Game Theory

Lecture Notes By

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Myerson Optimal Auction

Note: *This is a only a draft version, so there could be flaws. If you find any errors, please do send email to* `hari@csa.iisc.ernet.in`. *A more thorough version would be available soon in this space.*

1 Myerson Optimal Auction

A key problem that faces a social planner is to decide which direct revelation mechanism (or equivalently, social choice function) is *optimal* for a given problem. We now attempt to formalize the notion of optimality of social choice functions and optimal mechanisms. For this, we first define the concept of a *social utility function*.

Definition 1.1 (Social Utility Function) *Social utility function $w : \mathbb{R}^n \to \mathbb{R}$ that aggregates the profile $(u_1, \ldots, u_n) \in \mathbb{R}^n$ of individual utility values of the agents into a social utility.*

Consider a mechanism design problem and a direct revelation mechanism $\mathscr{D} = ((\Theta_i)_{i \in N}, f(\cdot))$ proposed for it. Let $(\theta_1, \ldots, \theta_n)$ be the actual type profile of the agents and assume for a moment that they will all reveal their true types when requested by the planner. In such a case, the social utility that would be realized by the social planner for a type profile θ of the agents is given by:

$$w(u_1(f(\theta), \theta_1), \ldots, u_n(f(\theta), \theta_n)). \tag{1}$$

However, recall the implicit assumption behind a mechanism design problem, namely, that the agents are autonomous, and they would report a type as dictated by their rational behavior. Therefore, the assumption that all the agents will report their true types is not true in general. In general, rationality implies that the agents report their types according to a strategy suggested by a Bayesian Nash equilibrium $s^*(\cdot) = (s_1^*(\cdot), \ldots, s_n^*(\cdot))$ of the underlying Bayesian game. In such a case, the social utility that would be realized by the social planner for a type profile θ of the agents is given by

$$w(u_1(f(s^*(\theta)), \theta_1), \ldots, u_n(f(s^*(\theta)), \theta_n)). \tag{2}$$

In some instances, the above Bayesian Nash equilibrium may turn out to be a dominant strategy equilibrium. Better still, truth revelation by all agents could turn out to be a Bayesian Nash equilibrium or a dominant strategy equilibrium.

1.1 Optimal Mechanism Design Problem

In view of the above notion of a social utility function, it is clear that the objective of a social planner would be to look for a social choice function $f(\cdot)$ that would maximize the expected social utility for a given social utility function $w(\cdot)$. However, being the social planner, it is always expected of him to be fair to all the agents. Therefore, the social planner would first put a few desirable constraints on the set of social choice functions from which he can probably choose. The desirable constraints may include any combination of all the previously studied properties of a social choice function, such as ex-post efficiency, incentive compatibility, and individual rationality. This set of social choice functions is known as a *set of feasible social choice functions* and is denoted by F. Thus, the problem of a social planner can now be cast as an optimization problem where the objective is to maximize the expected social utility, and the constraint is that the social choice function must be chosen from the feasible set F. This problem is known as the *optimal mechanism design* problem and the solution of the problem would be social choice function $f^*(\cdot) \in F$, which is used to define the optimal mechanism $\mathscr{D}^* = ((\Theta_i)_{i \in N}, f^*(\cdot))$ for the problem that is being studied.

Depending on whether the agents are loyal or autonomous rational entities, the optimal mechanism design problem may take two different forms.

$$\underset{f(\cdot) \in F}{\text{maximize}} \; E_\theta \left[w(u_1(f(\theta), \theta_1), \ldots, u_n(f(\theta), \theta_n)) \right] \tag{3}$$

$$\underset{f(\cdot) \in F}{\text{maximize}} \; E_\theta \left[w(u_1(f(s^*(\theta)), \theta_1), \ldots, u_n(f(s^*(\theta)), \theta_n)) \right] \tag{4}$$

The problem (13) is relevant when the agents are loyal and always reveal their true types whereas the problem (14) is relevant when the agents are rational. At this point of time, one may ask how to define the set of feasible social choice functions F. There is no unique definition of this set. The set of feasible social choice functions is a subjective judgment of the social planner. The choice of the set F depends on the desirable properties the social planner would wish to have in the optimal social choice function $f^*(\cdot)$. If we define

$$\begin{aligned}
F_{\text{DSIC}} &= \{f : \Theta \to X | f(\cdot) \text{ is dominant strategy incentive compatible}\} \\
F_{\text{BIC}} &= \{f : \Theta \to X | f(\cdot) \text{ is Bayesian incentive compatible}\} \\
F_{\text{EPIR}} &= \{f : \Theta \to X | f(\cdot) \text{ is ex-post individual rational}\} \\
F_{\text{IIR}} &= \{f : \Theta \to X | f(\cdot) \text{ is interim individual rational}\}
\end{aligned}$$

$$\begin{aligned}
F_{\text{EAIR}} &= \{f : \Theta \to X | f(\cdot) \text{ is ex-ante individual rational}\} \\
F_{\text{EAE}} &= \{f : \Theta \to X | f(\cdot) \text{ is ex-ante efficient}\} \\
F_{\text{IE}} &= \{f : \Theta \to X | f(\cdot) \text{ is interim efficient}\} \\
F_{\text{EPE}} &= \{f : \Theta \to X | f(\cdot) \text{ is ex post efficient}\}\,.
\end{aligned}$$

The set of feasible social choice functions F may be either any one of the above sets or intersection of any combination of the above sets. For example, the social planner may choose $F = F_{BIC} \bigcap F_{IIR}$. In the literature, this particular feasible set is known as *incentive feasible set* due to Myerson [1]. Also, note that if the agents are loyal then the sets F_{DSIC} and F_{BIC} will be equal to the whole set of all the social choice functions.

1.2 Myerson's Optimal Reverse Auction

We now consider the problem of procuring a single indivisible item from among a pool of suppliers and present Myerson's optimal auction that minimizes the expected cost of procurement subject to Bayesian incentive compatibility and interim individual rationality of all the selling agents. The classical Myerson auction [2] is for maximizing the expected revenue of a selling agent who wishes to sell an indivisible item to a set of prospective buying agents. We present it here for the reverse auction case.

Each bidder i's type lies in an interval $\Theta_i = [\underline{\theta_i}, \overline{\theta_i}]$. We impose the following additional conditions on the environment.

1. The auctioneer and the bidders are risk neutral.
2. Bidders' types are statistically independent, that is, the joint density $\phi(\cdot)$ has the form $\phi_1(\cdot) \times \ldots \times \phi_n(\cdot)$.
3. $\phi_i(\cdot) > 0 \; \forall \; i = 1, \ldots, n$.
4. We generalize the outcome set X by allowing a random assignment of the good. Thus, we now take $y_i(\theta)$ to be seller i's probability of selling the good when the vector of announced types is $\theta = (\theta_1, \ldots, \theta_n)$. Thus, the new outcome set is given by

$$X = \Big\{ (y_0, y_1 \ldots, y_n, t_0, t_1, \ldots, t_n) : y_0 \in [0,1], t_0 \leq 0, y_i \in [0,1], t_i \geq 0 \; \forall \; i = 1, \ldots, n, \\ \sum_{i=1}^{n} y_i \leq 1; \; \sum_{i=0}^{n} t_i = 0 \Big\}.$$

Recall that the utility functions of the agents in this example are given by, $\forall \; i = 1, \ldots, n$,

$$u_i(f(\theta), \theta_i) = u_i(y_0(\theta), \ldots, y_n(\theta), t_0(\theta), \ldots, t_n(\theta), \theta_i) = -\theta_i y_i(\theta) + t_i(\theta).$$

Thus, viewing $y_i(\theta) = v_i(k(\theta))$ in conjunction with the second and third conditions above, we can claim that the underlying environment here is linear.

In the above example, we assume that the auctioneer (buyer) is the social planner and he is looking for an optimal direct revelation mechanism to buy the good. Myerson's [2] idea was that the auctioneer must use a social choice function that is Bayesian incentive compatible and interim individual rational and at the same time minimizes the cost to the auctioneer. Thus, in this problem, the set of feasible social choice functions is given by $F = F_{BIC} \bigcap F_{IIR}$. The objective function in this case would be to minimize the total expected cost of the buyer, which would be given by

$$E_\theta \left[w(u_1(f(\theta), \theta_1), \ldots, u_n(f(\theta), \theta_n)) \right] = E_\theta \left[\sum_{i=1}^{n} t_i(\theta) \right].$$

Note that in the above objective function we have used $f(\theta)$ and not $f(s^*(\theta))$. This is because in the set of feasible social choice functions we are considering only BIC social choice functions, and for these functions we have $s^*(\theta) = \theta \;\; \forall \; \theta \in \Theta$. Thus, the Myerson's optimal auction design problem can be formulated as the following optimization problem:

$$\underset{f(\cdot) \in F}{\text{minimize}} \; E_\theta \left[\sum_{i=1}^{n} t_i(\theta) \right] \tag{5}$$

where

$$F = \{f(\cdot) = (y_0(\cdot), y_1(\cdot), \ldots, y_n(\cdot), t_0(\cdot), t_1(\cdot), \ldots, t_n(\cdot)) : f(\cdot) \text{ is BIC and interim IR}\}.$$

We have seen Myerson's Characterization Theorem (Theorem 2.12) for BIC SCFs in linear environment. Similarly, we can say that an SCF $f(\cdot)$ in the above context would be BIC iff it satisfies the following two conditions:

1. $\overline{y_i}(\cdot)$ is nonincreasing for all $i = 1, \ldots, n$.
2. $U_i(\theta_i) = U_i(\overline{\theta_i}) + \int_{\theta_i}^{\overline{\theta_i}} \overline{y_i}(s)ds \ \ \forall \, \theta_i \in \Theta_i; \ \forall \, i = 1, \ldots, n.$

Also, we can invoke the definition of interim individual rationality to claim that the an SCF $f(\cdot)$ in the above context would be interim IR iff it satisfies the following conditions:

$$U_i(\theta_i) \geq 0 \ \ \forall \theta_i \in \Theta_i; \ \forall \, i = 1, \ldots, n$$

where

- $\overline{t_i}(\hat{\theta}_i) = E_{\theta_{-i}}[t_i(\hat{\theta}_i, \theta_{-i})]$ is bidder i's expected transfer given that he announces his type to be $\hat{\theta}_i$ and that all the bidders $j \neq i$ truthfully reveal their types.
- $\overline{y_i}(\hat{\theta}_i) = E_{\theta_{-i}}[y_i(\hat{\theta}_i, \theta_{-i})]$ is the probability that object will be procured from bidder i given that he announces his type to be $\hat{\theta}_i$ and all bidders $j \neq i$ truthfully reveal their types.
- $U_i(\theta_i) = -\theta_i \overline{y_i}(\theta_i) + \overline{t_i}(\theta_i)$ (we can take unconditional expectation because types are independent).

Based on the above, problem (15) can be rewritten as follows:

$$\underset{(y_i(\cdot), U_i(\cdot))_{i \in N}}{\text{minimize}} \ \sum_{i=1}^{n} \int_{\underline{\theta_i}}^{\overline{\theta_i}} \left(\theta_i \overline{y_i}(\theta_i) + U_i(\theta_i)\right) \phi_i(\theta_i) d\theta_i \tag{6}$$

subject to

(i) $\overline{y_i}(\cdot)$ is nonincreasing $\ \forall \, i = 1, \ldots, n$

(ii) $y_i(\theta) \in [0, 1], \sum_{i=1}^{n} y_i(\theta) \leq 1 \ \forall i = 1, \ldots, n, \forall \, \theta \in \Theta$

(iii) $U_i(\theta_i) = U_i(\overline{\theta_i}) + \int_{\theta_i}^{\overline{\theta_i}} \overline{y_i}(s)ds \ \ \forall \, \theta_i \in \Theta_i; \ \forall \, i = 1, \ldots, n$

(iv) $U_i(\theta_i) \geq 0 \ \forall \, \theta_i \in \Theta_i; \ \forall \, i = 1, \ldots, n.$

We first note that if constraint (iii) is satisfied then constraint (iv) will be satisfied iff $U_i(\overline{\theta_i}) \geq 0 \ \ \forall \, i = 1, \ldots, n$. As a result, we can replace the constraint (iv) with

$$(\text{iv}') \quad U_i(\overline{\theta_i}) \geq 0 \ \forall \, i = 1, \ldots, n$$

Next, substituting for $U_i(\theta_i)$ in the objective function from constraint (iii), we get

$$\sum_{i=1}^{n} \int_{\underline{\theta_i}}^{\overline{\theta_i}} \left(\theta_i \overline{y_i}(\theta_i) + U_i(\overline{\theta_i}) + \int_{\theta_i}^{\overline{\theta_i}} \overline{y_i}(s)ds \right) \phi_i(\theta_i) d\theta_i.$$

Integrating by parts the above expression, the auctioneer's problem can be written as one of choosing the $y_i(\cdot)$ functions and the values $U_1(\overline{\theta_1}), \ldots, U_n(\overline{\theta_n})$ to minimize

$$\int_{\underline{\theta_1}}^{\overline{\theta_1}} \cdots \int_{\underline{\theta_n}}^{\overline{\theta_n}} \left[\sum_{i=1}^{n} y_i(\theta_i) J_i(\theta_i) \right] \left[\prod_{i=1}^{n} \phi_i(\theta_i) \right] d\theta_n \ldots d\theta_1 + \sum_{i=1}^{n} U_i(\overline{\theta_i})$$

subject to constraints (i), (ii), and (iv'), where

$$J_i(\theta_i) = \left(\theta_i + \frac{\Phi_i(\theta_i)}{\phi_i(\theta_i)} \right).$$

It is evident that the solution must have $U_i(\overline{\theta_i}) = 0$ for all $i = 1, \ldots, n$. Hence, the auctioneer's problem reduces to choosing functions $y_i(\cdot)$ to minimize

$$\int_{\underline{\theta_1}}^{\overline{\theta_1}} \cdots \int_{\underline{\theta_n}}^{\overline{\theta_n}} \left[\sum_{i=1}^{n} y_i(\theta_i) J_i(\theta_i) \right] \left[\prod_{i=1}^{n} \phi_i(\theta_i) \right] d\theta_n \ldots d\theta_1$$

subject to constraints (i) and (ii).

Let us ignore constraint (i) for the moment. Then inspection of the above expression indicates that $y_i(\cdot)$ is a solution to this relaxed problem iff for all $i = 1, \ldots, n$, we have

$$y_i(\theta) = \begin{cases} 0 & : \text{ if } J_i(\theta_i) > \min \left\{ \overline{\theta_0}, \min_{h \neq i} J_h(\theta_h) \right\} \\ 1 & : \text{ if } J_i(\theta_i) < \min \left\{ \overline{\theta_0}, \min_{h \neq i} J_h(\theta_h) \right\}. \end{cases} \tag{7}$$

Note that $J_i(\theta_i) = \min \left\{ \overline{\theta_0}, \min_{h \neq i} J_h(\theta_h) \right\}$ is a zero probability event.

In other words, if we ignore the constraint (i) then $y_i(\cdot)$ is a solution to this relaxed problem iff the good is allocated to a bidder who has the lowest nonnegative value for $J_i(\theta_i)$. Now, recall the definition of $\overline{y_i}(\cdot)$. It is easy to write down the following expression:

$$\overline{y_i}(\theta_i) \quad = \quad E_{\theta_{-i}} \left[y_i(\theta_i, \theta_{-i}) \right]. \tag{8}$$

Now, if we assume that $J_i(\cdot)$ is nondecreasing in θ_i then it is easy to see that the above solution $y_i(\cdot)$, given by (17), will be nonincreasing in θ_i, which in turn implies, by looking at expression (18), that $\overline{y_i}(\cdot)$ is nonincreasing in θ_i. Thus, the solution to this relaxed problem actually satisfies constraint (i) under the assumption that $J_i(\cdot)$ is nondecreasing. Assuming that $J_i(\cdot)$ is nondecreasing, the solution given by (17) seems to be the solution of the optimal mechanism design problem for single unit-single item procurement auction. The condition that $J_i(\cdot)$ is nondecreasing in θ_i is met by most of the distribution functions such as Uniform and Exponential.

So far we have computed the allocation rule for the optimal mechanism and now we turn our attention toward the payment rule. The optimal payment rule $t_i(\cdot)$ must be chosen in such a way that it satisfies

$$\overline{t_i}(\theta_i) = E_{\theta_{-i}}[t_i(\theta_i, \theta_{-i})] = U_i\left(\theta_i\right) + \theta_i \overline{y_i}\left(\theta_i\right) = \int_{\theta_i}^{\overline{\theta_i}} \overline{y_i}(s) ds + \theta_i \overline{y_i}\left(\theta_i\right). \tag{9}$$

Looking at the above formula, we can say that if the payment rule $t_i(\cdot)$ satisfies the following formula (20), then it would also satisfy the formula (19).

$$t_i(\theta_i, \theta_{-i}) = \int_{\theta_i}^{\overline{\theta_i}} y_i(s, \theta_{-i}) ds + \theta_i y_i\left(\theta_i, \theta_{-i}\right) \quad \forall\, \theta \in \Theta. \tag{10}$$

The above formula can be rewritten more intuitively as follows. For any vector θ_{-i}, let us define

$$z_i(\theta_{-i}) = \sup\left\{\theta_i : J_i(\theta_i) < \overline{\theta_0} \text{ and } J_i(\theta_i) \leq J_j(\theta_j)\; \forall\, j \neq i\right\}.$$

Then $z_i(\theta_{-i})$ is the supremum of all winning bids for bidder i against θ_{-i}, so

$$y_i(\theta_i, \theta_{-i}) = \begin{cases} 1 & : \text{ if } \theta_i < z_i(\theta_{-i}) \\ 0 & : \text{ if } \theta_i > z_i(\theta_{-i}). \end{cases}$$

This gives us

$$\int_{\theta_i}^{\overline{\theta_i}} y_i(s, \theta_{-i}) ds = \begin{cases} z_i(\theta_{-i}) - \theta_i & : \text{ if } \theta_i \leq z_i(\theta_{-i}) \\ 0 & : \text{ if } \theta_i > z_i(\theta_{-i}). \end{cases}$$

Finally, the formula (20) becomes

$$t_i(\theta_i, \theta_{-i}) = \begin{cases} z_i(\theta_{-i}) & : \text{ if } \theta_i \leq z_i(\theta_{-i}) \\ 0 & : \text{ if } \theta_i > z_i(\theta_{-i}). \end{cases}$$

That is, bidder i will receive payment only when the good is procured from him, and then he receives an amount equal to his highest possible winning bid.

We make a few interesting observations:

1. When the various bidders have differing distribution function $\Phi_i(\cdot)$ then, the bidder who has the smallest value of $J_i(\theta_i)$ is *not* necessarily the bidder who has bid the lowest amount for the good. Thus Myerson's optimal auction need not be allocatively efficient, and therefore, need not be ex-post efficient.

2. If the bidders are symmetric, that is,

   - $\Theta_1 = \ldots = \Theta_n = \Theta$
   - $\Phi_1(\cdot) = \ldots = \Phi_n(\cdot) = \Phi(\cdot)$,

then the allocation rule would be precisely the same as that of first-price reverse auction and second-price reverse auction. In such a case the object would be allocated to the lowest bidder. In such a situation, the optimal auction would also become allocatively efficient, and the payment rule described above would coincide with the payment rules in second-price reverse auction. In other words, the second price reverse auction would be an optimal auction when the bidders are symmetric. Therefore, many times, the optimal auction is also known as *modified Vickrey auction.*

Riley and Samuelson [3] also have studied the problem of design of an optimal auction for selling a single unit of a single item. They assume the bidders to be symmetric. Their work is less general than that of Myerson [2].

2 Optimal Mechanisms

An obvious problem that faces a social planner is to decide which direct revelation mechanism (or equivalently, social choice function) is *optimal* for a given problem. In the rest of this paper, our objective is to familiarize the reader with a couple of techniques which social planner can adopt to design an optimal direct revelation mechanism for a given problem at hand.

One notion of optimality in multi-agent systems is that of *Pareto efficiency*. We now define three different notions of efficiency: ex-ante, interim, and ex-post. These notions were introduced by Holmstorm and Myerson [4].

Definition 2.1 (Ex-Ante Efficiency) *For any given set of social choice functions F, and any member $f(\cdot) \in F$, we say that $f(\cdot)$ is ex-ante efficient in F if there is no other $\hat{f}(\cdot) \in F$ having the following two properties*

$$\begin{aligned} E_\theta[u_i(\hat{f}(\theta), \theta_i)] &\geq E_\theta[u_i(f(\theta), \theta_i)] \;\; \forall\, i = 1, \ldots, n \\ E_\theta[u_i(\hat{f}(\theta), \theta_i)] &> E_\theta[u_i(f(\theta), \theta_i)] \;\; \textit{for some } i \end{aligned}$$

Definition 2.2 (Interim Efficiency) *For any given set of social choice functions F, and any member $f(\cdot) \in F$, we say that $f(\cdot)$ is interim efficient in F if there is no other $\hat{f}(\cdot) \in F$ having the following two properties*

$$\begin{aligned} E_{\theta_{-i}}[u_i(\hat{f}(\theta), \theta_i)|\theta_i] &\geq E_{\theta_{-i}}[u_i(f(\theta), \theta_i)|\theta_i] \;\; \forall\, i = 1, \ldots, n, \; \forall\, \theta_i \in \Theta_i \\ E_{\theta_{-i}}[u_i(\hat{f}(\theta), \theta_i)|\theta_i] &> E_{\theta_{-i}}[u_i(f(\theta), \theta_i)|\theta_i] \;\; \textit{for some } i \textit{ and some } \theta_i \in \Theta_i \end{aligned}$$

Definition 2.3 (Ex-Post Efficiency) *For any given set of social choice functions F, and any member $f(\cdot) \in F$, we say that $f(\cdot)$ is ex-post efficient in F if there is no other $\hat{f}(\cdot) \in F$ having the following two properties*

$$\begin{aligned} u_i(\hat{f}(\theta), \theta_i) &\geq u_i(f(\theta), \theta_i) \;\; \forall\, i = 1, \ldots, n, \; \forall\, \theta \in \Theta \\ u_i(\hat{f}(\theta), \theta_i) &> u_i(f(\theta), \theta_i) \;\; \textit{for some } i \textit{ and some } \theta \in \Theta \end{aligned}$$

Using the above definition of ex-post efficiency, we can say that a social choice function $f(\cdot)$ is ex-post efficient in the sense of definition 5.1 in [5] if and only if it is ex-post efficient in the sense of definition 2.3 when we take $F = \{f : \Theta \to X\}$.

The following proposition establishes a relationship among these three different notions of efficiency.

Proposition 2.1 *Given any set of feasible social choice functions F and $f(\cdot) \in F$, we have*

$$f(\cdot) \text{ is ex-ante efficient} \Rightarrow f(\cdot) \text{ is interim efficient} \Rightarrow f(\cdot) \text{ is ex-post efficient}$$

For proof of the above proposition, refer to Proposition 23.F.1 of [6]. Also, compare the above proposition with the Proposition **??**.

With this setup, we now try to formalize the design objectives of a social planner. For this, we need to define the concept known as *social utility function*.

Definition 2.4 (Social Utility Function) *A social utility function is a function $w : \mathbb{R}^n \to \mathbb{R}$ that aggregates the profile $(u_1, \ldots, u_n) \in \mathbb{R}^n$ of individual utility values of the agents into a social utility.*

Consider a mechanism design problem and a direct revelation mechanism $\mathscr{D} = ((\Theta_i)_{i \in N}, f(\cdot))$ proposed for it. Let $(\theta_1, \ldots, \theta_n)$ be the actual type profile of the agents and assume for a moment that they will all reveal their true types when requested by the planner. In such a case, the social utility that would be realized by the social planner for every possible type profile θ of the agents is given by:

$$w(u_1(f(\theta), \theta_1), \ldots, u_n(f(\theta), \theta_n)) \tag{11}$$

However, recall the implicit assumption behind a mechanism design problem, namely, that the agents are autonomous and they would report a type as dictated by their rational behavior. Therefore, the assumption that all the agents will report their true types is not true in general. In general, rationality implies that the agents report their types according to a strategy suggested by a Bayesian Nash equilibrium $s^*(\cdot) = (s_1^*(\cdot), \ldots, s_n^*(\cdot))$ of the underlying Bayesian game. In such a case, the social utility that would be realized by the social planner for every possible type profile θ of the agents is given by

$$w(u_1(f(s^*(\theta)), \theta_1), \ldots, u_n(f(s^*(\theta)), \theta_n)) \tag{12}$$

In some instances, the above Bayesian Nash equilibrium may turn out to be a dominant strategy equilibrium. Better still, truth revelation by all agents could turn out to be a Bayesian Nash equilibrium or a dominant strategy equilibrium.

2.1 Optimal Mechanism Design Problem

In view of the above notion of social utility function, it is clear that the objective of a social planner would be to look for a social choice function $f(\cdot)$ that would maximize the expected social utility for a given social utility function $w(\cdot)$. However, being the social planner, it is always expected of him to be fair to all the agents. Therefore, the social planner would first put a few fairness constraints on the set of social choice functions which he can probably choose from. The fairness constraints may include any combination of all the previously studied properties of a social choice function, such as ex-post efficiency, incentive compatibility, and individual rationality. This set of social choice functions is known as *set of feasible social choice functions* and is denoted by F. Thus, the problem of a social planner can now be cast as an optimization problem where the objective is to maximize the expected social utility and the constraint is that the social choice function must be chosen from the feasible set F. This problem is known as the *optimal mechanism design* problem and the solution of the problem is some social choice function $f^*(\cdot) \in F$ which is used to define the optimal mechanism $\mathscr{D}^* = ((\Theta_i)_{i \in N}, f^*(\cdot))$ for the problem that is being studied.

Depending on whether the agents are loyal or autonomous entities, the optimal mechanism design problem may take two different forms.

$$\underset{f(\cdot) \in F}{\text{maximize}} \; E_\theta \left[w(u_1(f(\theta), \theta_1), \ldots, u_n(f(\theta), \theta_n)) \right] \tag{13}$$

$$\underset{f(\cdot) \in F}{\text{maximize}} \; E_\theta \left[w(u_1(f(s^*(\theta)), \theta_1), \ldots, u_n(f(s^*(\theta)), \theta_n)) \right] \tag{14}$$

The problem (13) is relevant when the agents are loyal and always reveal their true types whereas the problem (14) is relevant when the agents are rational. At this point of time, one may ask how to define the set of feasible social choice functions F. There is no unique definition of this set. The set of feasible social choice functions is a subjective judgment of the social planner. The choice of the set F depends on what all fairness properties the social planner would wish to have in the optimal social choice function $f^*(\cdot)$. If we define

$$\begin{aligned}
F_{DSIC} &= \{f : \Theta \to X | f(\cdot) \text{ is dominant strategy incentive compatible}\} \\
F_{BIC} &= \{f : \Theta \to X | f(\cdot) \text{ is Bayesian incentive compatible}\} \\
F_{ExPostIR} &= \{f : \Theta \to X | f(\cdot) \text{ is ex-post individual rational}\} \\
F_{IntIR} &= \{f : \Theta \to X | f(\cdot) \text{ is interim individual rational}\} \\
F_{ExAnteIR} &= \{f : \Theta \to X | f(\cdot) \text{ is ex-ante individual rational}\} \\
F_{Ex-AnteEff} &= \{f : \Theta \to X | f(\cdot) \text{ is ex-ante efficient}\} \\
F_{IntEff} &= \{f : \Theta \to X | f(\cdot) \text{ is interim efficient}\} \\
F_{Ex-PostEff} &= \{f : \Theta \to X | f(\cdot) \text{ is ex post efficient}\}
\end{aligned}$$

The set of feasible social choice functions F may be either any one of the above sets or intersection of any combination of the above sets. For example, the social planner may choose $F = F_{BIC} \bigcap F_{IntIR}$. In the literature, this particular feasible set is known as *incentive feasible set* due to Myerson [1]. Also, note that if the agents are loyal then the sets F_{DSIC} and F_{BIC} will be equal to the whole set of all the social choice functions.

If the environment is quasi-linear, then we can also define the set of allocatively efficient social choice functions F_{AE} and the set of budget balanced social choice functions F_{BB}. In such an environment, we will have $F_{Ex-PostEff} = F_{AE} \bigcap F_{BB}$.

2.2 Myerson's Optimal Auction: An Example of Optimal Mechanism

Let us consider Example 2.1 in [5], of single unit - single item auction without reserve price and discuss an optimal mechanism developed by Myerson [2]. The objective function here is to maximize the auctioneer's revenue.

Recall that each bidder i's type lies in an interval $\Theta_i = [\underline{\theta_i}, \overline{\theta_i}]$. We impose the following additional conditions on the environment.

1. The auctioneer and the bidders are risk neutral
2. Bidders' types are statistically independent, that is, the joint density $\phi(\cdot)$ has the form $\phi_1(\cdot) \times \ldots \times \phi_n(\cdot)$

3. $\phi_i(\cdot) > 0 \; \forall \; i = 1, \ldots, n$

4. We generalize the outcome set X relative to that considered in Example 2.1 in [5], by allowing a random assignment of the good. Thus, we now take $y_i(\theta)$ to be buyer i's probability of getting the good when the vector of announced types is $\theta = (\theta_1, \ldots, \theta_n)$. Thus, the new outcome set is given by

$$\begin{aligned} X \;\; = \;\; & \Bigg\{ (y_0, y_1 \ldots, y_n, t_0, t_1, \ldots, t_n) | y_0 \in [0,1], t_0 \geq 0, y_i \in [0,1], t_i \leq 0 \; \forall \; i = 1, \ldots, n, \\ & \sum_{i=1}^{n} y_i \leq 1 \sum_{i=0}^{n} t_i = 0 \Bigg\} \end{aligned}$$

[1] Recall that the utility functions of the agents in this example are given by

$$u_i(f(\theta), \theta_i) = u_i(y_0(\theta), \ldots, y_n(\theta), t_0(\theta), \ldots, t_n(\theta), \theta_i) = \theta_i y_i(\theta) + t_i(\theta) \; \forall \; i = 1, \ldots, n$$

Thus, viewing $y_i(\theta) = v_i(k(\theta))$ in conjunction with the second and third conditions above, we can claim that the underlying environment here is linear.

In the above example, we assume that the auctioneer is the social planner and he is looking for an optimal direct revelation mechanism to sell the good. Myerson's [2] idea was that the auctioneer must use a social choice function which is Bayesian incentive compatible and interim individual rational and at the same time fetches the maximum revenue to the auctioneer. Thus, in this problem, the set of feasible social choice functions is given by $F = F_{BIC} \bigcap F_{InterimIR}$. The objective function in this case would be to maximize the total expected revenue of the seller which would be given by

$$E_\theta \left[w(u_1(f(\theta), \theta_1), \ldots, u_n(f(\theta), \theta_n)) \right] = -E_\theta \left[\sum_{i=1}^{n} t_i(\theta) \right]$$

Note that in above objective function we have used $f(\theta)$ not $f(s^*(\theta))$. This is because in the set of feasible social choice functions we are considering only BIC social choice functions and for these functions we have $s^*(\theta) = \theta \quad \forall \; \theta \in \Theta$. Thus, Myerson's optimal auction design problem can be formulated as the following optimization problem.

$$\begin{array}{c} \text{maximize} \\ f(\cdot) \in F \end{array} \; - E_\theta \left[\sum_{i=1}^{n} t_i(\theta) \right] \tag{15}$$

where

$$F = \{ f(\cdot) = (y_1(\cdot), \ldots, y_n(\cdot), t_1(\cdot), \ldots, t_n(\cdot)) | f(\cdot) \text{ is BIC and interim IR} \}$$

By invoking Myerson's Characterization Theorem (Theorem 11.2 in [5]) for BIC SCF in linear environment, we can say that an SCF $f(\cdot)$ in the above context would be BIC iff it satisfies the following two conditions

1. $\overline{y_i}(\cdot)$ is non-decreasing for all $i = 1, \ldots, n$

[1] $\sum_{i=1}^{n} y_i < 1$ when there is no trade.

2. $U_i(\theta_i) = U_i(\underline{\theta_i}) + \int_{\underline{\theta_i}}^{\theta_i} \overline{y_i}(s)ds \;\; \forall\, \theta_i \in \Theta_i;\; \forall\, i = 1,\dots,n$

Also, we can invoke the definition of interim individual rationality to claim that the an SCF $f(\cdot)$ in the above context would be interim IR iff it satisfies the following conditions

$$U_i(\theta_i) \geq 0 \;\; \forall \theta_i \in \Theta_i;\; \forall\, i = 1,\dots,n$$

where

- $\overline{t_i}(\hat{\theta}_i) = E_{\theta_{-i}}[t_i(\hat{\theta}_i, \theta_{-i})]$ be bidder i's expected transfer given that he announces his type to be $\hat{\theta}_i$ and that all the bidders $j \neq i$ truthfully reveal their types.
- $\overline{y_i}(\hat{\theta}_i) = E_{\theta_{-i}}[y_i(\hat{\theta}_i, \theta_{-i})]$ is the probability that bidder i would receive the object given that he announces his type to be $\hat{\theta}_i$ and all bidders $j \neq i$ truthfully reveal their types.
- $U_i(\theta_i) = \theta_i \overline{y_i}(\theta_i) + \overline{t_i}(\theta_i)$ [2]

In view of the above paraphernalia, problem (15) can be rewritten as follows.

$$\underset{(y_i(\cdot), U_i(\cdot))_{i\in N}}{\text{maximize}} \;\; \sum_{i=1}^{n} \int_{\underline{\theta_i}}^{\overline{\theta_i}} \left(\theta_i \overline{y_i}(\theta_i) - U_i(\theta_i)\right) \phi_i(\theta_i) d\theta_i \tag{16}$$

subject to

(i) $\overline{y_i}(\cdot)$ is non-decreasing $\;\forall\, i = 1,\dots,n$
(ii) $y_i(\theta) \in [0,1], \sum_{i=1}^{n} y_i(\theta) \leq 1 \; \forall i = 1,\dots,n, \forall\, \theta \in \Theta$
(iii) $U_i(\theta_i) = U_i(\underline{\theta_i}) + \int_{\underline{\theta_i}}^{\theta_i} \overline{y_i}(s)ds \;\; \forall\, \theta_i \in \Theta_i;\; \forall\, i = 1,\dots,n$
(iv) $U_i(\theta_i) \geq 0 \;\forall\, \theta_i \in \Theta_i;\; \forall\, i = 1,\dots,n$

We first note that if constraint (iii) is satisfied then constraint (iv) will be satisfied iff $U_i(\underline{\theta_i}) \geq 0 \;\; \forall\, i = 1,\dots,n$. As a result, we can replace the constraint (iv) with
(iv') $U_i(\underline{\theta_i}) \geq 0 \; \forall\, i = 1,\dots,n$
Next, substituting for $U_i(\theta_i)$ in the objective function from constraint (iii), we get

$$\sum_{i=1}^{n} \int_{\underline{\theta_i}}^{\overline{\theta_i}} \left(\theta_i \overline{y_i}(\theta_i) - U_i(\underline{\theta_i}) - \int_{\underline{\theta_i}}^{\theta_i} \overline{y_i}(s)ds \right) \phi_i(\theta_i) d\theta_i$$

Integrating by parts the above expression, the auctioneer's problem can be written as one of choosing the $y_i(\cdot)$ functions and the values $U_1(\underline{\theta_1}), \dots, U_n(\underline{\theta_n})$ to maximize

$$\int_{\underline{\theta_1}}^{\overline{\theta_1}} \dots \int_{\underline{\theta_n}}^{\overline{\theta_n}} \left[\sum_{i=1}^{n} y_i(\theta_i) J_i(\theta_i) \right] \left[\prod_{i=1}^{n} \phi_i(\theta_i) \right] d\theta_n \dots d\theta_1 - \sum_{i=1}^{n} U_i(\underline{\theta_i})$$

[2] We can take unconditional expectation because types are independent

subject to constraints (i), (ii), and (iv'), where

$$J_i(\theta_i) = \left(\theta_i - \frac{1 - \Phi_i(\theta_i)}{\phi_i(\theta_i)}\right) = \left(\theta_i - \frac{\overline{\Phi_i}(\theta_i)}{\phi_i(\theta_i)}\right)$$

where, we define $\overline{\Phi_i}(\theta_i) = 1 - \Phi_i(\theta_i)$. It is evident that solution must have $U_i(\underline{\theta_i}) = 0$ for all $i = 1, \ldots, n$. Hence, the auctioneer's problem reduces to choosing functions $y_i(\cdot)$ to maximize

$$\int_{\underline{\theta_1}}^{\overline{\theta_1}} \cdots \int_{\underline{\theta_n}}^{\overline{\theta_n}} \left[\sum_{i=1}^{n} y_i(\theta_i) J_i(\theta_i)\right] \left[\prod_{i=1}^{n} \phi_i(\theta_i)\right] d\theta_n \ldots d\theta_1$$

subject to constraints (i) and (ii).

Let us ignore constraint (i) for the moment. Then inspection of the above expression indicates that $y_i(\cdot)$ is a solution to this relaxed problem iff for all $i = 1, \ldots, n$, we have

$$y_i(\theta) = \begin{cases} 0 & : \text{ if } J_i(\theta_i) < \max\{0, \max_{h \neq i} J_h(\theta_h)\} \\ 1 & : \text{ if } J_i(\theta_i) > \max\{0, \max_{h \neq i} J_h(\theta_h)\} \end{cases} \tag{17}$$

Note that $J_i(\theta_i) = \max\{0, \max_{h \neq i} J_h(\theta_h)\}$ is a zero probability event.

In other words, if we ignore the constraint (i) then $y_i(\cdot)$ is a solution to this relaxed problem iff the good is allocated to a bidder who has highest non-negative vale for $J_i(\theta_i)$. Now, recall the definition of $\overline{y_i}(\cdot)$. It is easy to write down the following expression

$$\overline{y_i}(\theta_i) \quad = \quad E_{\theta_{-i}}\left[y_i(\theta_i, \theta_{-i})\right] \tag{18}$$

Now, if we assume that $J_i(\cdot)$ is non-decreasing in θ_i then it is easy to see that above solution $y_i(\cdot)$, given by (17), will be non-decreasing in θ_i, which in turn implies, by looking at expression (18), that $\overline{y_i}(\cdot)$ is non-decreasing in θ_i. Thus, the solution to this relaxed problem actually satisfies constraint (i) under the assumption that $J_i(\cdot)$ is non-decreasing. Assuming that $J_i(\cdot)$ is non-decreasing, the solution given by (17) seems to be the solution of the optimal mechanism design problem for single unit- single item auction. The condition that $J_i(\cdot)$ is non-decreasing in θ_i is met by most of the distribution functions such as Uniform and Exponential.

So far we have computed the allocation rule for the optimal mechanism and now we turn out attention towards the payment rule. The optimal payment rule $t_i(\cdot)$ must be chosen in such a way that it satisfies

$$\overline{t_i}(\theta_i) = E_{\theta_{-i}}[t_i(\theta_i, \theta_{-i})] = U_i(\theta_i) - \theta_i \overline{y_i}(\theta_i) = \int_{\underline{\theta_i}}^{\theta_i} \overline{y_i}(s) ds - \theta_i \overline{y_i}(\theta_i) \tag{19}$$

Looking at the above formula, we can say that if the payment rule $t_i(\cdot)$ satisfies the following formula (20), then it would also satisfy the formula (19).

$$t_i(\theta_i, \theta_{-i}) = \int_{\underline{\theta_i}}^{\theta_i} y_i(s, \theta_{-i}) ds - \theta_i y_i(\theta_i, \theta_{-i}) \quad \forall\, \theta \in \Theta \tag{20}$$

The above formula can be rewritten more intuitively, as follows. For any vector θ_{-i}, let we define

$$z_i(\theta_{-i}) = \inf \{\theta_i | J_i(\theta_i) > 0 \text{ and } J_i(\theta_i) \geq J_j(\theta_j) \; \forall \; j \neq i\}$$

Then $z_i(\theta_{-i})$ is the infimum of all winning bids for bidder i against θ_{-i}, so

$$y_i(\theta_i, \theta_{-i}) = \begin{cases} 1 & : \text{ if } \theta_i > z_i(\theta_{-i}) \\ 0 & : \text{ if } \theta_i < z_i(\theta_{-i}) \end{cases}$$

This gives us

$$\int_{\underline{\theta_i}}^{\theta_i} y_i(s, \theta_{-i}) ds = \begin{cases} \theta_i - z_i(\theta_{-i}) & : \text{ if } \theta_i \geq z_i(\theta_{-i}) \\ 0 & : \text{ if } \theta_i < z_i(\theta_{-i}) \end{cases}$$

Finally, the formula (20) becomes

$$t_i(\theta_i, \theta_{-i}) = \begin{cases} -z_i(\theta_{-i}) & : \text{ if } \theta_i \geq z_i(\theta_{-i}) \\ 0 & : \text{ if } \theta_i < z_i(\theta_{-i}) \end{cases}$$

That is bidder i must pay only when he gets the good, and then he pays the amount equal to his lowest possible winning bid.

A few interesting observations are worth mentioning here.

1. When the various bidders have differing distribution function $\Phi_i(\cdot)$ then, the bidder who has the largest value of $J_i(\theta_i)$ is *not* necessarily the bidder who has bid the highest amount for the good. Thus Myerson's optimal auction need not be allocatively efficient and therefore, need not be ex-post efficient.

2. If the bidders are symmetric, that is,

   - $\Theta_1 = \ldots = \Theta_n = \Theta$
   - $\Phi_1(\cdot) = \ldots = \Phi_n(\cdot) = \Phi(\cdot)$

   then the allocation rule would be precisely the same allocation rule of first-price and second-price auctions. In such a case the object would be allocated to the highest bidder. In such a situation, the optimal auction would also become allocatively efficient. Also, note that in such a case the payment rule that we described above would coincide with the payment rules in second-price auction. In other words, the second price (Vickrey) auction would be the optimal auction when the bidders are symmetric. Therefore, many a time, the optimal auction is also known as *modified Vickrey auction.*

   Riley and Samuelson [3] also have studied the problem of design of an optimal auction for selling a single unit of a single item. They assume the bidders to be symmetric. Their work is less general than that of Myerson [2].

2.3 Extensions to Myerson's Auction

2.3.1 Efficient Optimal Auctions

Krishna and Perry [7] have argued in favor of an auction which will maximize the revenue subject to allocative efficiency (AE) and also DSIC and IIR constraints. The Green Laffont theorem (Theorem 10.2 in [5]) tells us that any DSIC and AE mechanism is necessarily a VCG mechanism. So, we have to look for a VCG mechanism which will maximize the revenue to the seller. Krishna and Perry [7] define, *social utility* as the value of an efficient allocation:

$$SW(\theta) = \sum_{j=1}^{j=n} v_j(k^*(\theta), \theta_j)$$

$$SW_{-i}(\theta) = \sum_{j \neq i} v_j(k^*(\theta), \theta_j)$$

With these functions, we can write the payment rule in Clarke's pivotal mechanism as

$$t_i(\theta) = SW_{-i}(0, \theta_{-i}) - SW_{-i}(\theta)$$

That is, payment by the agent i is the externality he is imposing by reporting type to be θ_i rather than zero. The authors of [7] generalize it. Fix a vector, $s = (s_1, s_2, \ldots, s_n) \in \Theta$ called as *basis* because, it defines the payment rule . The *VCG mechanism with basis* s is defined by

$$t_i(\theta|s_i) = SW(s_i, \theta_{-i}) - SW_{-i}(\theta)$$

It can be seen that this new mechanism is also DSIC. Now choosing an appropriate basis, one can always find an optimal auction in the class of VCG mechanisms. Krishna and Perry [7] have shown that the classical Vickrey auction is an optimal and efficient auction for a single indivisible item. They have also shown that the Vickrey auction is an optimal one among VCG mechanisms for multi-unit auctions, when all the bidders have downward sloping demand curves.

3 Problems

1. Consider a sealed bid auction with one seller and two buying agents. There is a single indivisible item which the seller wishes to sell. The bidders are symmetric with independent private values distributed uniformly over $[0, 1]$. Whoever bids higher will be allocated the item. For this auction:

    - What is the equilibrium bidding strategy of a bidder in the first price auction?
    - What is the expected revenue in the first price auction
    - What is the expected revenue in the second price auction
    - What is the expected revenue in the optimal auction

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