Game Theory

Lecture Notes By

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July 2012

Mechanism Design for Sponsored Search Auctions

Note: *This is a only a draft version, so there could be flaws. If you find any errors, please do send email to* `hari@csa.iisc.ernet.in`. *A more thorough version would be available soon in this space.*

The sponsored search auction problem was introduced briefly as an example in Chapter 1. In this chapter, we study this problem in more detail to illustrate a compelling application of mechanism design. We first describe a framework to model this problem as a mechanism design problem under a reasonable set of assumptions. Using this framework, we describe three well known mechanisms for sponsored search auctions — *Generalized First Price (GFP)*, *Generalized Second Price (GSP)*, and *Vickrey–Clarke–Groves (VCG)*. We then design an optimal auction mechanism by extending Myerson's optimal auction mechanism for a single indivisible good which was discussed in the previous chapter. For this, we impose the following well known requirements, which we feel are practical requirements for sponsored search auction, for any mechanism in this setting — *revenue maximization, individual rationality,* and *Bayesian incentive compatibility.* We call this mechanism the *Optimal (OPT)* mechanism. We then make a comparative study of three mechanisms, namely GSP, VCG, and OPT, along four different dimensions — *incentive compatibility, expected revenue earned by the search engine, individual rationality*, and *computational complexity.* This chapter is a detailed extension of the results presented in [1], [2].

1 Internet Advertising

The rapid growth of the Internet and the World Wide Web is transforming the way information is being accessed and used. Newer and innovative models for distributing, sharing, linking, and marketing the information are appearing. As with any major medium, a major way of financially supporting this growth has been advertising (popularly known as *Internet advertising* or *web advertising*). The advertisers-supported web site is one of the successful business models in the emerging web landscape.

The newspapers, magazines, radio, and TV channels built advertising revenue into their business models long ago. Television advertisements have long been accepted as a way of providing funding for programming and constitute a multi billion-dollar industry even today. After these traditional media of advertising, the World Wide Web seems to be the first new way of reaching a potentially worldwide audience in a personalized and meaningful way. Advertisers have realized that Internet advertising is a move from complex, expensive, multinational campaigns to the ones that are targeted at the individuals wherever they are. Moreover, the traditional media do not have an exact methodology to measure the success or failure of an advertising campaign, i.e., return on investment (ROI). Because of these artifacts, Internet advertising has become an obvious choice for advertisers today.

The rise of Internet advertising has witnessed a range of advertising formats and pricing models. In what follows, we provide a brief summary of the most common Internet advertising formats and the various pricing models that have emerged so far.

1.1 Internet Advertising Formats

Banner Ads or Display Ads

This is the oldest form of Internet advertising and also happens to be a popular format for advertising on the Internet. In this format, an ad is a long thin strip of information that may be either static or may include a hyperlink to the advertiser's web page. The advertiser pays an online company in return for space to display the banner ad on one or more of the online company's web pages.

Rich Media

Due to advancement in technology, the static and simple banner ads started becoming richer in terms of user interaction. In this format of advertising, the banner ads are integrated with some components of streaming video and/or audio and interactivity that can allow users to view and interact with products and services — for example, flash or Java script ads, a multimedia product description, a virtual test drive, etc. "Interstitials" have been consolidated within the rich media category and represent full- or partial-page text and image server-push advertisements that appear in the transition between two pages of content. Forms of interstitials can include splash screens, pop-up windows, and superstitials.

E-mail

In this format, an advertiser pays a fee to the commercial e-mail service providers, e.g., `Rediff`, `Yahoo!` etc., to include the banner or rich media ad in e-mail newsletters, e-mail marketing campaigns, and other commercial e-mail communication.

Classifieds

In this format, an advertiser pays on-line companies to list specific products or services. For example, online job boards and employment listings, real estate listings, automotive listings, yellow pages, etc.

Referrals

In this format, an advertiser pays online companies that refer qualified leads or purchase inquiries. For example, automobile dealers pay a fee in exchange for receiving a qualified purchase inquiry online, fees paid when users register or apply for credit card, contest or other service.

Search

In today's web advertising industry, this is the highest revenue generating format among all the other Internet advertising formats. In this format, advertisers pay on-line companies to list and/or link their company site domain names to a specific search word or phrase. The other popular terminologies used for this format are *paid placement, paid listing, paid links, sponsored links, and sponsored search links.* In this format, the text links appear at the top or side of the search results for specific keywords. The more the advertiser pays, the higher the position it gets. The paid placement format is extremely popular among those online companies that provide services based on the indexing-retrieval technologies, such as pure Web search engines (e.g., `AskJeeves, Google,` and `LookSmart`), information portals with search functionality (e.g., `About, AOL, MSN, Rediff,` and `Yahoo!`), meta search engines (e.g., `Metacrawler`), and comparison shopping engines (e.g., `Amazon, eBay, MySimon, MakeMyTrip,` and `Shopping`).

1.2 Pricing Models for Internet Advertising

Each one of the previously mentioned advertising formats would use one or more of the following three pricing schemes for the purpose of charging the advertisement fee.

Pay-Per-Impression (PPI) Models

This is the earliest pricing model to sell the space for banner ads or display ads over the World Wide Web. In early 1994, Internet advertisements were largely sold on a per-impression basis. Advertisers used to pay flat fee to show their ads for a fixed number of times (typically, one thousand showings, or "impressions"). Contracts were negotiated on a case-by-case basis, minimum contracts for advertising purchases were large (typically, a few thousand dollars per month), and entry was slow. The commonly used term for this pricing model is CPM (Cost-Per-Million-Impressions) rather than PPI. Even today, PPI based pricing models are quite popular for major Internet portals, such as `AOL, MSN,` and `Yahoo!` to display the ads in the form of a banner.

Pay-Per-Transaction (PPT) Models

Ideally, an advertiser would always prefer a pricing model in which the advertiser pays only when a customer actually completes a transaction. The PPT models were born out of this contention. A prominent example of PPT models is `Amazon.com`'s Associates Program.[1] Under this program, a website that sends customers to `Amazon.com` receives a percentage of customers' purchases.

Pay-Per-Click (PPC) Models

It is easy to see that an advertiser would always favor the PPT model to pay for its advertising spending, whereas a World Wide Web owner would always prefer the PPI model to charge the advertisers. The PPC models seem to be a compromise between these two. In these models, an advertiser pays only when a user clicks on the ad. These models were originally introduced by `Overture` in 1997, and today they have almost become a standard pricing model for the specific sector of Internet advertising, so called *search*. Today, all the major Internet search engine companies, including `Google, MSN,` and `Yahoo!`, use this pricing model to sell their advertising space [2]. The PPC models for the search engines

[1] See `http://www.amazon.com/gp/browse.html?&node=3435371.`

[2] The PPC models are susceptible to click fraud which is a major drawback of these models [3].

basically rely on some or other form of the auction models. There are many terms currently used in practice to refer these auction models, e.g., *search auctions, Internet search auctions, sponsored search auctions, paid search auctions, paid placement auctions, AdWord auctions, slot auctions, etc.*

1.3 An Analysis of Internet Advertising

Internet advertisement is roughly 10 years old now. In just a relatively short time, advertising on the Internet has become a common activity embraced by advertisers and marketers across all industry sectors, and today there is no dearth of activity going on in the web advertising industry. This industry appears to be growing much faster than traditional mass media advertising vehicles such as print, radio, and TV. The total Internet advertising revenue touched $21.1 billion in the year of 2007. The following table gives a quick estimate of the size of the market dominated by Internet advertising and the pace with which it is growing (Source: Interactive Advertising Bureau. URL: `http://www.iab.net/resources/ad_revenue.asp` accessed on April 09, 2008) All the data in this table are in billion-dollars. The columns *Q1* through *Q4* represent the revenue generated from Internet advertising in all four quarters of each of the past 11 years. The *Annual Revenue* and *Year/Year* columns give the annual revenue generated and year-by-year growth of the Internet advertising industry. The last two columns are the most important in the sense that they give an estimate of the market share of two major formats of the Internet advertising — sponsored search and display ads.

Year	Q1	Q2	Q3	Q4	Annual Revenue	Year/Year Growth	Market Share of Sponsored Search	Market Share of Display Ads
2007	2.802	2.985	3.1	3.6	21.1	+25%	-	-
2006	2.802	2.985	3.1	3.6	16.879	+35%	40%	22%
2005	2.802	2.985	3.1	3.6	12.542	+30.3%	41%	20%
2004	2.230	2.369	2.333	2.694	9.626	+33%	40%	19%
2003	1.632	1.660	1.793	2.182	7.267	+21%	35%	21%
2002	1.520	1.458	1.451	1.580	6.010	-16%	15%	29%
2001	1.872	1.848	1.773	1.641	7.134	-12%	04%	36%
2000	1.922	2.091	1.951	2.123	8.087	+75%	01%	48%
1999	0.693	0.934	1.217	1.777	4.621	+141%	-	56%
1998	0.351	0.423	0.491	0.656	1.920	+112%	-	56%
1997	0.130	0.214	0.227	0.336	0.907	+239%	-	55%

Table 1: Historical revenue performance of the Internet advertising industry

It is evident from the Table 1 that sponsored search is a key factor in deciding the revenue performance of any search engine company. Therefore, our interest lies in studying the different kinds of mechanisms for sponsored search auction. In particular, we want to investigate

- Which mechanisms satisfy desirable economic properties, such as *incentive compatibility,* and/or *individual rationality*?

- Which mechanisms are the most favorable choice for advertisers, i.e., which mechanisms yield the minimum advertising expense for the advertisers?
- Which mechanisms are the most favorable choice for the search engines, i.e., which mechanisms yield the maximum revenue for the search engines?

With these objectives in mind, in the next section we first explain how a typical sponsored search auction works, and then we discuss different mechanisms for it.

2 Sponsored Search Auction

When an Internet user (which we will sometimes refer to as the user, searcher, or customer) enters a keyword (i.e., a search term) into a search engine, he gets back a page with results, containing both the links most relevant to the query and the sponsored links, i.e., paid advertisements. When a user clicks on a sponsored link, he is sent to the respective advertiser's web page. The advertiser then pays the search engine for sending the user to its web page. Figure 1 depicts the result of a search performed on Google using the keyword auctions. There are two different stacks — the left stack contains the links that are most relevant to the query term, and the right stack contains the sponsored links. Sometimes, a few sponsored links are placed on top of the search results as shown. These sponsored links are clearly distinguishable from the actual search results. However, the visibility of a sponsored search link depends on its location (slot) on the result page. Typically, a number

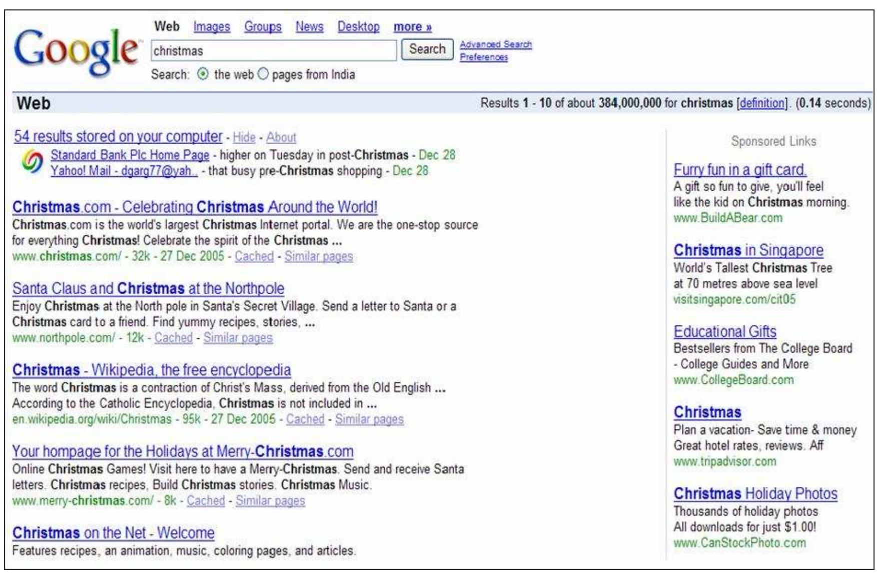

Figure 1: Result of a search performed on Google

of merchants (advertisers) are interested in advertising alongside the search results of a keyword. However, the number of slots available to display the sponsored links is limited. Therefore, against every search performed by the user, the search engine faces the problem of matching the advertisers to the slots. In addition, the search engine also needs to decide on a price to be charged to each advertiser. Note that each advertiser has different desirability for different slots on the search result page. The visibility of an ad shown at the top of the page is much better than an ad shown at the bottom, and, therefore, it is more likely to be clicked by the user. Therefore, an advertiser naturally prefers a slot with higher visibility. Hence, search engines need a system for allocating the slots to advertisers and deciding on a price to be charged to each advertiser. Due to increasing demands for advertising space, most search engines are currently using auction mechanisms for this purpose. These auctions are called sponsored search auctions. In a typical sponsored search auction, advertisers are invited to submit bids on keywords, i.e., the maximum amount they are willing to pay for an Internet user clicking on the advertisement. This is typically referred by the term Cost-Per-Click (CPC). Based on the bids submitted by the advertisers for a particular keyword, the search engine (which we will sometimes refer to as the auctioneer or the seller) picks a subset of advertisements along with the order in which to display them. The actual price charged also depends on the bids submitted by the advertisers. There are many terms currently used in practice to refer to these auctions models, e.g., search auctions, Internet search auctions, sponsored search auctions, paid search auctions, paid placement auctions, AdWord auctions, slot auctions, etc.

3 Sponsored Search Auction as a Mechanism Design Problem

Consider a search engine that has received a query from an Internet user, and it immediately faces the problem of conducting an auction for selling its advertising space among the available advertisers for this particular query word. Let us assume that

1. There are n advertisers interested in this particular keyword, and $N = \{1, 2, \ldots, n\}$ represents the set of these advertisers. Also, there are m slots available with search engine to display the ads and $M = \{1, 2, \ldots, m\}$ represents the set of these advertising slots.

2. α_{ij} is the probability that a user will click on the i^{th} advertiser's ad if it is displayed in j^{th} position (slot), where the first position refers to the top most position. We assume that α_{ij} satisfy the following condition:

$$1 \geq \alpha_{i1} \geq \alpha_{i2} \geq \ldots \geq \alpha_{im} \geq 0 \; \forall i \in N. \tag{1}$$

   Note, here we are assuming that click probability α_{ij} does not depend on which other advertiser has been allocated to what other position. We refer to this assumption as *absence of allocative externality* among the advertisers.

3. Each advertiser precisely knows the value derived out of each click performed by the user on his ad[3] but does not know the value derived out of a single user-click by the other advertisers. Formally, this is modeled by supposing that advertiser i observes a parameter, or signal θ_i that represents his value for each user click. The parameter θ_i is referred to as advertiser i's *type*. The set of possible types of advertiser i is denoted by Θ_i.

4. Each advertiser perceives any other advertiser's valuation as a draw from some probability distribution. Similarly, he knows that the other advertisers regard his own valuation as a draw from some probability distribution. More precisely, for advertiser i, $i = 1, 2, \ldots, n$, there is some probability distribution $\Phi_i(\cdot)$ from which he draws his valuation θ_i. Let $\phi_i(\cdot)$ be the

[3]Note this value should be independent of the position of the ad and should only depend on whether a user clicks on the ad or not.

corresponding PDF. We assume that the θ_i takes values from a closed interval $[\underline{\theta_i}, \overline{\theta_i}]$ of the real line. That is, $\Theta_i = [\underline{\theta_i}, \overline{\theta_i}]$. We also assume that any advertiser's valuation is statistically independent from any other advertiser's valuation. That is, $\Phi_i(\cdot), i = 1, 2, \ldots, n$ are mutually independent. We refer to this assumption as *independent private values assumption.* Note that probability distribution $\Phi_i(\cdot)$ can be viewed as the distribution of a random variable that gives the profit earned by advertiser i when a random customer clicks on advertiser's Ad.

5. Each advertiser i is rational and intelligent in the sense of [4]. This fact is modeled by assuming that the advertisers always try to maximize a Bernoulli utility function $u_i : X \times \Theta_i \to \mathbb{R}$, where X is the set of outcomes, which will be defined shortly.

6. The probability distribution functions $\Phi_i(\cdot)$, the type sets $\Theta_1, \ldots, \Theta_n$, and the utility functions $u_i(\cdot)$ are assumed to be common knowledge among the advertisers. Note that utility function $u_i(\cdot)$ of advertiser i depends on both the outcome x and the type θ_i. Although the type θ_i is not common knowledge, by saying that $u_i(\cdot)$ is common knowledge we mean that for any given type θ_i, the auctioneer (that is, search engine in this case) and every other advertiser can evaluate the utility function of advertiser i.

In view of the above modeling assumptions, the sponsored search auction problem can now be restated as follows. For any query word, each interested advertiser i, bids an amount $b_i \geq 0$, which depends on his actual type θ_i. Now each time the search engine receives this query word, it first retrieves the information from its database of all the advertisers who are interested in displaying their ads against the search result of this query and their corresponding bid vector $b = (b_1, \ldots, b_n)$. The search engine then decides the winning advertisers along with the order in which their ads will be displayed against the search results and the amount that will be paid by each advertiser if the user clicks on his ad. These are called *allocation* and *payment rules*, respectively. Depending on what allocation and payment rules are employed by the search engine, it may take different forms. It is easy to verify that this is a perfect setting to formulate the sponsored search auction problem as a mechanism design problem. A sponsored search auction can be viewed as an *indirect mechanism* $\mathcal{M} = ((B_i)_{i \in N}, g(\cdot))$, where $B_i \subset \mathbb{R}^+$ is the set of bids that an advertiser i can ever report to the search engine and $g(\cdot)$ is the allocation and payment rule. Note, if we assume that for each advertiser i, the set of bids B_i is the same as type set Θ_i, then indirect mechanism $\mathcal{M} = ((B_i)_{i \in N}, g(\cdot))$ becomes a direct revelation mechanism $\mathcal{D} = ((\Theta_i)_{i \in N}, f(\cdot))$, where $f(\cdot)$ becomes the allocation and payment rule. In the rest of this chapter, we will assume that $B_i = \Theta_i \;\; \forall \, i = 1, \ldots, n$. Thus, in view of this assumption, we can regard a sponsored search auction as a direct revelation mechanism.

The various components of a typical sponsored search mechanism design problem are listed below.

3.1 Outcome Set X

An outcome in the case of a sponsored search auction may be represented by a vector $x = (y_{ij}, p_i)_{i \in N, j \in M}$, where y_{ij} is the probability that advertiser i is allocated to the slot j, and p_i denotes the price-per-click charged from advertiser i. The set of feasible alternatives is then

$$X = \left\{ (y_{ij}, p_i)_{i \in N, j \in M} \,\middle|\, y_{ij} \in [0,1] \; \forall i \in N, \; \forall j \in M, \sum_{i=1}^{n} y_{ij} \leq 1 \; \forall j \in M, \sum_{j=1}^{m} y_{ij} \leq 1 \; \forall i \in N, \right. \\ \left. p_i \geq 0 \; \forall i \in N \right\}.$$

Note that the randomized outcomes are also included in the above outcome set. This implies that the randomized mechanisms are also part of the design space.

3.2 Utility Function of Advertisers $u_i(\cdot)$

The utility function of advertiser i can be given, for $x = (y_{ij}, p_i)_{i \in N, j \in M}$, by

$$u_i(x, \theta_i) \quad = \quad \left(\sum_{j=1}^{m} y_{ij} \alpha_{ij} \right) (\theta_i - p_i).$$

3.3 Social Choice Function $f(\cdot)$ (Allocation and Payment Rules)

The general structure of the allocation and payment rule for this case is

$$f(b) = (y_{ij}(b), p_i(b))_{i \in N, j \in M}$$

where $b = (b_1, \ldots, b_n)$ is a bid vector of the advertisers. The functions $y_{ij}(\cdot)$ form the allocation rule, and the functions $p_i(\cdot)$ form the payment rule.

3.4 Linear Environment

Through a slight modification in the definition of allocation rule, payment rule, and utility functions, we can show that a sponsored search auction is indeed a direct revelation mechanism in a linear environment. To transform the underlying environment to a linear one, we redefine the allocation and payment rule as below.

$$f(b) = (y(b), t_i(b))_{i \in N, j \in M}$$

where $y(b) = (y_{ij}(b))_{i \in N, j \in M}$ and $t_i(b) = \left(\sum_{j=1}^{m} y_{ij}(b) \alpha_{ij} \right) p_i(b)$. The quantity $t_i(b)$ can be viewed as the average payment made by the advertiser i to the search engine against every search query received by the search engine, and when the bid vector of the advertisers is $b = (b_1, \ldots, b_n)$.

Now, we can rewrite the utility functions in following manner:

$$u_i(f(b), \theta_i) \quad = \quad \theta_i v_i(y(b)) - t_i(b)$$

where $v_i(y(b)) = \left(\sum_{j=1}^{m} y_{ij}(b) \alpha_{ij} \right)$. The quantity $v_i(y(b))$ can be interpreted as the probability that advertiser i will receive a user click whenever there is a search query received by the search engine and when the bid vector of the advertisers is $b = (b_1, \ldots, b_n)$. Now, it is easy to verify that the underlying environment here is *linear.*

In what follows, we illustrate three basic mechanisms for sponsored search auctions with respect to the above model.

- Generalized First Price (GFP) Mechanism
- Generalized Second Price (GSP) Mechanism (also called Next Price Auction)
- Vickrey–Clarke–Groves (VCG) Mechanism

For each of these mechanisms, we describe the allocation rule $y_{ij}(\cdot)$ and payment rule $p_i(\cdot)$.

4 Generalized First Price (GFP) Mechanism

In 1997, `Overture` (formerly `GoTo`; recently acquired by `Yahoo!`) introduced the first auction mechanism ever used for sponsored search. The term *Generalized First Price Auction* was coined by Edelman, Ostrovsky, and Schwarz [5]. The allocation and payment rules under this mechanism are the following.

4.1 Allocation Rule

In this mechanism, the m advertising slots are allocated to advertisers in *descending order of their bids* [5]. Let $b^{(k)}$ be the k^{th} highest element in $(b_1, \ldots, b_n)$. Similarly, let $(b_{-i})^{(k)}$ be the k^{th} highest element in $(b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n)$. In view of these definitions, we can say that if $b = (b_1, b_2, \ldots, b_n)$ is the profile of bids received from the n advertisers, then the first slot is allocated to the advertiser whose bid is equal to $b^{(1)}$. Similarly, the second slot is allocated to the advertiser whose bid is equal to $b^{(2)}$, and so on. [4] That is, for all $i \in N$ and all $j \in M$,

$$y_{ij}(b) = \begin{cases} 1 & : \text{ if } b_i = b^{(j)} \\ 0 & : \text{ otherwise.} \end{cases} \tag{2}$$

4.2 Payment Rule

Every time a user clicks on a sponsored link, an advertiser's account is automatically billed *the amount of the advertiser's bid*. That is, if $b = (b_1, b_2, \ldots, b_n)$ is the profile of bids received from the n advertisers then, for all $i \in N$,

$$p_i(b) = \begin{cases} b_i & : \text{ if advertiser } i\text{'s Ad is displayed} \\ 0 & : \text{ otherwise.} \end{cases} \tag{3}$$

The ease of use, very low entry costs, and transparency of the mechanism quickly led to the success of `Overture`'s paid search platform as the advertising provider for major search engines including `MSN` and `Yahoo!`.

5 Generalized Second Price (GSP) Mechanism

The primary motivation for this auction mechanism was instability of the GFP mechanism. In particular, it has been shown by Edelman, Ostrovsky, and Schwarz [5] that under the GFP mechanism, truth-telling is not an equilibrium bidding strategy for the advertisers, and this fact leads to instability in the system, which in turn leads to inefficient investments on behalf of the advertisers. The GFP mechanism also creates volatile prices, which in turn cause allocative inefficiencies. [5] `Google` realized these problems and tried fixing the problems by introducing its own new program, called *AdWord Select*, in February 2002.

Recognizing the tangible advantages, `Yahoo!/Overture` also switched to the GSP mechanism. The payment rules are the same in both `Google`'s version of the GSP mechanism and the `Yahoo!/Overture` version of the GSP mechanism. However, the allocation rules are slightly different in these two

[4] If two advertisers have the same bid, then the tie can be broken by an appropriate rule. It is easy to verify that two advertisers having the same bid value is a zero probability event.

[5] See [5] for a detailed discussion about demerits of the GFP mechanism.

versions of the GSP mechanisms. The `Yahoo!/Overture` version of the GSP mechanism follows the same allocation rule as the GFP mechanism but the allocation rule in `Google`'s version of the GSP mechanism is more general. In what follows we describe the different allocation rules for the GSP mechanism and investigate their relationships.

5.1 Allocation Rule

1. `Yahoo!/Overture`**'s Allocation Rule:** This rule is the same as the allocation rule of GFP mechanisms, that is, the slots are allocated to the advertisers in descending order of their bids.

2. **Greedy Allocation Rule:** The primary motivation for this rule is *allocative efficiency*, which we will discuss later. In this rule, the first slot is allocated to the advertiser $i \in N$ for whom the quantity $\alpha_{i1} b_i$ is the maximum. If there is a tie then it is broken by appropriate rule. Now this advertiser is removed from the set N, and an advertiser among the remaining ones is chosen for whom $\alpha_{i2} b_i$ is the maximum. The second slot is allocated to this advertiser. In similar fashion, all the other slots are allocated.

3. `Google`**'s Allocation Rule:** `Google` had also realized the fact that unappealing and poorly targeted ads attract relatively few clicks and thus provide less revenue for `Google`, and, therefore, the probability that the searcher will click on an ad link must be taken into account. In practice, `Google` uses some sort of stylized version of the *greedy* allocation rule. In `Google`'s actual version of GSP mechanisms, for each advertiser `Google` computes its estimated *Click-Through-Rate (CTR)*, which is the ratio of the number of clicks received by the ad to the number of times the Ad was displayed against the search results — popularly known as number of *impressions*. Now the advertisers are ranked in decreasing order of the *ranking scores*, where ranking score of an advertiser is defined as the product of the advertiser's bid and estimated *CTR*. [6]

In order to understand the relationship among these three allocation rules, we need to first understand the relationship between click probability and CTR.

5.2 Relationship between Click Probability and CTR

The notions of click probability and CTR seem to be quite similar in nature. The objective here is to develop a better understanding of these two quantities. Recall the following definitions that we presented earlier:

$$\begin{aligned} \alpha_{ij} &= \text{Probability that user will click on } i^{th} \text{ advertiser's ad if it is displayed in } j^{th} \text{ position} \\ \text{CTR}_i &= \text{Probability that user will click on } i^{th} \text{ advertiser's ad if it is displayed} \\ y_{ij} &= \text{Probability that advertiser } i\text{'s ad is displayed in position } j. \end{aligned}$$

In view of above definitions, it is easy to verify that

$$\begin{aligned} \text{CTR}_i &= \sum_{j=1}^{m} y_{ij}\alpha_{ij} \ \forall \ i \in N \\ \text{CTR}_i &\leq \sum_{j=1}^{m} \alpha_{ij} \ \forall \ i \in N. \end{aligned}$$

[6]See `Google`'s online AdWord demo on *bidding and ranking*. URL: `https://adwords.google.com/select/library/index.html` (accessed on November 21, 2005).

In practice, the click probabilities (α_{ij}) and CTR are learned by means of available data. In what follows, we present three different ways in which one can learn these quantities. This discussion is taken from [3].

5.2.1 Average over Fixed Time Window

In this method, we fix a time interval T and consider an advertiser i. Let I_i be the number of times advertiser i's ad was displayed (irrespective of its position) against search results during the interval T. Further, let I_{ij} impressions of the ad were made at position j out of I_i impressions. Let $C_i \leq I_i$ be the total number of clicks received on this ad during the interval T. Moreover, out of these C_i clicks, C_{ij} clicks were received when the ad was displayed in position j. Now we can define the click probabilities and CTR in following manner.

$$\begin{aligned} \alpha_{ij} &= \frac{C_{ij}}{I_{ij}} \; \forall \, i \in N, \; \forall \, j \in M \\ \text{CTR}_i &= \frac{C_i}{I_i} \; \forall \, i \in N. \end{aligned}$$

5.2.2 Average over Fixed Impression Window

This method is the same as the previous one except that here we fix the number of impressions instead of time. Let us fix the total number of past impression equal to some positive integer constant, say I, and then count the total number of click C_i that were received against these impressions. Similarly, we fix the total number of past impression made at position j equal to same positive integer constant I and then count the total number of click C_{ij} that were received against these impressions. Now, the click probabilities and CTR can be defined in following manner:

$$\begin{aligned} \alpha_{ij} &= \frac{C_{ij}}{I} \; \forall \, i \in N, \; \forall \, j \in M \\ \text{CTR}_i &= \frac{C_i}{I} \; \forall \, i \in N. \end{aligned}$$

5.2.3 Average over Fixed Click Window

In this method, we fix the total number of clicks received so far equal to some positive integer constant, say C, and then count the total number of impressions I_i that were made in order to receive these clicks. Similarly, fix the total number of clicks received for position j so far to the same constant C, and then count the total number of impressions I_{ij} that were made in order to receive these clicks. Now, the click probabilities and CTR can be defined in following manner:

$$\begin{aligned} \alpha_{ij} &= \frac{C}{I_{ij}} \; \forall \, i \in N, \; \forall \, j \in M \\ \text{CTR}_i &= \frac{C}{I_i} \; \forall \, i \in N. \end{aligned}$$

5.3 Relationship Among Different Allocation Rules

Before moving on to the payment rule for GSP, we would like to explore the relationship among the allocation rules. In order to understand the relationship, let us assume that $b = (b_1, b_2, \ldots, b_n)$ is the profile of bids received from the n advertisers. Consider the following optimization problem:

Maximize

$$\sum_{i=1}^{n} b_i v_i(y(b)) = \sum_{i=1}^{n} \sum_{j=1}^{m} (b_i \alpha_{ij}) y_{ij}(b)$$

subject to

$$\begin{aligned}
\sum_{i=1}^{n} y_{ij}(b) &\leq 1 \ \forall j \in M \\
\sum_{j=1}^{m} y_{ij}(b) &\leq 1 \ \forall i \in N \\
y_{ij}(b) &\leq 0 \ \forall i \in N, \ \forall j \in M.
\end{aligned}$$

It is easy to see that for given click probabilities α_{ij}, where these probabilities satisfy the condition (1), the greedy allocation rule basically provides a solution to the above optimization problem. Such an allocation would be an *efficient* allocation. `Yahoo!/Overture`'s allocation rule and `Google`'s allocation rule become special cases of the greedy allocation rule under certain conditions that are summarized in the following propositions.

Proposition 5.1 *Given any bid vector $b = (b_1, \ldots, b_n)$, the greedy allocation rule and `Yahoo!/Overture`'s allocation rule result in the same allocation if the following two conditions are satisfied:*

1. *The click probabilities satisfy the assumption of absence of allocative externality among the advertisers, that is, $1 \geq \alpha_{i1} \geq \alpha_{i2} \geq \ldots \geq \alpha_{im} \geq 0 \ \forall i \in N$.*
2. *The click probabilities depend only on the positions of the advertisements and are independent of the identities of the advertisers, that is, $\alpha_{1j} = \alpha_{2j} = \ldots = \alpha_{nj} = \alpha_j \ \forall j \in M$.*

Proposition 5.2 *Given any bid vector $b = (b_1, \ldots, b_n)$, both the greedy allocation rule and `Google`'s allocation rule result in the same allocation if the following two conditions are satisfied:*

1. *The click probabilities satisfy the assumption of absence of allocative externality among the advertisers, that is, $1 \geq \alpha_{i1} \geq \alpha_{i2} \geq \ldots \geq \alpha_{im} \geq 0 \ \forall i \in N$.*
2. *The click probabilities depend only on the identities of the advertisers and are independent of the positions of the advertisements, that is, $\alpha_{i1} = \alpha_{i2} = \ldots = \alpha_{im} = \alpha_i = CTR_i \ \forall i \in N$.*

In the rest of the chapter, we will work with the following assumptions:

1. Click probabilities depend only on the positions of the ads and are independent of the identities of the advertisers. That is, $\alpha_{1j} = \alpha_{2j} = \ldots = \alpha_{nj} = \alpha_j \ \forall j \in M$.
2. The allocation rule in a GSP mechanism is the same as the greedy allocation rule, which would be the same as `Yahoo!/Overture`'s allocation rule because of the previous assumption.

5.4 Payment Rule

In this auction mechanism, every time a user clicks on a sponsored link, an advertiser's account is automatically billed *the amount of the advertiser's bid who is just below him in the ranking of the displayed ads plus a minimum increment (typically $0.01).* The advertiser whose ad appears at the bottom-most position is charged the amount of the highest bid among the disqualified bids plus the minimum increment. If there is no such bid then he is charged nothing. If $b = (b_1, b_2, \ldots, b_n)$ is the profile of bids received from the n advertisers, then because of the assumptions we made earlier regarding the allocation rule in the GSP mechanism, the price per click that is charged to an advertiser i would be given by

$$p_i(b) \quad = \quad \begin{cases} \sum_{j=1}^{m} \left(b^{(j+1)} y_{ij}(b)\right) & : \text{ if either } m < n \text{ or } n \leq m \text{ but } b_i \neq b^{(n)} \\ 0 & : \text{ otherwise} \end{cases}$$

where $b^{(j+1)}$ is the $(j+1)^{th}$ highest bid which is the same as the bid of an advertiser whose Ad is allocated to position $(j+1)$.[7]

6 Vickrey–Clarke–Groves (VCG) Mechanism

On the face of it, the GSP mechanism appears to be a generalized version of the well known Vickrey auction, which is used for selling a single indivisible good. But as shown by Edelman, Ostrovsky, and Schwarz [5], and also shown in the later part of this chapter, the GSP mechanism is indeed not a generalization of the classical Vickrey auction to the setting where a set of ranked objects is being sold. The generalization of the Vickrey auction is the Clarke mechanism, which we introduced in the previous chapter. In this section, our objective is to develop the Clarke mechanism for the sponsored search auction. We refer to this as the VCG mechanism, following standard practice.

6.1 Allocation Rule

By definition, the VCG mechanism is allocatively efficient. Therefore, in the case of a sponsored search auction, the allocation rule $y^*(\cdot)$ in the VCG mechanism is

$$y^*(\cdot) = \underset{y(\cdot)}{\arg\max} \sum_{i=1}^{n} b_i v_i(y(b)) = \underset{y_{ij}(\cdot)}{\arg\max} \sum_{i=1}^{n} \sum_{j=1}^{m} (b_i \alpha_{ij}) y_{ij}(b). \tag{4}$$

In the previous section, we have already seen that the greedy allocation rule is a solution to (4). Moreover, under the assumption that click probabilities are independent of advertisers' identities, the allocation $y^*(\cdot)$ allocates the slots to the advertisers in the decreasing order of their bids. That is, if $b = (b_1, b_2, \ldots, b_n)$ is the profile of bids received from the n advertisers then $y^*(\cdot)$ must satisfy the following condition:

$$y^*_{ij}(b) \quad = \quad \begin{cases} 1 & : \quad b_i = b^{(j)} \\ 0 & : \quad \text{otherwise.} \end{cases} \tag{5}$$

We state below an interesting observation regarding GFP and GSP mechanisms, which is based on the above observations.

[7] We have ignored the small increment $0.01 because all the future analysis and results are insensitive to this amount.

Proposition 6.1 *If click probabilities depend only on the positions of the ads and are independent of the identities of the advertisers, then*

1. *The GFP mechanism is allocatively efficient.*
2. *The GSP mechanism is allocatively efficient if it uses the greedy allocation rule, which is the same as* `Yahoo!/Overture`*'s allocation rule.*
3. *The allocation rule for the VCG mechanism, which is an efficient allocation, is given by (5). Moreover, this allocation rule is precisely the same as the GFP allocation rule and* `Yahoo!/Overture`*'s allocation rule.*

6.2 Payment Rule

As per the definition of the VCG mechanism, the expected payment $t_i(b)$ made by an advertiser i, when the profile of the bids submitted by the advertisers is $b = (b_1, \ldots, b_n)$, must be calculated using the following Clarke's payment rule:

$$t_i(b) = \left[\sum_{j \neq i} b_j v_j(y^*(b))\right] - \left[\sum_{j \neq i} b_j v_j(y^*_{-i}(b))\right] \tag{6}$$

where $y^*_{-i}(\cdot)$ is an efficient allocation of the slots among the advertisers when advertiser i is removed from the scene. Substituting value of $y^*(\cdot)$ from Equation (5) and making use of the fact that $v_i(y^*(b)) = \sum_{j=i}^{m} y^*_{ij}(b)\alpha_j$, Equation (6) can be written as follows:

Case 1 ($m < n$):

$$\begin{aligned} t^{(j)}(b) &= \alpha_j p^{(j)}(b) \\ &= \begin{cases} \beta_j b^{(j+1)} + t^{(j+1)}(b) & : \text{ if } 1 \leq j \leq (m-1) \\ \alpha_m b^{(m+1)} & : \text{ if } j = m \\ 0 & : \text{ if } m < j \leq n \end{cases} \end{aligned} \tag{7}$$

where

- $t^{(j)}(b)$ is the expected payment made by the advertiser whose ad is displayed in j^{th} position, for every search query received by the search engine and when the bid profile of the advertisers is $b = (b_1, \ldots, b_n)$,
- $p^{(j)}(b)$ is the payment made by the advertiser, whose ad is displayed in j^{th} position, for every click made by the user and when the bid profile of the advertisers is $b = (b_1, \ldots, b_n)$,
- and $\beta_j = (\alpha_j - \alpha_{j+1})$,
- $b^{(j)}$ has its usual interpretation.

Case 2 ($n \leq m$):

$$\begin{aligned} t^{(j)}(b) &= \alpha_j p^{(j)}(b) \\ &= \begin{cases} \beta_j b^{(j+1)} + t^{(j+1)}(b) & : \text{ if } 1 \leq j \leq (n-1) \\ 0 & : \text{ if } j = n. \end{cases} \end{aligned} \tag{8}$$

Unfolding Equations (7) and (8) result in the following expressions:

Case 1 ($m < n$):

$$p^{(j)}(b) = \frac{1}{\alpha_j} t^{(j)}(b) \quad = \quad \begin{cases} \frac{1}{\alpha_j}\left[\sum_{k=j}^{m-1} \beta_k b^{(k+1)}\right] + \frac{\alpha_m}{\alpha_j} b^{(m+1)} & : \text{ if } 1 \leq j \leq (m-1) \\ b^{(m+1)} & : \text{ if } j = m \\ 0 & : \text{ if } m < j \leq n. \end{cases} \tag{9}$$

Case 2 ($n \leq m$):

$$p^{(j)}(b) = \frac{1}{\alpha_j} t^{(j)}(b) \quad = \quad \begin{cases} \frac{1}{\alpha_j} \sum_{k=j}^{n-1} \beta_k b^{(k+1)} & : \text{ if } 1 \leq j \leq (n-1) \\ 0 & : \text{ if } j = n. \end{cases} \tag{10}$$

Thus, we can say that Equation (5) describes the allocation rule for the VCG mechanism and Equations (9) and (10) describe the payment rule for the VCG mechanism.

So far we have discussed three basic mechanisms for sponsored search auction — GFP, GSP, and VCG. The GFP and GSP are the two mechanisms that have been used by the search engines in practice. The VCG mechanism, though not implemented so far by any search engine, is another mechanism that is worth studying because of its unique property, namely dominant strategy incentive compatibility. In fact, at least theoretically, there are infinitely many ways in which a sponsored search auction can be conducted where each mechanism has its own pros and cons. We will now focus on an another important mechanism, namely, Optimal (OPT) mechanism. This mechanism is interesting from a theoretical as well as a practical stand-point because it maximizes the search engine's revenue while ensuring individual rationality and incentive compatibility for the advertisers. Myerson first studied a similar auction mechanism in the context of selling a single indivisible good [6] (see Section 2.20). Myerson called such an auction mechanism an *optimal auction.* Following the same terminology, we would prefer to call a similar mechanism for the sponsored search auction an *optimal mechanism* for sponsored search auction (OPT mechanism for short). The readers are suggested to refer to the previous chapter (see Section 2.20) for details regarding this mechanism. It may be noted that the discussion in Section 2.20 is on optimal procurement auction (cost minimization) whereas here, it would be on optimal auctions for selling (revenue maximization).

7 Optimal (OPT) Mechanism

In this section, our goal is to compute the allocation and payment rule $f(\cdot)$ that results in an optimal mechanism for the sponsored search auction. This calls for extending Myerson's optimal auction to the case of the sponsored search auction. We follow a line of attack that is similar to that of Myerson [6]. Recall that we formulated the sponsored search auction as a direct revelation mechanism $\mathcal{D} = ((\Theta_i)_{i \in N}, f(\cdot))$ in a linear environment, where the Bernoulli utility function of an advertiser i is given by

$$\begin{aligned} u_i(f(b), \theta_i) &= \left(\sum_{j=1}^{m} y_{ij}(b)\alpha_j\right)(\theta_i - p_i(b)) \\ &= v_i(y(b))(\theta_i - p_i(b)) \\ &= \theta_i v_i(y(b)) - t_i(b) \end{aligned}$$

where

- $v_i(y(b)) = \left(\sum_{j=1}^{m} y_{ij}(b)\alpha_j\right)$ is the valuation function of the advertiser i and can be interpreted as the probability that advertiser i will receive a user click whenever there is a search query received by the search engine and when the bid vector of the advertisers is b.
- $t_i(b) = v_i(y(b))p_i(b)$ can be viewed as the average payment made by advertiser i to the search engine against every search query received by the search engine and when the bid vector of the advertisers is b.

7.1 Allocation Rule

It is convenient to define

- $\overline{t}_i(b_i) = E_{\theta_{-i}}[t_i(b_i, \theta_{-i})]$ is the expected payment made by advertiser i when he bids an amount b_i and all the advertisers $j \neq i$ bid their true types.
- $\overline{v}_i(b_i) = E_{\theta_{-i}}\left[v_i(y(b_i, \theta_{-i}))\right]$ is the probability that advertiser i will receive a user click if he bids an amount b_i and all the advertisers $j \neq i$ bid their true types.
- $U_i(\theta_i) = \theta_i \overline{v}_i(\theta_i) - \overline{t}_i(\theta_i)$ gives advertiser i's expected utility from the mechanism conditional on his type being θ_i when he and all other advertisers bid their true types.

The problem of designing an optimal mechanism for the sponsored search auction can now be written as one of choosing functions $y_{ij}(\cdot)$ and $U_i(\cdot)$ to solve:

Maximize

$$\sum_{i=1}^{n} \int_{\underline{\theta_i}}^{\overline{\theta_i}} \left(\theta_i \overline{v}_i(\theta_i) - U_i(\theta_i)\right) \phi_i(\theta_i) d\theta_i \tag{11}$$

subject to

(i) $\overline{v}_i(\cdot)$ is non-decreasing $\quad \forall i \in \ N$

(ii) $y_{ij}(\theta) \in [0,1], \sum_{j=1}^{m} y_{ij}(\theta) \leq 1, \sum_{i=1}^{n} y_{ij}(\theta) \leq 1 \;\; \forall i \in N, \; \forall j \in M, \; \forall \theta \in \Theta$

(iii) $U_i(\theta_i) = U_i(\underline{\theta_i}) + \int_{\underline{\theta_i}}^{\theta_i} \overline{v}_i(s) ds \;\; \forall i \in N, \; \forall \theta_i \in \Theta_i$

(iv) $U_i(\theta_i) \geq 0 \; \forall i \in N, \; \forall \theta_i \in \Theta_i$.

In the above formulation, the objective function is the total expected payment received by the search engine from all the advertisers. Note that constraints (iv) are the advertisers' interim individual rationality constraints while constraint (ii) is the feasibility constraint. Constraints (i) and (iii) are the necessary and sufficient conditions for the allocation and payment rule $f(\cdot) = (y_{ij}(\cdot), t_i(\cdot))_{i \in N, j \in M}$ to be Bayesian incentive compatible. These constraints are taken from [6].

We have a critical observation to make here. Note that in the above optimization problem, we have replaced the bid b_i by the actual type θ_i. This is because we are imposing the Bayesian incentive compatibility constraints on the allocation and payment rule, and, hence, every advertiser will bid his true type. Thus, while dealing with the OPT mechanism, we can safely interchange θ_i and b_i for any $i \in N$.

Note first that if constraint (iii) is satisfied, then constraint (iv) will be satisfied iff $U_i(\underline{\theta_i}) \geq 0 \; \forall i \in N$. As a result, we can replace the constraint (iv) with

$$\text{(iv')} \; U_i(\underline{\theta_i}) \geq 0 \; \forall i \in N.$$

Next, substituting for $U_i(\theta_i)$ in the objective function from constraint (iii), we get

$$\sum_{i=1}^{n} \int_{\underline{\theta_i}}^{\overline{\theta_i}} \left(\overline{v}_i(\theta_i)\theta_i - U_i(\underline{\theta_i}) - \int_{\underline{\theta_i}}^{\theta_i} \overline{v}_i(s)ds \right) \phi_i(\theta_i)d\theta_i.$$

Integrating by parts the above expression, the search engine's problem can be written as one of choosing the $y_{ij}(\cdot)$ functions and the values $U_1(\underline{\theta_1}), \ldots, U_n(\underline{\theta_n})$ to maximize

$$\int_{\underline{\theta_1}}^{\overline{\theta_1}} \ldots \int_{\underline{\theta_n}}^{\overline{\theta_n}} \left[\sum_{i=1}^{n} v_i(y(\theta_i, \theta_{-i}))J_i(\theta_i) \right] \left[\prod_{i=1}^{n} \phi_i(\theta_i) \right] d\theta_n \ldots d\theta_1 - \sum_{i=1}^{n} U_i(\underline{\theta_i})$$

subject to constraints (i), (ii), and (iv'), where

$$J_i(\theta_i) = \theta_i - \frac{1 - \Phi_i(\theta_i)}{\phi_i(\theta_i)}.$$

It is evident that the solution must have $U_i(\underline{\theta_i}) = 0$ for all $i = 1, 2, \ldots, n$. Hence, the search engine's problem reduces to choosing functions $y_{ij}(\cdot)$ to maximize

$$\int_{\underline{\theta_1}}^{\overline{\theta_1}} \ldots \int_{\underline{\theta_n}}^{\overline{\theta_n}} \left[\sum_{i=1}^{n} v_i(y(\theta_i, \theta_{-i}))J_i(\theta_i) \right] \left[\prod_{i=1}^{n} \phi_i(\theta_i) \right] d\theta_n \ldots d\theta_1$$

subject to constraints (i) and (ii).

Let us ignore the constraint (i) for the moment. Then an inspection of the above expression indicates that $y_{ij}(\cdot)$ is a solution to this relaxed problem iff for all $i = 1, 2, \ldots, n$ we have

$$y_{ij}(\theta) = \begin{cases} 0 \quad \forall j = 1, 2, \ldots, m & : \text{ if } J_i(\theta_i) < 0 \\ 1 \quad \forall j = 1, 2, \ldots, m < n & : \text{ if } J_i(\theta_i) = J^{(j)} \\ 1 \quad \forall j = 1, 2, \ldots, n \leq m & : \text{ if } J_i(\theta_i) = J^{(j)} \\ 0 & : \text{ otherwise} \end{cases} \tag{12}$$

where $J^{(j)}$ is the j^{th} highest value among $J_i(\theta_i)$s.

In other words, if we ignore the constraint (i) then $y_{ij}(\cdot)$ is a solution to this relaxed problem if and only if no slot is allocated to any advertiser having negative value $J_i(\theta_i)$, and the rest of the advertisers' ads are displayed in the same order as the values of $J_i(\theta_i)$. That is, the first slot is allocated to the advertiser who has the highest nonnegative value for $J_i(\theta_i)$, the second slot is allocated to the advertiser who has the second highest nonnegative value for $J_i(\theta_i)$, and so on.

Now, recall the definition of $\overline{v}_i(\cdot)$. It is easy to write down the following expression:

$$\overline{v}_i(\theta_i) = E_{\theta_{-i}} \left[v_i(y(\theta_i, \theta_{-i})) \right] \tag{13}$$

$$= E_{\theta_{-i}} \left[\sum_{j=1}^{m} y_{ij}(\theta_i, \theta_{-i})\alpha_j \right]. \tag{14}$$

Now if we assume that $J_i(\cdot)$ is nondecreasing in θ_i, it is easy to see that the above solution $y_{ij}(\cdot)$, given by (12), will be nondecreasing in θ_i, which in turn implies, by looking at expression (14), that $\overline{v}_i(\cdot)$ is nondecreasing in θ_i. Thus, the solution to this relaxed problem actually satisfies constraint (i) under the assumption that $J_i(\cdot)$ is nondecreasing. Assuming that $J_i(\cdot)$ is nondecreasing, the solution given by (12) appears to be the solution of the optimal mechanism design problem for sponsored search auction. Note that in Equation (12), we have written the allocation rule $J_i(\cdot)$ as a function of actual type profile θ of the advertisers rather than the bid vector b. This is because in an OPT mechanism, each advertiser bids his true type, and we have $b_i = \theta_i \;\; \forall \, i = 1, \ldots, n$.

The condition that $J_i(\cdot)$ is nondecreasing in θ_i is met by most distribution functions such as uniform and exponential. In the rest of this chapter, we will work with the assumption that for every advertiser i, $J_i(\cdot)$ is nondecreasing in θ_i.

It is interesting to note that in the above allocation rule, the condition $J_i(.) > 0$ for qualifying an advertiser to display his/her ad can be expressed more conveniently in the form of reserve price in the following sense. For each advertiser i, we first compute the value of θ_i at which we have $J_i(\theta_i) = 0$. This value we call the reserve price of advertiser i. Now the allocation rule says that we first discard all those advertisers whose bid is less than their corresponding reserve price. Among the remaining advertisers, we allocate the advertisers in decreasing order of their $J_i(\theta_i)$ values. Further, if the advertisers are symmetric then the reserve price will be the same for all the advertisers, and moreover if $J_i(.)$ is nondecreasing in θ_i then among the qualified advertisers, the allocation rule will be the same as the GFP allocation rule. This interpretation of the allocation rule is the same as the allocation rule in [7] under the parallel case. This observation leads to the following proposition.

Proposition 7.1 *If the advertisers have nonidentical distribution functions $\Phi_i(\cdot)$ then the advertiser who has the k^{th} largest value of $J_i(b_i)$ is* not *necessarily the advertiser who has bid the k^{th} highest amount. Thus the OPT mechanism need not be allocatively efficient and, therefore, need not be ex post efficient.*

Proposition 7.2 *If the advertisers are symmetric in following sense*

- $\Theta_1 = \ldots = \Theta_n = \Theta$
- $\Phi_1(\cdot) = \ldots = \Phi_n(\cdot) = \Phi(\cdot)$

and for every advertiser i, we have $J_i(\cdot) > 0$ and $J_i(\cdot)$ is nondecreasing, then

- $J_i(\cdot) = \ldots = J_n(\cdot) = J(\cdot)$.
- *The rank of an advertiser in the decreasing order sequence of $J_1(b_1), \ldots, J_n(b_n)$ is precisely the same as the rank of the same advertiser in the decreasing order sequence of $b_1, \ldots, b_n$.*
- *For a given bid vector b, the OPT mechanism results in the same allocation as suggested by the GFP, the GSP, and the VCG mechanisms.*
- *The OPT mechanism is allocatively efficient.*

7.2 Payment Rule

Now we compute the payment rule. Again, following Myerson's line of attack, the optimal expected payment rule $t_i(\cdot)$ must be chosen in such a way that it satisfies

$$\bar{t}_i(\theta_i) = E_{\theta_{-i}}[t_i(\theta_i, \theta_{-i})] = \theta_i \bar{v}_i(\theta_i) - U_i(\theta_i) = \theta_i \bar{v}_i(\theta_i) - \int_{\underline{\theta_i}}^{\theta_i} \bar{v}_i(s) ds. \tag{15}$$

Looking at the above formula, we can say that if the payment rule $t_i(\cdot)$ satisfies the following formula (16) then it would also satisfy formula (15).

$$t_i(\theta_i, \theta_{-i}) = \theta_i v_i(y(\theta_i, \theta_{-i})) - \int_{\underline{\theta_i}}^{\theta_i} v_i(s, \theta_{-i}) ds \;\; \forall\, \theta \in \Theta \tag{16}$$

The above formula can be rewritten in a more intuitive way, for which we need to define the following quantities for any vector θ_{-i}:

Case 1 ($m < n$):

$$\begin{aligned}
z_{i1}(\theta_{-i}) &= \inf\left\{\theta_i | J_i(\theta_i) > 0 \text{ and } J_i(\theta_i) \geq J_{-i}^{(1)}\right\} \\
z_{i2}(\theta_{-i}) &= \inf\left\{\theta_i | J_i(\theta_i) > 0 \text{ and } J_{-i}^{(1)} > J_i(\theta_i) \geq J_{-i}^{(2)}\right\} \\
\vdots &= \vdots \\
z_{im}(\theta_{-i}) &= \inf\left\{\theta_i | J_i(\theta_i) > 0 \text{ and } J_{-i}^{(m-1)} > J_i(\theta_i) \geq J_{-i}^{(m)}\right\}.
\end{aligned}$$

Case 2 ($n \leq m$):

$$\begin{aligned}
z_{i1}(\theta_{-i}) &= \inf\left\{\theta_i | J_i(\theta_i) > 0 \text{ and } J_i(\theta_i) \geq J_{-i}^{(1)}\right\} \\
z_{i2}(\theta_{-i}) &= \inf\left\{\theta_i | J_i(\theta_i) > 0 \text{ and } J_{-i}^{(1)} > J_i(\theta_i) \geq J_{-i}^{(2)}\right\} \\
\vdots &= \vdots \\
z_{in}(\theta_{-i}) &= \inf\left\{\theta_i | J_i(\theta_i) > 0 \text{ and } J_{-i}^{(n-1)} > J_i(\theta_i)\right\}.
\end{aligned}$$

In the above definitions, $J_{-i}^{(k)}$ is the k^{th} highest value among the following $(n-1)$ values

$$J_1(\theta_1), \ldots, J_{i-1}(\theta_{i-1}), J_{i+1}(\theta_{i+1}), \ldots, J_n(\theta_n).$$

The quantity $z_{ik}(\theta_{-i})$ is the infimum of all the bids for advertisers i that can make him win the k^{th} slot against the bid vector θ_{-i} from the other advertisers. In view of the above definitions, we can write

Case 1 ($m < n$):

$$v_i(y(\theta_i, \theta_{-i})) = \begin{cases}
\alpha_1 & : \text{ if } \theta_i \geq z_{i1}(\theta_{-i}) \\
\alpha_2 & : \text{ if } z_{i1}(\theta_{-i}) > \theta_i \geq z_{i2}(\theta_{-i}) \\
\vdots & : \vdots \\
\alpha_m & : \text{ if } z_{i(m-1)}(\theta_{-i}) > \theta_i \geq z_{im}(\theta_{-i}) \\
0 & : \text{ if } z_{im}(\theta_{-i}) > \theta_i.
\end{cases}$$

Case 2 $(n \leq m)$**:**

$$v_i(y(\theta_i,\theta_{-i})) = \begin{cases} \alpha_1 & : \text{ if } \theta_i \geq z_{i1}(\theta_{-i}) \\ \alpha_2 & : \text{ if } z_{i1}(\theta_{-i}) > \theta_i \geq z_{i2}(\theta_{-i}) \\ \vdots & : \vdots \\ \alpha_n & : \text{ if } z_{i(n-1)}(\theta_{-i}) > \theta_i. \end{cases}$$

This gives us the following expressions for $\int_{\underline{\theta_i}}^{\theta_i} v_i(s,\theta_{-i})ds$. In these expressions, r is the position of advertiser i's ad.

Case 1 $(m < n)$**:**

$$\int_{\underline{\theta_i}}^{\theta_i} v_i(y(s,\theta_{-i}))ds = \begin{cases} \alpha_r(\theta_i - z_{ir}(\theta_{-i})) + \sum\limits_{j=(r+1)}^{m} \alpha_j \left(z_{i(j-1)}(\theta_{-i}) - z_{ij}(\theta_{-i})\right) & : \text{ if } 1 \leq r \leq (m-1) \\ \alpha_m(\theta_i - z_{im}(\theta_{-i})) & : \text{ if } r = m \\ 0 & : \text{ otherwise.} \end{cases}$$

Case 2 $(n \leq m)$**:**

$$\int_{\underline{\theta_i}}^{\theta_i} v_i(y(s,\theta_{-i}))ds = \begin{cases} \alpha_r(\theta_i - z_{ir}(\theta_{-i})) + \sum\limits_{j=(r+1)}^{n} \alpha_j \left(z_{i(j-1)}(\theta_{-i}) - z_{ij}(\theta_{-i})\right) & : \text{ if } 1 \leq r \leq (n-1) \\ \alpha_n(\theta_i - z_{in}(\theta_{-i})) & : \text{ if } r = n. \end{cases}$$

Substituting the above value for $\int_{\underline{\theta_i}}^{\theta_i} v_i(y(s,\theta_{-i}))ds$ in formula (16), we get

Case 1 $(m < n)$**:**

$$p_i(\theta_i,\theta_{-i}) = \frac{1}{\alpha_r} t_i(\theta_i,\theta_{-i}) = \begin{cases} \frac{\alpha_m}{\alpha_r} z_{im}(\theta_{-i}) + \frac{1}{\alpha_r} \sum\limits_{j=r}^{m-1} \beta_j z_{ij}(\theta_{-i}) & : \text{ if } 1 \leq r \leq (m-1) \\ z_{im}(\theta_{-i}) & : \text{ if } r = m \\ 0 & : \text{ otherwise.} \end{cases} \tag{17}$$

Case 2 $(n \leq m)$**:**

$$p_i(\theta_i,\theta_{-i}) = \frac{1}{\alpha_r} t_i(\theta_i,\theta_{-i}) = \begin{cases} \frac{\alpha_n}{\alpha_r} z_{in}(\theta_{-i}) + \frac{1}{\alpha_r} \sum\limits_{j=r}^{n-1} \beta_j z_{ij}(\theta_{-i}) & : \text{ if } 1 \leq r \leq (n-1) \\ z_{in}(\theta_{-i}) & : \text{ if } r = n \\ 0 & : \text{ otherwise.} \end{cases} \tag{18}$$

The above relations say that an advertiser i must pay only when his ad receives a click, and he pays the amount equal to $p_i(\theta)$. Note that in above relations, we have expressed the payment rule $p_i(\cdot)$ as a function of actual type profile θ of the advertisers rather than the bid vector b. This is because in an OPT mechanism, each advertiser bids his true type, and we have $b_i = \theta_i \ \forall \, i = 1, \ldots, n$.

Thus, we can say that Equation (12) describes the allocation rule for the OPT mechanism and Equations (17) and (18) describe the the payment rule for the OPT mechanism.

In what follows, we discuss an important special case of the OPT mechanism when the advertisers are symmetric.

7.3 OPT Mechanism and Symmetric Advertisers

Let us assume that advertisers are symmetric in the following sense:

- $\Theta_1 = \ldots = \Theta_n = \Theta = [L, U]$.
- $\Phi_1(\cdot) = \ldots = \Phi_n(\cdot) = \Phi(\cdot)$.

Also, we assume that

- $J(\cdot)$ is non-decreasing over the interval $[L, U]$.
- $J(x) > 0 \;\; \forall \, x \in [L, U]$.

Note that if $J(L) > 0$ then we must have $L > 0$.

Proposition 7.2 shows that if the advertisers are symmetric then the allocation rule under the OPT mechanism is the same as the GFP, the GSP, and the VCG mechanisms. Coming to the payment rule, it is easy to verify that if advertiser i is allocated the slot r for the bid vector (θ_i, θ_{-i}) then we should have

Case 1 ($m < n$)**:**

$$z_{ij}(\theta_{-i}) = \begin{cases} \theta^{(j)} & : \text{ if } 1 \leq j \leq (r-1) \\ \theta^{(j+1)} & : \text{ if } r \leq j \leq m. \end{cases} \tag{19}$$

Case 2 ($n \leq m$)**:**

$$z_{ij}(\theta_{-i}) = \begin{cases} \theta^{(j)} & : \text{ if } 1 \leq j \leq (r-1) \\ \theta^{(j+1)} & : \text{ if } r \leq j \leq (n-1) \\ L & : \text{ if } j = n. \end{cases} \tag{20}$$

If we substitute Equations (19) and (20) into Equations (17) and (18), then we get the following payment rule for the OPT mechanism when the advertisers are symmetric:

Case 1 ($m < n$)**:**

$$p_i(\theta_i, \theta_{-i}) = \frac{1}{\alpha_r} t_i(\theta_i, \theta_{-i}) = \begin{cases} \frac{\alpha_m}{\alpha_r}\theta^{(m+1)} + \frac{1}{\alpha_r}\sum_{j=r}^{m-1} \beta_j \theta^{(j+1)} & : \text{ if } 1 \leq r \leq (m-1) \\ \theta^{(m+1)} & : \text{ if } r = m \\ 0 & : \text{ otherwise.} \end{cases} \tag{21}$$

Case 2 ($n \leq m$)**:**

$$p_i(\theta_i, \theta_{-i}) = \frac{1}{\alpha_r} t_i(\theta_i, \theta_{-i}) = \begin{cases} \frac{\alpha_n}{\alpha_r} L + \frac{1}{\alpha_r}\sum_{j=r}^{n-1} \beta_j \theta^{(j+1)} & : \text{ if } 1 \leq r \leq (n-1) \\ L & : \text{ if } r = n \\ 0 & : \text{ otherwise.} \end{cases} \tag{22}$$

Compare the above equations with the payment rule of the VCG mechanism given by Equations (9) and (10). This comparison leads to the following proposition:

Proposition 7.3 *If the advertisers are symmetric in following sense*

- $\Theta_1 = \ldots = \Theta_n = \Theta = [L, U]$
- $\Phi_1(\cdot) = \ldots = \Phi_n(\cdot) = \Phi(\cdot)$

and for every advertiser i, we have $J_i(\cdot) > 0$ and $J_i(\cdot)$ is non-decreasing over the interval $[L, U]$, then

- *the payment rule for Case 1 coincides with the corresponding payment rule in the VCG mechanism,*
- *and the payment rule for the Case 2 differs from the corresponding payment rule of the VCG mechanism just by a constant amount L.*

Note that L cannot be zero because of the assumption that $J(L) > 0$.

8 Problems

1. Consider 5 bidders $\{1, 2, 3, 4, 5\}$ with valuations $v_1 = 20; v_2 = 15; v_3 = 12; v_4 = 10; v_5 = 6$ participating in a sponsored search auction where there are 3 sponsored slots. What will be the result of applying the GSP and VCG auctions here.
2. In the above problem, instead of three sponsored slots, let there be three identical items on auction. Let the demand by agent 1 be 2 units with the rest of agents having unit demand. Determine the winners and their payments when Clarke mechanism is applied here.

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