Game Theory

Lecture Notes By

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COOPERATIVE GAME THEORY
The Core

Note: *This is a only a draft version, so there could be flaws. If you find any errors, please do send email to* `hari@csa.iisc.ernet.in`. *A more thorough version would be available soon in this space.*

Given a transferable utility game (N, v), there are two key questions that are of fundamental interest: (1) Which coalition(s) should form? (2) How should a coalition that forms divide its winnings among its members? An answer to the second question has implications for the first question. In this chapter, we start studying the answers to these questions using one of the central notions, the core, in cooperative game theory.

1 Preliminary Definitions

Let $N = \{1, \ldots, n\}$ be a set of players and let (N, v) be a TU game.

- *A payoff allocation* $x = (x_1, \ldots, x_n)$ is any vector in $\mathbb{R}^n$. x_i is the utility payoff to player i, $i \in N$. An allocation $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ is said to be *feasible* for a coalition C iff

$$\sum_{i \in C} y_i \leq v(C)$$

  If an allocation y is feasible for C, the players in C can achieve their components in this allocation by dividing among themselves the worth $v(C)$ that they can get by cooperating together.

- A payoff allocation $x = (x_1, \ldots, x_n)$ is said to be *individually rational* if $x_i \geq v(\{i\})\ \forall i \in N$. It is said to be *collectively rational* if $\sum_{i \in N} x_i = v(N)$. It is said to be *coalitionally rational* if $\sum_{i \in C} x_i \geq v(C)\ \ \forall C \subseteq N$. Note that coalitional rationality implies individual rationality.

- We have, in the previous chapter, defined the notion of an *imputation*. It can be seen that an imputation is a payoff allocation that is individually rational and collectively rational.

2 The Core

The core of a TU game (N, v) is the set of all payoff allocations that are individually rational, coalitionally rational, and collectively rational. In other words, the core is the set of all imputations that are coalitionally rational. Thus we have

$$\text{Core } (N, v) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = v(N); \;\; \sum_{i \in C} x_i \geq v(C) \;\; \forall C \subseteq N\}$$

A coalition C can *improve on* an allocation $x = (x_1, \ldots, x_n) \in$
$mathbbR^n$ iff

$$v(C) > \sum_{i \in C} x_i$$

This implies that C can improve on x iff there exists some allocation y such that y is feasible for C and the players in C all get strictly higher payoff in y than in x. We also say that the coalition C *blocks* x. An allocation x is said to be in the core of (N, v) iff x is feasible for N and no coalition can improve upon it. This implies that x is in the core of (N, v) iff

$$\begin{aligned} \sum_{i \in N} x_i &= v(N) \\ \sum_{i \in C} x_i &\geq v(C) \;\; \forall C \subseteq N \end{aligned}$$

We can make the following observations on the concept of the core.

- If a feasible allocation x is not in the core, then there would exist some coalition C such that the players in C could all do strictly better than in x by cooperating together and dividing the worth $v(C)$ among themselves.
- To apply the core as a solution concept given a strategic form game Γ with transferable utility, the minimax representation in coalition form is an appropriate way to derive a characteristic function from Γ. When v is derived using the minimax representation, the players in C can guarantee themselves payoffs that are strictly better than in x (where x is a feasible allocation not in the core), no matter what the other players in $N \backslash C$ do.
- The core is an appealing solution concept in the light of the assumption that all coalitions can negotiate effectively.

2.1 The Core: Some Examples

Divide the Dollar Game

For the Divide-the-Dollar game (Version 1), $core(N, v)$ is given by.

$$\{(x_1, x_2, x_3) \in mathbbR^3 : x_1 + x_2 + x_3 = 300, x_1 \geq 0, x_2 \geq 0, x_3 \geq 0\}$$

The core treats the three players symmetrically and includes all individually rational Pareto efficient allocations. In Version 2 of the game (where players 1 and 2 can get 300 together), the core will be

$$\{(x_1, x_2, x_3) \in mathbbR^3 : x_1 + x_2 = 300, x_1 \geq 0, x_2 \geq 0, x_3 = 0\}$$

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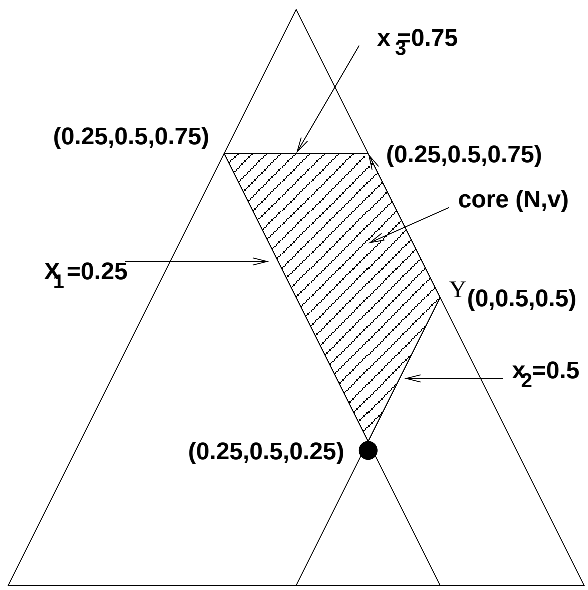

Figure 1: Core of the game is a trapezoid

If the players 1 and 2 can get 300 together without player 3 or the players 1 and 3 can get 300 together without player 2 (Version 3 of the game), it can be seen that the core consists of just one element, namely $(300, 0, 0)$.

If the players get a nonzero allocation whenever any two players suggest the same allocation (Version 4 of the game - three person majority game), it can be seen that the core is empty. The following insights justify the emptiness of the core in this case:

- When any player i gets a positive payoff in a feasible allocation, the other two players must get less than 300 which they could get by themselves.
- No matter what the final allocation is, there is always a coalition that could gain if it gets one more final opportunity to negotiate effectively against this allocation.

A Three Player TU Game

This example is taken from [1]. Here, $N = \{1, 2, 3\}$ and $v(1) = v(2) = v(3) = 0; v(12) = 0.25; v(13) = 0.5; v(23) = 0.75; v(123) = 1$. Note that $(x_1, x_2, x_3) \in$ *core* (N, v) iff

$$x_1 \geq 0;\ x_2 \geq 0;\ x_3 \geq 0$$

$$x_1 + x_2 \geq 0.25;\ x_1 + x_3 \geq 0.5;\ x_2 + x_3 \geq 0.75;$$

$$x_1 + x_2 + x_3 = 1$$

This would mean $x_1 \leq 0.25;\ x_2 \leq 0.5;\ x_3 \leq 0.75$. The core is depicted in Figure 1 and happens to be a trapezoid with vertices at $(0, 0.25, 0.75), (0, 0.5, 0.5), (0.25, 0, 0.75)$, and $(0.25, 0.5, 0.25)$.

House Allocation

This example is originally from Von Neumann and Morgenstern [2]. Player 1 has a house which she values at Rs 1 million and wishes to sell the house. There are 2 potential buyers, player 2 and player 3 who have a valuation of Rs 2 million each and who also have with them Rs 2 million each. Suppose player 1 sells the house to player 2 at a price p where $1 \leq p \leq 2$. Then utility of player 1 is p; utility of player 2 is $(2-p)+p = 4-p$; and the utility of player 3 is 2. Similar would be the case if player 1 sells house to player 3 at a price p, where $1 \leq p \leq 2$. We thus get the characteristic form of the game as:

$$
\begin{gathered}
v(1) = 1;\ v(2) = 2;\ v(3) = 2 \\
v(12) = v(13) = v(23) = 4 \\
v(123) = 6
\end{gathered}
$$

An allocation $(x_1, x_2, x_3) \in core\ (N, v)$ iff

$$
\begin{gathered}
x_1 \geq 1;\ x_2 \geq 2;\ x_3 \geq 2 \\
x_1 + x_2 \geq 4;\ x_2 + x_3 \geq 4;\ x_1 + x_3 \geq 4; \\
x_1 + x_2 + x_3 = 6
\end{gathered}
$$

The only allocation that satisfies the above equations is $(2, 2, 2)$ which is the only element in the core. This corresponds to player 1 selling her house at the maximum possible price of Rs. 2 million.

A Glove Market

This example shows that core may be non-empty in some pathological cases. First we consider a simple case and then generalize it. Let there be 5 suppliers of gloves. Of these the first two players can each supply one left glove and the other three players can supply one right glove each. Suppose $N_L = \{1, 2\}$ is the set of left glove suppliers and $N_R = \{3, 4, 5\}$ the set of right glove suppliers. Suppose the worth of each coalition is the number of matched pairs that it can assemble. For example, if $C = \{1, 3\}$, the worth of C is 1, whereas if $C = \{3, 4\}$, the worth of C is 0. In general, given $C \subset N$,

$$v(C) = \min\{|C \cap N_L|,\ |C \cap N_R|\}$$

The core of this game can be seen to be the singleton set $\{(1, 1, 0, 0, 0)\}$.

On the other hand, if $N_L = \{1, 2, 3\}$ and $N_R = \{4, 5\}$, the core of the game would be the singleton set $\{(0, 0, 0, 1, 1)\}$. If the cardinality of the two sets is the same, then the core takes a different form altogether. For example, if $N_L = \{1, 2\}$ and $N_R = \{3, 4\}$, the core of the game would be the singleton set $\{(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})\}$.

We will now generalize the example to $n = 2,000,001$. Of these 1,000,000 players can each supply one left glove and the other 1,000,001 players can each supply one right glove. The core of this game consists of the unique allocation x such that

$$
\begin{aligned}
x_i &= 1 \quad \text{if } i \in N_L \\
&= 0 \quad \text{if } i \in N_R
\end{aligned}
$$

The reason for this is:

$$\sum_{i \in N_L} x_i + \sum_{j \in N_R \setminus \{k\}} x_j \geq 1,000,000 \quad \forall k \in N_R$$

- Suppose that some right glove supplier has some positive payoff in a feasible allocation. Then the total payoff to the other 2,000,000 players must be less than 1,000,000, which they could improve by making 1,000,000 matched pairs among themselves, without this distinguished right glove supplier.

- The economic interpretation is as follows. Since right gloves are in excess supply, they have a market price of 0. If we add just two more left glove suppliers, the unique core allocation would switch to

$$\begin{aligned} x_i &= 0 \quad \text{if } i \in N_L \\ &= 1 \quad \text{if } i \in N_R \end{aligned}$$

  in which every left glove supplier gets zero payoff while every right glove supplier gets payoff 1.

- It would be interesting to see what the core would be if there are 1,000,000 left glove suppliers and 1,000,000 right glove suppliers.

2.2 Epsilon Core

The instability displayed by the core for large games (such as the glove market example) can be somewhat overcome using the notion of ϵ-core or *epsilon*-approximate core. Given any number ϵ, an allocation x is in the ϵ-core of the coalitional game (N, v) iff

$$\begin{aligned} \sum_{i \in N} x_i &= v(N) \\ \sum_{i \in C} x_i &\geq v(C) - \epsilon|C| \ \forall C \subseteq N \end{aligned}$$

If x is in the ϵ-core, then no coalition C would be able to guarantee all its members more than ϵ more than what they get in x.

Example: Epsilon Core of Glove Market

For the glove market example, an allocation x is in the ϵ-core iff

$$\min\{x_i : i \in N_L\} + \min\{x_j : j \in N_R\} \geq 1 - 2\epsilon$$

This means the worst-off left glove supplier and the worst-off right glove supplier together must get at least $1 - 2\epsilon$. With 1,000,000 left glove suppliers and 1,000,001 right glove suppliers, if $\epsilon > 0.0000005$, then for any number λ such that $0 \leq \lambda \leq 1$, the allocation x such that

$$\begin{aligned} x_i &= 1 - \lambda \qquad \forall i \in N_L \\ x_j &= \frac{\lambda}{1.000001} \forall j \in N_R \end{aligned}$$

is in the ϵ-core.

2.3 The Concept of the Core: Some Implications

Myerson [3] has discussed some important implications of the concept of the core. We provide a summary of the discussion of these here.

- If an allocation x does not belong to the core, then it would imply that there exists some coalition C such that players in C can all do better than in x by sharing $v(C)$ among themselves.

- If an allocation x belongs to the core, then it implies that for each player, a unilateral deviation will not make the player strictly better off. This means x is a Nash equilibrium of an appropriate contract signing game.

- If the core is empty, then we are unable to draw any conclusions about the game. At the same time, if the core consists of a large number of elements, then also, we have difficulty in preferring any particular allocation in the core.

- Given a game (N, v) and a feasible allocation x not in the core of v, it is not possible to conclude that the players will not agree to the allocation x. This can be argued in the following way. There are two cases.

  1. In case 1, the coalitions can make binding agreements that its members cannot subsequently break without the consent of all others in the coalition. In this case even if there is a coalition that can improve upon x, members may be prevented from joining this (better) coalition by invoking such agreements.
  2. In case 2, coalitions are not allowed to make binding agreements of the type discussed in case 1. Even if there exists a coalition that can improve upon x, the members of the original coalition may not be interested in the better allocation due to the apprehension that some future negotiations might actually leave them worse off than in the original coalition.

- If an allocation x is in the core of v and coalitional agreements are binding, there would still exist certain forms of coalitional bargaining under which an attempt to achieve x might be blocked by some coalition.

3 Characterization of Games with Non-Empty Core

The classic, independent findings of Shapley and Bondereva provides a characterization of games with non-empty core, using linear programming [4, 5]. To understand this characterization, we first look into the following linear program, given a TU game (N, v).

$$\begin{array}{rc} \text{minimize} & x_1 + x_2 + \ldots + x_n \\ \text{subject to} & \sum_{i \in C} x_i \geq v(C) \ \ \forall C \subseteq N \\ & (x_1, \ldots, x_n) \in \\ mathbbR^n & \end{array}$$

The above LP determines the least amount of transferable utility which is necessary for an allocation so that no coalition can improve upon it. Note that this LP definitely has a solution if it is feasible.

This is because all the inequalities are of the $\geq$ type and and also there is a structure which makes it feasible. Let $(x_1^*, x_2^*, \ldots, x_n^*)$ be an optimal solution of this LP. Then we know

$$\sum_{i \in C} x_i^* \geq v(C) \;\; \forall C \subseteq N$$

In particular,

$$x_1^* + \ldots + x_n^* \geq v(N)$$

There are two possibilities:

1. $x_1^* + \ldots + x_n^* = v(N)$. In this case, all solutions of this LP will constitute the core of (N, v). In fact, the core will consist precisely of the solutions of this LP.

2. $x_1^* + \ldots + x_n^* > v(N)$. In this case, it is clear that the core is empty.

3.1 Examples: Divide the Dollar Game

Recall Version 4 of the divide-the-dollar game where $N = \{1, 2, 3\}$ and $v(1) = v(2) = v(3) = 0; v(12) = v(23) = v(13) = v(123) = 300$. The linear program would be:

$$\begin{aligned}
minimize \quad & x_1 + x_2 + x_3 \\
s.t \quad & x_1 \geq 0; x_2 \geq 0; x_3 \geq 0 \\
& x_1 + x_2 \geq 300 \\
& x_2 + x_3 \geq 300 \\
& x_1 + x_3 \geq 300 \\
& x_1 + x_2 + x_3 \geq 300 \\
& x_1, x_2, x_3 \in \\
mathbbR &
\end{aligned}$$

An optimal solution of the primal LP is:

$$x_1^* = x_2^* = x_3^* = 150$$

Since $x_1^* + x_2^* + x_3^* = 450 > 300 = v(\{1, 2, 3\})$, the core is empty.

Now consider Version 3 of the Divide the Dollar game. Recall that $N = \{1, 2, 3\}$ and $v(1) = v(2) = v(3) = v(23) = 0; v(12) = v(13) = v(123) = 300$. The LP here is:

$$\begin{aligned}
minimize \quad & x_1 + x_2 + x_3 \\
s.t \quad & x_1 \geq 0; x_2 \geq 0; x_3 \geq 0 \\
& x_1 + x_2 \geq 300 \\
& x_1 + x_3 \geq 300 \\
& x_2 + x_3 \geq 0 \\
& x_1 + x_2 + x_3 \geq 300
\end{aligned}$$

An optimal solution here is:

$$x_1^* = 300; \;\; x_2^* = x_3^* = 0$$

The optimal value is 300. The core consists of a single element $\{(300, 0, 0)\}$.

3.2 Duals of the Linear Programs for Divide the Dollar

Let us first examine the dual of the LP for the majority voting game. The objective function of the dual LP is:

$$\alpha(1)[v(1)] + \alpha(2)[v(2)] + \alpha(3)[v(3)] + \alpha(12)[v(12)] + \alpha(23)[v(23)] + \alpha(13)[v(13)] + \alpha(123)[v(123)]$$

The dual LP will be:

$$\begin{aligned} maximize \quad & 300[\alpha(12) + \alpha(23) + \alpha(13) + \alpha(123)] \\ s.t \quad & \alpha(1) + \alpha(12) + \alpha(13) + \alpha(123) = 1 \\ & \alpha(2) + \alpha(12) + \alpha(23) + \alpha(123) = 1 \\ & \alpha(3) + \alpha(13) + \alpha(23) + \alpha(123) = 1 \\ & \alpha(1) \geq 0; \quad \alpha(2) \geq 0; \quad \ldots \ \alpha(123) \geq 0 \end{aligned}$$

An optimal solution of the dual LP:

$$\begin{aligned} \alpha(12) = \alpha(13) = \alpha(23) = \frac{1}{2} \\ \alpha(1) = \alpha(2) = \alpha(3) = \alpha(123) = 0 \end{aligned}$$

Here the core is empty. Now we examine the dual of the LP for Version 3 of the game. The dual LP is:

$$\begin{aligned} maximize \quad & 300[\alpha(12) + \alpha(13) + \alpha(123)] \\ s.t \quad & \alpha(1) + \alpha(12) + \alpha(13) + \alpha(123) = 1 \\ & \alpha(2) + \alpha(12) + \alpha(23) + \alpha(123) = 1 \\ & \alpha(3) + \alpha(13) + \alpha(23) + \alpha(123) = 1 \\ & \alpha(1) \geq 0; \quad alpha(2) \geq 0; \quad \ldots \ \alpha(123) \geq 0 \end{aligned}$$

The optimal value of the dual is again, obviously, 300.

3.3 Shapley - Bondereva Characterization

In general, the primal LP is:

$$\begin{array}{rc} \text{minimize} & \sum_{i \in N} x_i \\ \text{subject to} & \sum_{i \in C} x_i \geq v(C) \quad \forall C \subseteq N \\ & x_i \in \\ mathbbR \ \forall i \in N & \end{array}$$

The dual LP is given by:

$$\text{maximize} \sum_{C \subseteq N} \alpha(C) v(C)$$

$$\text{subject to} \sum_{C \supseteq \{i\}} \alpha(C) = 1 \quad \forall i \in N$$

$$\alpha(C) \geq 0 \ \ \forall C \subseteq N$$

Let us apply the *strong duality theorem* here (See Appendix of Chapter 12 on matrix games) - if the primal (dual) has an optimal solution, then the dual (primal) has an optional solution and the optimal values are the same. In the present case, we know that an optimal solution exists. Hence by applying the strong duality theorem, given a TU game (N, v), there exists an allocation $x^* \in \mathbb{R}^n$ and a vector $\alpha^*(C)_{C \subseteq N}$ such that

$$\sum_{i \in C} x_i^* \geq v(C)$$

$$\alpha^*(C) \geq 0 \ \ \forall C \subseteq N$$

$$\sum_{C \supseteq \{i\}} \alpha^*(C) = 1 \ \ \forall i \in N$$

This is illustrated by both the examples presented above. Let us now impose the following condition:

$$\sum_{C \supseteq \{i\}} \alpha(C) = 1 \ \ \forall i \subseteq N \Rightarrow \sum_{C \subseteq \{i\}} \alpha(C) v(C) \leq v(N)$$

That is, feasibility of the dual implies that the objective function of the dual is $\leq v(N)$. We know that the optimal value of the primal $\geq v(N)$. Since the LP has a solution, the optimal solution therefore will have value $v(N)$. This ensures that the core is non-empty. This means we have a necessary and sufficient condition for the non-emptiness of the core. This necessary and sufficient condition is called *balancedness*.

3.4 Balanced TU Games

A TU game (N, v) is said to be balanced iff

$$\sum_{C \supseteq \{i\}} \alpha(C_i) = 1 \ \ \forall i \in N \Rightarrow \sum_{C \subseteq N} \alpha(C) v(C) \leq v(N)$$

The components of any optimal solution $(x_1^*, x_2^*, \ldots, x_n^*)$ are called *balanced aspirations* of the players. The phrase is appropriate since

$$\sum_{i \in C} x_i^* \geq v(C) \ \ \forall C \subseteq N$$

The aspirations are balanced by the fact that $(x_1 + x_2 + \ldots + x_n)$ is minimized. Let us interpret the condition for $N = \{1, 2\}$.

$$\begin{aligned} \alpha(1) + \alpha(12) &= 1 \text{ and} \\ \alpha(2) + \alpha(12) &= 1 \\ \alpha(1)v(1) + \alpha(2)v(2) + \alpha(12)v(12) &\leq v(12) \end{aligned}$$

The above means that

- Player 1 can be part of the coalitions $\{1\}, \{1, 2\}, \ldots, \{1, 2, \ldots, n\}$
- Similarly, Player 2 can be part of the coalitions $\{2\}, \{1, 2\}, \ldots,$

- $\alpha(1) + \alpha(12) + \ldots + \alpha(12, \ldots, n)$ implies that at any point of time, Player 1 is part of exactly one of these coalitions.

We are looking at $\alpha(1)v(1)+\alpha(2)v(2)+\alpha(3)v(3)+\alpha(12)v(12)+\alpha(23)v(23)+\alpha(13)v(13)+\alpha(123)v(123)$. This weighted sum $\leq v(123)$ means that the formation of coalitions, is balanced such that the net value generated is bounded by what the grand coalition can generate. Coalition formation will balance itself in such a way that together they cannot generate more than $v(N)$. If they are able to generate such a formation will not be stable.

4 To Probe Further

The discussion presented in this chapter follows that of Myerson [3] and Straffin [1]. The foundations for the results on the core have been laid in the classic papers by Gerard Debreu and Herbert Scarf [6] and Herbert Scarf [7]. The balancedness characterization of the core is by Bondereva [4] and Shapley [5].

5 Problems

1. (Straffin 1993) [1]. A constant sum TU game (N, v) is one in which
$$v(C) + v(N \setminus C) = c$$
where c is some constant. Show that the core of any essential, constant sum game is empty.

2. Consider the glove market example. What will be the core of this game if there are 1,000,000 left glove suppliers and 1,000,000 right glove suppliers?

3. (Straffin 1993) [1]. Consider the following variant of the real estate example. Player 1 has a value of Rs. 1 million; player 2 has value of Rs. 2 million; and player 3 has a value of Rs. 3 million for the house. Player 2 has Rs. 3 million cash, so also player 3. Formulate an appropriate TU game and compute the core.

4. (Straffin 1993) [1]. Consider a three person superadditive game with $v(1) = v(2) = v(3) = 0; v(12) = a; v(13) = b; v(23) = c; v(123) = d$ where $0 \leq a, b, c \leq d$. Compute the core for this game. When is the core non-empty for this game?

5. Find the core of the *communication satellites game* [1] defined as follows:
$$v(1) = v(2) = v(3) = 0$$
$$v(12) = 5.2; \;\; v(13) = 2.5; \;\; v(23) = 3; \;\; v(123) = 5.2$$

6. Compute the core of the logistics game discussed in the previous chapter.

7. Consider a game with five players where player 1 is called a big player and the others are called small players. The big player with one or more small players can earn a worth of 1. The four

small players together can also earn 1. Let

$$\begin{aligned} N &= \{1,2,3,4,5\} && (1) \\ v(C) &= 1 \text{ if } 1 \in C \text{ and } |C| \geq 2 && (2) \\ &= 1 \text{ if } |C| \geq 4 && (3) \\ &= 0 \text{ otherwise} && (4) \end{aligned}$$

Compute the core for this game.

8. Let us consider a version of divide the dollar problem with 4 players and total worth equal to 400. Suppose that any coalition with three or more players will be able to achieve the total worth. Also, a coalition with two players will be able to achieve the total worth only if player 1 is a part of the two player coalition. Set up a characteristic function for this TU game and compute the core.

9. There are four players $\{1,2,3,4\}$ who are interested in a wealth of 400 (real number). Any coalition containing at least two players and having player 1 would be able to achieve the total wealth of 400. Similarly, any coalition containing at least three players and containing player 2 also would be able to achieve the total wealth of 400. Set up a characteristic form game for this situation and compute the core.

References

[1] Philip D. Straffin Jr. *Game Theory and Strategy*. The Mathematical Association of America, 1993.

[2] John von Neumann and Oskar Morgenstern. *Theory of Games and Economic Behavior*. Princeton University Press, 1944.

[3] Roger B. Myerson. *Game Theory: Analysis of Conflict*. Harvard University Press, Cambridge, Massachusetts, USA, 1997.

[4] O.N. Bondereva. Some applications of linear programming methods to the theory of cooperative games. *(In Russian)*, 1963.

[5] Lloyd S. Shapley. On balanced sets and the core. *Naval Research Logistics Quarterly*, 14:435–460, 1967.

[6] Gerard Debreu and Herbert Scarf. A limit theorem on the core of an economy. *International Economic Review*, 4:235–246, 1963.

[7] Herbert E. Scarf. The core of an n-person game. *Econometrica*, 35:50–69, 1967.