nodes A and B are two intermediate stages in the supply chain process. Each path in the network from S to T corresponds to a particular way in which the supply chain network can be formed. For example, (SP_1, SP_5, SP_4) is a path from S to T and this is one possible solution to the supply chain formation problem. Each service provider has a service cost that is private information of the provider. The problem facing the supply chain planner here is to determine the network of service providers who will enable forming the supply chain at the least cost. Since the private values of the service providers are not known, the supply chain planner is faced with a mechanism design problem.

A situation such as described above occurs in many other settings. The first example is a logistics network scenario where S denotes the source, T denotes the destination, and the nodes A and B denote two separate logistics hubs. The edges denote the connectivity between logistics hubs, the source, and the destination. The service providers here are logistics providers who own the transportation service on the edges. Another example is a procurement network scenario where the service providers correspond to suppliers. One more example is that of a telecom network where the service providers could correspond to Internet/bandwidth service providers, and the nodes correspond to cities.

For this example, we have $N = \{SCP, SP_1, \ldots, SP_6\}$. For brevity, we will call this set $\{0, 1, 2, 3, 4, 5, 6\}$, where 0 is the supply chain planner and $1, 2, \ldots, 6$ are the service providers. Let us assume the type sets to be:

$$\Theta_0 = \{\theta_0\}\,; \quad \Theta_i = \left[\underline{\theta_i}, \bar{\theta}_i\right] \subset \mathbb{R} \;\; \forall\, i \in \{1, 2, \ldots, 6\}.$$

An outcome is of the form $(y_0, y_1, \ldots, y_6, t_0, t_1, \ldots, t_6)$ where

$$\begin{aligned} y_0 &= 1 \quad \text{if the supply chain planner buys a path from } S \text{ to } T \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

For $i = 1, \ldots, 6$,

$$\begin{aligned} y_i &= 1 \quad \text{if } SP_i \text{ is part of the supply chain formed} \\ &= 0 \quad \text{otherwise} \\ t_i &= \quad\;\; \text{payment received by service provider } SP_i. \end{aligned}$$

Note that $-t_0$, the payment required to be made by the supply chain planner is just the sum of $t_1, \ldots, t_6$. The set of outcomes in this example is given by

$$\begin{aligned} X = \; & \{(y_0, y_1, ..., y_6, t_0, t_1, ..., t_6) : y_i \in \{0, 1\} \;\forall i \in N; \\ & \{SP_i \in N \setminus \{0\} \;:\; y_i = 1\} \text{ forms a path from S to T;} \\ & t_i \in \mathbb{R} \;\forall i \in N;\; t_0 = -\textstyle\sum_{i=1}^{6} t_i\}. \end{aligned}$$

A social choice function in this problem is given by

$$f(\theta) = (y_0(\theta), y_1(\theta), \ldots, y_6(\theta), t_0(\theta), t_1(\theta), \ldots, t_6(\theta))$$

$$y_0(\theta) = 1 \;\; \forall \theta \in \Theta.$$

For $i = 1, 2, \ldots, 6$, $\forall \theta \in \Theta$,

$$\begin{aligned} y_i(\theta) &= 1 \quad \text{if } SP_i \text{ is part of a shortest cost path} \\ & \qquad\;\; \text{from } S \text{ to } T \text{ selected by the supply chain planner} \\ &= 0 \quad \text{otherwise} \\ t_i(\theta) &= \quad\;\; \text{payment received by } SP_i;\; i = 1, 2, ..., 6 \\ t_0(\theta) &= \quad\;\; -\textstyle\sum_{i=1}^{6} t_i(\theta) \\ u_i(f(\theta), \theta_i) &= \quad\;\; u_i(y_0(\theta), \ldots, y_6(\theta), t_0(\theta), \ldots, t_6(\theta); \theta_i) \\ &= \quad\;\; t_i(\theta) - y_i(\theta)\theta_i. \end{aligned}$$

θ_i is to be viewed as the willingness to sell of the agent i where $i = 1, 2, \ldots, 6$.

2 Implementation of Social Choice Functions

In the preceding section, we have seen a series of examples of social choice functions. In this section, we motivate, through illustrative examples, the concept of implementation of social choice functions. We then formally define the notion of implementation through direct mechanisms and indirect mechanisms.

2.1 Implementation Through Direct Mechanisms

We first provide examples to motivate implementation by direct mechanisms.

Example 7 (Technology driven Supplier Selection) Recall Example 1 where $N = \{1, 2\}$; $\Theta_1 = \{a_1\}$; , $\Theta_2 = \{a_2, b_2\}$ and $\Theta = \{(a_1, a_2), (a_1, b_2)\}$. Suppose the social planner (in this case, the buyer) wishes to implement the social choice function f with $f(a_1, a_2) = y$ and $f(a_1, b_2) = x$. Announcing this as the social choice function, let us say the social planner asks the agents to reveal their types. Agent 1 has nothing to reveal since his type is common knowledge (as his type set is a singleton). We will now check whether agent 2 would be willing to truthfully reveal its type.

- If $\theta_2 = a_2$, then, because $f(a_1, a_2) = y$ and $f(a_1, b_2) = x$ and $u_2(y, a_2) > u_2(x, a_2)$, agent 2 is happy to reveal a_2 as its type.
- However if $\theta_2 = b_2$, then because $u_2(y, b_2) > u_2(x, b_2)$ and $f(a_1, b_2) = x$, agent 2 would wish to lie and claim that its type is a_2 and not b_2.

Thus though the social planner would like to implement an SCF $f(\cdot)$, the social planner would be unable to implement the above SCF since one of the agents (in this case agent 2) does not find it in his best interest to reveal his true type.

On the other hand, let us say the social planner wishes to implement the social choice function f given by $f(a_1, a_2) = z$ and $f(a_1, b_2) = y$. One can show in this case that the SCF can be implemented. Table 1 lists all the nine SCFs and their implementability.

SCF		Implementable
$f(a_1, a_2)$	$f(a_1, b_2)$	
x	x	$\surd$
x	y	$\times$
x	z	$\times$
y	x	$\times$
y	y	$\surd$
y	z	$\times$
z	x	$\times$
z	y	$\surd$
z	z	$\surd$

Table 1: Social choice functions and their implementability

Example 8 (Implementability of SCF-SELL1) Let us say we ask the question whether SCF-SELL1 is implementable. The answer for this question is **No**. The following analysis shows why.

Let us say buyer 2 always announces his true value θ_2. Let us say the valuation of buyer 1 is θ_1 and he announces $\hat{\theta}_1$.

If $\theta_2 \leq \hat{\theta}_1$, then buyer 1 is the winner and his utility will be $\theta_1 - \hat{\theta}_1$. If $\theta_2 > \hat{\theta}_1$, then buyer 2 is the winner and buyer 1's utility is zero.

Since buyer 1 wishes to maximize his expected utility he solves the problem

$$\max_{\hat{\theta}_1} (\theta_1 - \hat{\theta}_1)\ P\{\theta_2 \leq \hat{\theta}_1\}$$

Since θ_2 is uniformly distributed on [0,1],

$$P\{\theta_2 \leq \hat{\theta}_1\} = \hat{\theta}_1$$

Thus buyer 1 tries to solve the problem:

$$\max_{\hat{\theta}_1} (\theta_1 - \hat{\theta}_1)\hat{\theta}_1$$

This problem has the solution

$$\hat{\theta}_1 = \frac{\theta_1}{2}$$

Thus if buyer 2 announces his true valuation, then the best response for buyer 1 is to announce $\frac{\theta_1}{2}$.

Similarly if buyer 1 always announces his true valuation θ_1, then the best response of buyer 2 is to announce $\frac{\theta_2}{2}$.

Thus there is no incentive for the buyers to announce their true valuations.

So, a social planner who wishes to realize the above social choice function finds the rational players will not reveal their true private values.

Since the social planner is unable to elicit the true private values, this social choice function cannot be implemented.

Example 9 (Implementability of SCF-SELL2) Let us say buyer 2 announces his valuation as $\hat{\theta}_2$. There are two cases.

1. $\theta_1 \geq \hat{\theta}_2$
2. $\theta_1 < \hat{\theta}_2$

Case 1: $\theta_1 \geq \hat{\theta}_2$

Let $\hat{\theta}_1$ be the announcement of buyer 1. Here there are two cases.

- If $\hat{\theta}_1 \geq \hat{\theta}_2$, then the payoff for buyer 1 is $\theta_1 - \hat{\theta}_2 \geq 0$.
- If $\hat{\theta}_1 < \hat{\theta}_2$, then the payoff for buyer 1 is 0.
- Thus in this case, the maximum payoff possible is $\theta_1 - \hat{\theta}_2 \geq 0$.

If $\hat{\theta}_1 = \theta_1$ (that is, buyer 1 announces his true valuation), then payoff for buyer 1 $= \theta_1 - \hat{\theta}_2$, which happens to be the maximum possible payoff as shown above. Thus announcing θ_1 is a best response to buyer 1 whatever the announcement of buyer 2.

Case 2: $\theta_1 < \hat{\theta}_2$

Here again there are two cases: $\hat{\theta}_1 \geq \hat{\theta}_2$ and $\hat{\theta}_1 < \hat{\theta}_2$.

- If $\hat{\theta}_1 > \hat{\theta}_2$, then the payoff for buyer 1 is $\theta_1 - \hat{\theta}_2$ which is negative.
- If $\hat{\theta}_1 < \hat{\theta}_2$, then buyer 1 does not win and payoff for him is zero.
- Thus in this case, the maximum payoff possible is 0.

If $\hat{\theta}_1 = \theta_1$, payoff for buyer 1 is 0. By announcing $\hat{\theta}_1 = \theta_1$, his true valuation, buyer 1 gets zero payoff, which in this case is a best response.

We can now make the following observations about this example.

- Revealing his true valuation is optimal for buyer 1 regardless of what buyer 2 announces.
- Similarly announcing his true valuation is optimal for buyer 2 whatever the announcement of buyer 1.
- More formally, truth telling is a weakly dominant strategy for each player.
- Thus this social choice function can be implemented even though the valuations are private values. We simply ask each buyer to report his type and then we choose $f(\theta)$.

The questions to ask now are the following:

1. What social choice functions can be implemented when the types of agents are private information?
2. What do we mean exactly by implementing a social choice function?
3. What is a mechanism?

Example 10 (Implementability of SCF-PROC1) Recall the social choice function SCF-PROC1 that we introduced in the context of procurement of a single indivisible resource (Example 3). Recall the definition of SCF-PROC1:

$$\begin{aligned} y_0(\theta) &= 0 \quad \forall\theta \\ y_1(\theta) &= 1 \quad \text{if } \theta_1 \leq \theta_2 \\ &= 0 \quad \text{if } \theta_1 > \theta_2 \\ y_2(\theta) &= 1 \quad \text{if } \theta_1 > \theta_2 \\ &= 0 \quad \text{if } \theta_1 \leq \theta_2 \end{aligned}$$

$$\begin{aligned} t_1(\theta) &= \theta_1 \, y_1(\theta) \\ t_2(\theta) &= \theta_2 \, y_2(\theta) \\ t_0(\theta) &= -(t_1(\theta) + t_2(\theta)). \end{aligned}$$

We note that the social choice function is very attractive to the buyer since the buyer will capture all of the consumption benefits that are generated by the good. We assume that θ_1 and θ_2 are drawn independently from a uniform distribution over $[0, 1]$. Now we ask the question: Can we implement this social choice function? The answer for this question is *no*. The following analysis shows why.

Let us say seller 2 announces his true value θ_2. Suppose the valuation of seller 1 is θ_1, and he announces $\hat{\theta}_1$. If $\theta_2 \geq \hat{\theta}_1$, then seller 1 is the winner, and his utility will be $\hat{\theta}_1 - \theta_1$. If $\theta_2 < \hat{\theta}_1$, then seller 2 is the winner, and seller 1's utility is zero. Since seller 1 wishes to maximize his expected utility he solves the problem

$$\max_{\hat{\theta}_1} (\hat{\theta}_1 - \theta_1) \; P\{\theta_2 \geq \hat{\theta}_1\}.$$

Since θ_2 is uniformly distributed on [0,1],

$$P\{\theta_2 \geq \hat{\theta}_1\} = 1 - P\{\theta_2 < \hat{\theta}_1\} = 1 - \hat{\theta}_1.$$

Thus seller 1 tries to solve the problem:

$$\max_{\hat{\theta}_1}(\hat{\theta}_1 - \theta_1)(1 - \hat{\theta}_1).$$

This problem has the solution

$$\hat{\theta}_1 = \frac{1 + \theta_1}{2}.$$

Thus if seller 2 announces his true valuation, then the best response for seller 1 is to announce $\frac{1+\theta_1}{2}$.

Similarly if seller 1 announces his true valuation θ_1, then the best response of seller 2 is to announce $\frac{1+\theta_2}{2}$. Thus there is no incentive for the sellers to announce their true valuations. So, a social planner who wishes to realize the above social choice function finds the rational players will not report their true private values. Thus the social choice function cannot be implemented through a direct mechanism.

Example 11 (Implementability of SCF-PROC2) Recall the social choice function SCF-PROC2, again in the context of procurement of a single indivisible resource (Example 3):

$$\begin{aligned} y_0(\theta) &= 0 \quad \forall \theta \\ y_1(\theta) &= 1 \quad \text{if } \theta_1 \leq \theta_2 \\ &= 0 \quad \text{if } \theta_1 > \theta_2 \\ y_2(\theta) &= 1 \quad \text{if } \theta_1 > \theta_2 \\ &= 0 \quad \text{if } \theta_1 \leq \theta_2 \end{aligned}$$

$$\begin{aligned} t_1(\theta) &= \theta_2 \, y_1(\theta) \\ t_2(\theta) &= \theta_1 \, y_2(\theta) \\ t_0(\theta) &= -(t_1(\theta) + t_2(\theta)). \end{aligned}$$

We now show that the function SCF-PROC2 can be implemented. Let us say seller 2 announces his valuation as $\hat{\theta}_2$. There are two cases.

1. $\theta_1 \leq \hat{\theta}_2$
2. $\theta_1 > \hat{\theta}_2$.

Case 1: $\theta_1 \leq \hat{\theta}_2$

Let $\hat{\theta}_1$ be the announcement of seller 1. Here there are two cases.

- If $\hat{\theta}_1 \leq \hat{\theta}_2$, then the payoff for seller 1 is $\hat{\theta}_2 - \theta_1 \geq 0$.
- If $\hat{\theta}_1 > \hat{\theta}_2$, then the payoff for seller 1 is 0.
- Thus in this case, the maximum payoff possible is $\hat{\theta}_2 - \theta_1 \geq 0$.

If $\hat{\theta}_1 = \theta_1$ (that is, seller 1 announces his true valuation), then payoff for seller 1 is $\hat{\theta}_2 - \theta_1$, which happens to be the maximum possible payoff as shown above. Thus announcing θ_1 is a best response to seller 1 whatever the announcement of seller 2.

Case 2: $\theta_1 > \hat{\theta}_2$

Here again there are two cases: $\hat{\theta}_1 \leq \hat{\theta}_2$ and $\hat{\theta}_1 > \hat{\theta}_2$.

- If $\hat{\theta}_1 \leq \hat{\theta}_2$, then the payoff for seller 1 is $\hat{\theta}_2 - \theta_1$, which is negative.
- If $\hat{\theta}_1 > \hat{\theta}_2$, then seller 1 does not win, and payoff for him is zero.
- Thus in this case, the maximum payoff possible is 0.

If $\hat{\theta}_1 = \theta_1$, payoff for seller 1 is 0. By announcing $\hat{\theta}_1 = \theta_1$, his true valuation, seller 1 gets zero payoff, which in this case is a best response.
We can now make the following observations about this example.

- Revealing his true valuation is optimal for seller 1 regardless of what seller 2 announces.
- Similarly, announcing his true valuation is optimal for seller 2 whatever the announcement of seller 1.
- More formally, truth revelation is a weakly dominant strategy for each player.
- Thus this social choice function can be implemented even though the valuations are private values. We simply ask each seller to report his type and then we choose $f(\theta)$.

2.2 Implementation Through Indirect Mechanisms

The examples above have shown us a possible way in which to try to implement a social choice function. The protocol we followed for implementing the social choice functions was:

- Ask each agent to reveal his or her types θ_i;
- Given the announcements $(\hat{\theta_1},\ \ldots,\ \hat{\theta_n})$, choose the outcome $x = f(\hat{\theta}_1,\ \ldots,\ \hat{\theta}_n) \in X$.

Such a method of trying to implement an SCF is referred to as a *direct revelation mechanism*. Another approach to implementing a social choice function is the *indirect way.* Here the mechanism makes the agents interact through an institutional framework in which there are rules governing the actions the agents would be allowed to play and in which there is a way of transforming these actions into a social outcome. The actions the agents choose will depend on their private values and become the strategies of the players. Auctions provide an example of *indirect mechanisms*. We provide an example right away.

Example 12 (First Price Sealed Bid Auction for Selling) Consider an auctioneer or a seller and two potential buyers as before. Here each buyer submits a sealed bid, $b_1 \geq 0 (i = 1, 2)$. The sealed bids are looked at and the buyer with the higher bid is declared the winner. If there is a tie, buyer 1 is declared the winner. The winning buyer pays to the seller an amount equal to his bid. The losing bidder does not pay anything.

Note that there is a subtle difference between the situations in Example 2 and Example 4. In Example 2 (direct mechanism), each buyer is asked to announce his type, whereas in Example 4 (indirect mechanism), each buyer is asked to submit a bid. The bid submitted may (and will) of course depend on the type. Based on the type the buyer has a strategy for bidding. So it becomes a game.

In this case, only one bid is submitted. It could be as well as that there are multiple rounds of bidding. Also, here we have sealed bids. We may as well have open cry bids. These variations lead to different indirect mechanisms.

Let us make the following assumptions:

1. θ_1, θ_2 are independently drawn from the uniform distribution on $[0, 1]$.

2. The sealed bid of buyer i takes the form $b_i(\theta_i) = \alpha_i \theta_i$, where $\alpha_i \in [0, 1]$.

Buyer 1's problem is now to bid in a way to maximize his payoff:

$$\max_{b_1 \geq 0} (\theta_1 - b_1) P\{b_2(\theta_2) \leq b_1\}$$

Since the bid of player 2 is $b_2(\theta_2) = \alpha_2 \theta_2$ and $\theta_2 \in [0, 1]$, the maximum bid of buyer 2 is α_2. Buyer 1 knows this and therefore $b_1 \in [0, \alpha_2]$. Also,

$$\begin{aligned} P\{b_2(\theta_2) \leq b_1\} &= P\{\alpha_2 \theta_2 \leq b_1\} \\ &= P\{\theta_2 \leq \frac{b_1}{\alpha_2}\} \\ &= \frac{b_1}{\alpha_2} \text{ since } \theta_2 \text{ is uniform over } [0, 1] \end{aligned}$$

Thus buyer 1's problem is:

$$\max_{b_1 \in [0, \alpha_2]} (\theta_1 - b_1) \frac{b_1}{\alpha_2}$$

The solution to this problem is

$$b_1(\theta_1) = \begin{cases} \frac{1}{2}\theta_1 & \text{if } \frac{1}{2}\theta_1 \leq \alpha_2 \\ \alpha_2 & \text{if } \frac{1}{2}\theta_1 > \alpha_2 \end{cases}$$

We can show on similar lines that

$$b_2(\theta_2) = \begin{cases} \frac{1}{2}\theta_2 & \text{if } \frac{1}{2}\theta_2 \leq \alpha_1 \\ \alpha_1 & \text{if } \frac{1}{2}\theta_2 > \alpha_1 \end{cases}$$

Let $\alpha_1 = \alpha_2 = \frac{1}{2}$. Then we get

$$\begin{aligned} b_1(\theta_1) &= \frac{\theta_1}{2} \qquad \forall\ \theta_1 \in \Theta_1 = [0, 1] \\ b_2(\theta_2) &= \frac{\theta_2}{2} \qquad \forall\ \theta_2 \in \Theta_2 = [0, 1] \end{aligned}$$

Note that if $b_2(\theta_2) = \frac{\theta_2}{2}$, the best response of buyer 1 is $b_1(\theta_1) = \frac{\theta_1}{2}$ and vice-versa. Hence the profile $\left(\frac{\theta_1}{2}, \frac{\theta_2}{2}\right)$ is a Bayesian Nash equilibrium of an underlying Bayesian game.

This means there is a Bayesian Nash equilibrium of an underlying game (induced by the indirect mechanism called the first price sealed bid auction) that indirectly yields the outcome

$$f(\theta) = (y_0(\theta), y_1(\theta), y_2(\theta), t_0(\theta), t_1(\theta), t_2(\theta))$$

such that

$$\begin{aligned}
y_0(\theta) &= 0 \qquad \forall\, \theta \in \Theta \\
y_1(\theta) &= 1 \qquad \text{if } \theta_1 \geq \theta_2 \\
&= 0 \qquad \text{else} \\
y_2(\theta) &= 1 \qquad \text{if } \theta_1 < \theta_2 \\
&= 0 \qquad \text{else} \\
t_1(\theta) &= -\frac{1}{2}\theta_1 y_1(\theta) \\
t_2(\theta) &= -\frac{1}{2}\theta_2 y_2(\theta) \\
t_0(\theta) &= -(t_1(\theta) + t_2(\theta))
\end{aligned}$$

Example 13 (First Price Procurement Auction) Consider an auctioneer or a buyer and two potential sellers as before. Here each seller submits a sealed bid, $b_i \geq 0$ $(i = 1, 2)$. The sealed bids are examined and the seller with the lower bid is declared the winner. If there is a tie, seller 1 is declared the winner. The winning seller receives an amount equal to his bid from the buyer. The losing seller does not receive anything.

Note that there is a subtle difference between the situations in Example 10 and Example 13. In Example 10 (direct mechanism), each seller is asked to announce his type, whereas in Example 13 (indirect mechanism), each seller is asked to submit a bid. The bid submitted may (and will) of course depend on the type. Based on the type, the seller has a strategy for bidding. So it becomes a game.

Let us make the following assumptions:

1. θ_1, θ_2 are independently drawn from the uniform distribution on $[0, 1]$.

2. The sealed bid of seller i takes the form $b_i(\theta_i) = \alpha_i\theta_i + \beta_i$, where $\alpha_i \in [0, 1], \beta_i \in [0, 1 - \alpha_i]$. He has to make sure that $b_i \in [0, 1]$. The term β_i is like a fixed cost whereas $\alpha_i\theta_i$ indicates a fraction of the true cost.

Seller 1's problem is now to bid in a way to maximize his payoff:

$$\max_{1 \geq b_1 \geq 0} (b_1 - \theta_1) P\{b_2(\theta_2) \geq b_1\}$$

$$\begin{aligned}
P\{b_2(\theta_2) \geq b_1\} &= 1 - P\{b_2(\theta_2) < b_1\} \\
&= 1 - P\{\alpha_2\theta_2 + \beta_2 < b_1\} \\
&= 1 - \frac{b_1 - \beta_2}{\alpha_2} \text{ if } b_1 \geq \beta_2 \qquad (1) \\
&\quad\ \text{since } \theta_2 \text{ is uniform over } [0, 1] \\
&= 1 \text{ if } b_1 < \beta_2. \qquad (2)
\end{aligned}$$

Thus seller 1's problem is:

$$\max_{b_1 \geq \beta_2} (b_1 - \theta_1)(1 - \frac{b_1 - \beta_2}{\alpha_2}).$$

The solution to this problem is

$$b_1(\theta_1) = \frac{\alpha_2 + \beta_2}{2} + \frac{\theta_1}{2}. \qquad (3)$$

We can show on similar lines that

$$b_2(\theta_2) = \frac{\alpha_1 + \beta_1}{2} + \frac{\theta_2}{2}. \tag{4}$$

As the bid of seller i takes the form $b_i(\theta_i) = \alpha_i\theta_i + \beta_i$, where $\alpha_i \in [0,1], \beta_i \in [0, 1-\alpha_i]$, from the equations (3) and (4), we obtain $\alpha_1 = \alpha_2 = \frac{1}{2}$. As the goal of each seller is to maximize the profit and $\beta_i \in [0, 1-\alpha_i]$, $\beta_1 = \beta_2 = \frac{1}{2}$. Then we get

$$\begin{aligned} b_1(\theta_1) &= \frac{1+\theta_1}{2} \qquad \forall\ \theta_1 \in \Theta_1 = [0,1] \\ b_2(\theta_2) &= \frac{1+\theta_2}{2} \qquad \forall\ \theta_2 \in \Theta_2 = [0,1]. \end{aligned}$$

Note that if $b_2(\theta_2) = \frac{1+\theta_2}{2}$, the best response of seller 1 is $b_1(\theta_1) = \frac{1+\theta_1}{2}$ and vice-versa. Hence the profile $\left(\frac{1+\theta_1}{2}, \frac{1+\theta_2}{2}\right)$ is a Bayesian Nash equilibrium of an underlying Bayesian game. In other words, there is a Bayesian Nash equilibrium of an underlying game (induced by the indirect mechanism called the first price procurement auction) that (indirectly) yields the outcome

$$f(\theta) = (y_0(\theta), y_1(\theta), y_2(\theta), t_0(\theta), t_1(\theta), t_2(\theta))$$

such that

$$\begin{aligned} y_0(\theta) &= 0 \qquad \forall\ \theta \in \Theta \\ y_1(\theta) &= 1 \qquad \text{if } \theta_1 \leq \theta_2 \\ &= 0 \qquad \text{else} \\ y_2(\theta) &= 1 \qquad \text{if } \theta_1 > \theta_2 \\ &= 0 \qquad \text{else} \\ t_1(\theta) &= \frac{1+\theta_1}{2} y_1(\theta) \\ t_2(\theta) &= \frac{1+\theta_2}{2} y_2(\theta) \\ t_0(\theta) &= -(t_1(\theta) + t_2(\theta)). \end{aligned}$$

The above SCF is precisely SCF-PROC3 that we had introduced in Example 3.

Example 14 (Second Price Sealed Bid Auction for Selling) Here, each buyer is asked to submit a sealed bid $b_i \geq 0$. The bids are examined and the buyer with higher bid is declared the winner. In case there is a tie, buyer 1 is declared the winner. The winning buyer will pay to the seller an amount equal to the second highest bid. The losing bidder does not pay anything.

In this case, we can show that $b_i(\theta_i) = \theta_i$ for $i = 1, 2$ constitutes a weakly dominant strategy for each player. The arguments are identical to those in Example 3.

Thus the game induced by the indirect mechanism second price sealed bid auction has a weakly dominant strategy in which the following social choice function is implemented.

$$f(\theta) = (y_0(\theta), y_1(\theta), y_2(\theta), t_0(\theta), t_1(\theta), t_2(\theta))$$

such that

$$y_0(\theta) = 0 \quad \forall 0 \in \theta$$

$$
\begin{aligned}
y_1((\theta)) &= 1 \quad \theta_1 \geq \theta_2 \\
&= 0 \quad \theta_1 < \theta_2 \\
y_2(\theta) &= 1 \quad \theta_1 < \theta_2 \\
&= 0 \quad \theta_1 \geq \theta_2 \\
t_1(\theta) &= -\theta_2 y_2(\theta) \quad \forall \theta \in \Theta \\
t_2(\theta) &= -\theta_1 y_2(\theta) \quad \forall \theta \in \Theta \\
t_0(\theta) &= -(t_1(\theta) + t_2(\theta))
\end{aligned}
$$

Example 15 (Second Price Procurement Auction) Here, each seller is asked to submit a sealed bid $b_i \geq 0$. The bids are examined, and the seller with the lower bid is declared the winner. In case there is a tie, seller 1 is declared the winner. The winning seller receives as payment from the buyer an amount equal to the second lowest bid. The losing bidder does not receive anything. In this case, we can show that $b_i(\theta_i) = \theta_i$ for $i = 1, 2$ constitutes a weakly dominant strategy for each player. The arguments are identical to those in Example 11.

Thus the game induced by the indirect mechanism second price procurement auction has a weakly dominant strategy in which the social choice function SCF-PROC2 is implemented.

We can summarize the findings of the current section so far in the following way.

- The function SCF-PROC1 cannot be implemented.
- The function SCF-PROC2 can be implemented in dominant strategies by a direct mechanism. Also, the indirect mechanism, namely second price procurement auction, implements SCF-PROC2 in dominant strategies.
- The function SCF-PROC3 is implemented in Bayesian Nash equilibrium by an indirect mechanism, the first price procurement auction.

2.3 Bayesian Game Induced by First Price Sealed Bid Auction

First, note that $N = \{0, 1, 2\}$. Type sets are $\Theta_0, \Theta_1, \Theta_2$ and the common prior is $\phi \in \Delta(\Theta)$. The set of outcomes is

$$
\begin{aligned}
X = \{&(y_0, y_1, y_2, t_0, t_1, t_2) : \\
&y_i \in \{0, 1\}, y_0 + y_1 + y_2 = 1, \\
&t_i \in \Re, t_0 + t_1 + t_2 \leq 0\}
\end{aligned}
$$

$$
u_i((y_0, y_1, y_2, t_0, t_1, t_2), \theta_i) = \theta_i y_i + (t_i + \bar{m}_i)
$$

$$
\begin{aligned}
S_1 &= \Re_+ \\
S_2 &= \Re_+
\end{aligned}
$$

$$
\begin{aligned}
g(b_0, b_1, b_2) = \ &(y_0(b_0, b_1, b_2), \\
&y_1(b_0, b_1, b_2), y_2(b_0, b_1, b_2), \\
&t_0(b_0, b_1, b_2), t_1(b_0, b_1, b_2), \\
&t_2(b_0, b_1, b_2))
\end{aligned}
$$

such that

$$
\begin{aligned}
y_0(b_0, b_1, b_2) &= 0 \qquad \forall\; b_0, b_1, b_2 \\
y_1(b_0, b_1, b_2) &= 1 \qquad \text{if } b_1 \geq b_2 \\
&= 0 \qquad \text{if } b_1 < b_2
\end{aligned}
$$

$$
\begin{aligned}
y_2(b_0, b_1, b_2) &= 1 \quad \text{if } b_1 < b_2 \\
&= 0 \quad \text{if } b_1 \geq b_2 \\
t_1(b_0, b_1, b_2) &= -b_1 y_1(b_0, b_1, b_2) \\
t_2(b_0, b_1, b_2) &= -b_2 y_2(b_0, b_1, b_2) \\
t_0(b_0, b_1, b_2) &= -(t_1(b_0, b_1, b_2) + t_2(b_0, b_1, b_2))
\end{aligned}
$$

The game induced by the second price sealed bid auction will be similar except for appropriate changes in t_1, t_2, and t_0.

Strategies in the Induced Bayesian Game

A strategy s_i for an agent i in the induced Bayesian game is a function $s_i : \Theta_i \rightarrow S_i$. Thus, given a private value $\theta_i \in \Theta_i$, $s_i(\theta_i)$ will give the action of player i. The strategy $s_i(.)$ will specify actions corresponding to private values. In the auction scenario, the bid b_i of player i is a function of his valuation θ_i. $b_i(\theta_i) = \alpha_i \theta_i$ is a particular strategy for player i.

2.4 Bayesian Nash Incentive Compatibility of First Price Auction

We have seen that the first price sealed bid auction implements the following social choice function:

$$f(\theta) = (y_0(\theta), y_1(\theta), y_2(\theta), t_0(\theta), t_1(\theta), t_2(\theta))$$

with

$$
\begin{aligned}
y_0(\theta) &= 0 \quad \forall \theta \in \Theta \\
y_1(\theta) &= 1 \quad \text{if } \theta_1 \geq \theta_2 \\
&= 0 \quad \text{otherwise} \\
t_1(\theta) &= -\frac{\theta_1}{2} y_1(\theta) \\
t_2(\theta) &= -\frac{\theta_2}{2} y_2(\theta) \\
t_0(\theta) &= -(t_1(\theta) + t_2(\theta))
\end{aligned}
$$

If buyer 1 has type θ_1, then his optimal bid $\hat{\theta}_1$ is obtained by solving

$$\max_{\hat{\theta}_1} \left(\theta_1 - \frac{\hat{\theta}_1}{2} \right) P\{\theta_2 \leq \hat{\theta}_1\}$$

This is the same as

$$\max_{\hat{\theta}_1} \left(\theta_1 - \frac{\hat{\theta}_1}{2} \right) \hat{\theta}_1$$

This yields $\hat{\theta}_1 = \theta_1$. Thus it is optimal for buyer 1 to reveal his true private value if buyer 2 reveals his true value. The same situation applies to buyer 2. This implies that the social choice function is Bayesian Nash incentive compatible (since the equilibrium involved is Bayesian Nash equilibrium).

2.5 Dominant Strategy Incentive Compatibility of Second Price Auction

It is easy to see that the social choice function implemented by the second price auction is weakly dominant strategy incentive compatible.

2.6 Bayesian Game Induced by a Mechanism

Recall that a mechanism is an institution or a framework with a set of rules that prescribe the actions available to players and specify how these action profiles are transformed into outcomes. A mechanism specifies an action set for each player. The outcome function gives the rule for obtaining outcomes from action profiles. Given:

1. a set of agents $\{1, 2, \ldots, n\}$,
2. type sets $\Theta_1, \ldots, \Theta_n$,
3. a common prior $\phi \in \Delta(\Theta)$,
4. a set of outcomes X,
5. utility functions $u_1, \ldots, u_n$, with $u_i : X \times \Theta_i \to \mathbb{R}$,

a mechanism $M = (S_1, \ldots, S_n, g(.))$ induces a Bayesian game

$$(N, (\Theta_i), (S_i), (p_i), (U_i))$$

among the players where

$$U_i(\theta_1, \ldots, \theta_n, s_1, \ldots, s_n) = u_i(g(s_1, \ldots, s_n), \theta_i).$$

Strategies in the Induced Bayesian Game

A strategy s_i for an agent i in the induced Bayesian game is a function $s_i : \Theta_i \to S_i$. Thus, given a private value $\theta_i \in \Theta_i$, $s_i(\theta_i)$ will give the action of player i. The strategy $s_i(.)$ will specify actions corresponding to private values. In the auction scenario, the bid b_i of player i is a function of his valuation θ_i. For example, $b_i(\theta_i) = \alpha_i \theta_i + \beta_i$ is a particular strategy for player i.

Figure 5 captures the idea behind an indirect mechanism and the Bayesian game that is induced by an indirect mechanism.

Example 16 (Bayesian Game Induced by First Price Procurement Auction) First, note that $N = \{0, 1, 2\}$. The type sets are $\Theta_0, \Theta_1, \Theta_2$, and the common prior is $\phi \in \Delta(\Theta)$. The set of outcomes is

$$X = \{(y_0, y_1, y_2, t_0, t_1, t_2) : y_i \in \{0, 1\}, y_0 + y_1 + y_2 = 1, t_i \in \mathbb{R}, t_0 + t_1 + t_2 \leq 0\}$$

$$u_i((y_0, y_1, y_2, t_0, t_1, t_2), \theta_i) = -\theta_i y_i + t_i \quad i = 1, 2$$

$$u_0((y_0, y_1, y_2, t_0, t_1, t_2), \theta_0) = \theta_0 y_0 + t_0$$

$$S_1 = \mathbb{R}_+ \;; \quad S_2 = \mathbb{R}_+$$

$$\begin{aligned} g(b_0, b_1, b_2) \quad = \quad & (y_0(b_0, b_1, b_2), y_1(b_0, b_1, b_2), y_2(b_0, b_1, b_2), \\ & t_0(b_0, b_1, b_2), t_1(b_0, b_1, b_2), t_2(b_0, b_1, b_2)) \end{aligned}$$

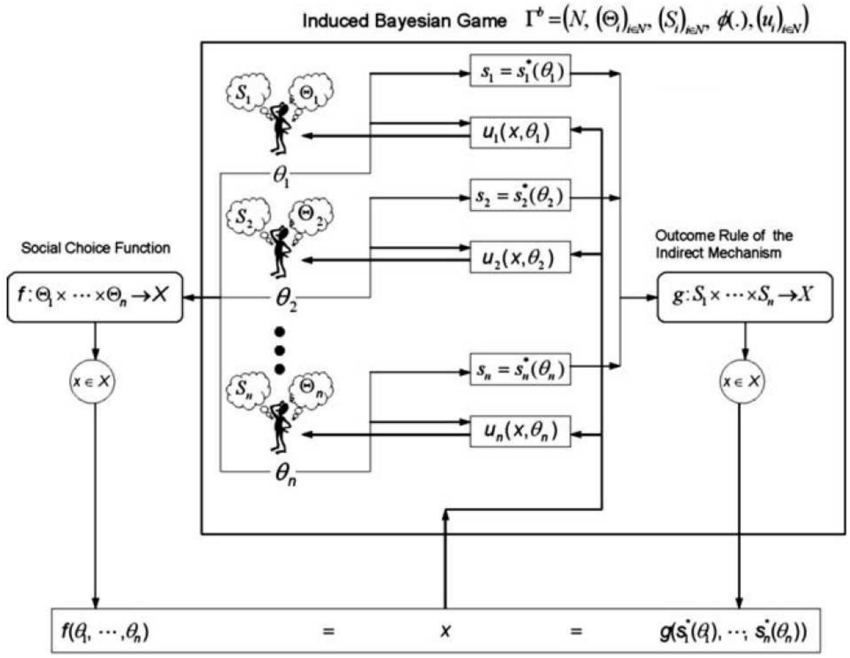

Figure 5: The idea behind implementation by an indirect mechanism

such that

$$
\begin{aligned}
y_0(b_0, b_1, b_2) &= 0 && \forall\, b_0, b_1, b_2 \\
y_1(b_0, b_1, b_2) &= 1 && \text{if } b_1 \leq b_2 \\
&= 0 && \text{if } b_1 > b_2 \\
y_2(b_0, b_1, b_2) &= 1 && \text{if } b_1 > b_2 \\
&= 0 && \text{if } b_1 \leq b_2 \\
t_1(b_0, b_1, b_2) &= b_1 y_1(b_0, b_1, b_2) \\
t_2(b_0, b_1, b_2) &= b_2 y_2(b_0, b_1, b_2) \\
t_0(b_0, b_1, b_2) &= -(t_1(b_0, b_1, b_2) + t_2(b_0, b_1, b_2)).
\end{aligned}
$$

The game induced by the second price procurement auction will be similar except for appropriate changes in t_1 and t_2.

2.7 Implementation of a Social Choice Function by a Mechanism

We now formalize the notion of implementation of a social choice function by a mechanism.

Definition 1 (Implementation of an SCF) *We say that a mechanism $\mathscr{M} = ((S_i)_{i\in N}, g(\cdot))$ implements the social choice function $f(\cdot)$ if there is a pure strategy equilibrium $s^*(\cdot) = (s_1^*(\cdot), \ldots, s_n^*(\cdot))$ of the Bayesian game Γ^b induced by $\mathscr{M}$ such that $g\left(s_1^*(\theta_1), \ldots, s_n^*(\theta_n)\right) = f\left(\theta_1, \ldots, \theta_n\right)\ \forall \left(\theta_1, \ldots, \theta_n\right) \in \Theta$.*

Figure 5 explains the idea behind a mechanism implementing a social choice function. Depending on the nature of the underlying equilibrium, two ways of implementing an SCF $f(\cdot)$ are standard in the literature.

Definition 2 (Implementation in Dominant Strategies) *We say that a mechanism* $\mathscr{M} = ((S_i)_{i \in N}, g(\cdot))$ *implements the social choice function* $f(\cdot)$ *in dominant strategy equilibrium if there is a weakly dominant strategy equilibrium* $s^*(\cdot) = (s_1^*(\cdot), \ldots, s_n^*(\cdot))$ *of the game* Γ^b *induced by* $\mathscr{M}$ *such that*

$$g\left(s_1^*(\theta_1), \ldots, s_n^*(\theta_n)\right) = f\left(\theta_1, \ldots, \theta_n\right) \ \forall \left(\theta_1, \ldots, \theta_n\right) \in \Theta.$$

Note 1 *Since a strongly dominant strategy equilibrium is automatically a weakly dominant strategy equilibrium, the above definition applies to the strongly dominant case also. In the latter case, we could say the implementation is in strongly dominant strategy equilibrium. It is worth recalling that there could exist multiple weakly dominant strategy equilibria whereas a strongly dominant strategy equilibrium, if it exists, is unique.*

Definition 3 (Implementation in Bayesian Nash Equilibrium) *We say that a mechanism* $\mathscr{M} = ((S_i)_{i \in N}, g(\cdot))$ *implements the social choice function* $f(\cdot)$ *in Bayesian Nash equilibrium if there is a pure strategy Bayesian Nash equilibrium* $s^*(\cdot) = (s_1^*(\cdot), \ldots, s_n^*(\cdot))$ *of the game* Γ^b *induced by* $\mathscr{M}$ *such that*

$$g\left(s_1^*(\theta_1), \ldots, s_n^*(\theta_n)\right) = f\left(\theta_1, \ldots, \theta_n\right) \ \forall \left(\theta_1, \ldots, \theta_n\right) \in \Theta.$$

Note 2 *In the definition, what is implicitly implied is a pure strategy Bayesian Nash equilibrium. Such an equilibrium may or may not exist, but we implicitly assume that such an equilibrium exists.*

Note 3 *The game* Γ^b *induced by the mechanism* $\mathscr{M}$ *may have more than one equilibrium, but the above definition requires only that* one of them *induces outcomes in accordance with the SCF* $f(\cdot)$*. Implicitly, then, the above definition assumes that, if multiple equilibria exist, the agents will play the equilibrium that the mechanism designer (social planner) wants. This is an extremely important problem that is addressed by a theory called* implementation theory*. A brief idea about implementation theory will be provided in Section 2.21.*

Note 4 *Another implicit assumption of the above definition is that the game induced by the mechanism is a simultaneous move game, that is all the agents, after learning their types, choose their actions simultaneously.*