Game Theory

Lecture Notes By

Y. Narahari

Department of Computer Science and Automation

Indian Institute of Science

Bangalore, India

October 2012

COOPERATIVE GAME THEORY
The Two Person Bargaining Problem

Note: *This is a only a draft version, so there could be flaws. If you find any errors, please do send email to* `hari@csa.iisc.ernet.in`. *A more thorough version would be available soon in this space.*

The Nash bargaining problem represents one of the earliest and most influential results in cooperative game theory. Given two rational and intelligent players and a set of feasible allocations from among which a unique allocation is to be chosen, the Nash bargaining theory proposes an elegant axiomatic approach to solve the problem. This chapter describes the problem and proves the Nash bargaining result.

1 Nash Program

Cooperation refers to coalitions of two or more players acting together with a specific common purpose in mind. Since rationality and intelligence are two fundamental assumptions in game theory, any cooperation between players must take into account the objective of maximizing their own individual payoffs. As we have seen in the previous chapter, the notion of cooperation which is closely tied with the notion of correlated strategies can be developed without abandoning the individual decision theoretic foundations underlying game theory. This has been emphasized by John Nash himself [1, 2]. According to Nash, cooperative actions can be considered as the culmination of a certain process of bargaining among the cooperating players and consequently, cooperation between players can be studied using core concepts of non-cooperative game theory. In this bargaining process, we can expect each player to behave according to some bargaining strategy that satisfies the original utility-maximization criterion as in standard game theory.

The ingenious idea of Nash is to define a *cooperative transformation* that will transform a strategic form game into another strategic form game that has an extended strategy space for each player. The extended strategy set for a player has all the strategies of the original game and also additional strategies that capture bargaining with the other players to jointly plan cooperative strategies. This is on the lines of what we have studied in the previous chapter on correlated strategies. We will illustrate this with the standard example of the prisoner's dilemma problem which provides a classic example for illustrating the benefits of cooperation.

	NC	C
NC	-2, -2	-10, -1
C	-1, -10	-5, -5

Clearly, the rationality of the players suggests the equilibrium (C, C) which is obviously not an optimal option for the players. The two players here have a strong incentive to bargain with each other or sign a contract to transform this game into one which has equilibria that are better for both the players. This has already been brought in the previous chapter. The concept of a *Nash program* for cooperative game theory is to define cooperative solution concepts such that a cooperative solution for any given game is a Nash equilibrium of some cooperative transformation of the original non-cooperative game. The notion of Nash program was introduced by Nash in his classic paper [2]. If we carefully describe all the possibilities that are feasible when the players bargain or sign contracts with each other, we may end up with a game that has numerous equilibria. This means the Nash program may not lead to a unique cooperative solution. This automatically means that we should have a credible theory of cooperative equilibrium selection. This is what the Nash bargaining theory offers in an axiomatic and rigorous, yet intuitive way.

2 The Two Person Bargaining Problem

We now study the classic two person bargaining problem enunciated by Nash in [1]. According to Nash, the term *bargaining* refers to a situation in which

- individuals have the possibility of concluding a mutually beneficial agreement
- there is a conflict of interests about which agreement to conclude
- no agreement may be imposed on any player without that player's approval.

The following assumptions are implicit in Nash's formulation: when two players negotiate or an impartial arbitrator arbitrates, the payoff allocations that the two players ultimately get should depend only on:

- the payoffs they would expect if the negotiation or arbitration were to fail to reach a settlement, and
- on the set of payoff allocations that are jointly feasible for the two players in the process of negotiation or arbitration.

The two person bargaining problem has been applied in many important contexts including:

- *Management labor arbitration* where the management negotiates contracts with the labor union
- *International relations*, where two Nations or two groups of Nations work out agreements on issues such as nuclear disarmament, military cooperation, anti-terrorist strategy, bilateral emission control initiatives, etc.
- *Duopoly market games* where two major competing companies work out adjustments on their production to maximize their profits

- *Supply chain contracts* where a buyer and a supplier work out a mutually beneficial contract that facilitates a long-term relationship

- *Negotiation protocols* in multi-agent systems

2.1 The Bargaining Problem

The two person bargaining problem consists of a pair (F, v) where F is called the feasible set and v is called the disagreement point.

- F, the feasible set of allocations, is a closed, convex subset of $\mathbb{R}^2$.

- The disagreement point $v = (v_1, v_2) \in \mathbb{R}^2$ represents the disagreement payoff allocation for the two players. It is also called the *status-quo point* or the *default point*. This gives the payoffs for the two players in the event that the negotiations fail. It may be noted that v is invariably chosen to belong to the feasible set F though it is not a mandatory technical requirement.

- The set $F \cap \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq v_1; x_2 \geq v_2\}$ is assumed to be non-empty and bounded.

Justification for the Assumptions

- F is assumed to be convex. This can be justified as follows. Assume that the players can agree to jointly randomized strategies (correlated strategies). Consequently, if the utility allocations $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are feasible and $0 \leq \alpha \leq 1$, then the expected utility allocation $\alpha x + (1 - \alpha)y$ can be achieved by planning to implement x with probability α and to implement y with probability $(1 - \alpha)$.

- F is assumed to be closed (that is, any convergent sequence in F will converge to a point that belongs to F). This is a natural topological requirement. If we have a sequence of allocations belonging to F and the limiting allocation does not belong to F, then we have an undesirable situation that is not acceptable.

- The set $F \cap \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq v_1, x_2 \geq v_2\}$ is assumed to be non-empty and bounded. This assumption implies that there exists some feasible allocation that is at least as good as disagreement for both players, but unbounded gains over the disagreement point are not possible. Both these requirements are reasonable.

2.2 Connection to Two Player Non-Cooperative Games

Suppose $\Gamma = \langle \{1, 2\}, S_1, S_2, u_1, u_2 \rangle$ is a two person strategic form game. If the strategies of the players can be regulated by binding contracts, then one possibility for the feasible set F is the set of all allocations under correlated strategies:

$$F = \{(u_1(\alpha), u_2(\alpha)) : \alpha \in \Delta(S_1 \times S_2)\}$$

where

$$u_i(\alpha) = \sum_{s \in S} \alpha(s) u_i(s)$$

We could also choose the subset of allocations under individually rational correlated strategies (recall that the payoffs for players under individually rational correlated strategies will be at least the respective minmax values). If the players' strategies cannot be regulated by binding contracts then a possibility for F would be the set of all allocations under correlated equilibria:

$$F = \{(u_1(\alpha), u_2(\alpha)) : \alpha \text{ is a correlated equilibrium of } \Gamma\}$$

There are several possibilities for the disagreement point v. The first possibility is to let v_i be the minmax value for player i.

$$v_1 = \min_{\sigma_2 \in \Delta(S_2)} \max_{\sigma_1 \in \Delta(S_1)} u_1(\sigma_1, \sigma_2)$$

$$v_2 = \min_{\sigma_1 \in \Delta(S_1)} \max_{\sigma_2 \in \Delta(S_2)} u_2(\sigma_1, \sigma_2)$$

The above choice is quite reasonable and scientific because the minmax value for a player is the minimum guaranteed payoff to the player even in a strictly competitive setting where the other player always tries to hurt this player most.

A second possibility is to choose some focal Nash equilibrium (σ_1, σ_2) of Γ and let

$$v_1 = u_1(\sigma_1, \sigma_2); \quad v_2 = u_2(\sigma_1, \sigma_2)$$

A focal Nash equilibrium is one which becomes a natural choice for the players due to some external factors prevailing at the time of allocation.

A third possibility is to derive $v = (v_1, v_2)$ from some *rational threats*. The theory of rational threats has been proposed using rationality-based arguments and will be explained in an appendix of this chapter.

A sound theory of negotiation or arbitration must allow us to identify, given any bargaining problem (F, v), a unique allocation vector in $\mathbb{R}^2$ that would be selected as a result of negotiation or arbitration. Let us denote this unique allocation by $f(F, v)$. Thus the bargaining problem involves finding an appropriate solution function $f(.)$ from the set of all two-person bargaining problems into $\mathbb{R}^2$, which is the set of payoff allocations.

3 The Axioms of Nash

John Nash used a brilliant axiomatic approach to solve this problem. He first came up with a list of properties an ideal bargaining solution function is expected to satisfy and then proved that there exists a unique solution that satisfies all of these properties. The following are the five axioms of Nash:

1. Strong Efficiency
2. Individual Rationality
3. Scale Covariance
4. Independence of Irrelevant Alternatives (IIA)
5. Symmetry

Let us use the notation $f(F, v) = (f_1(F, v), f_2(F, v))$ to denote the Nash bargaining solution for the bargaining problem (F, v).

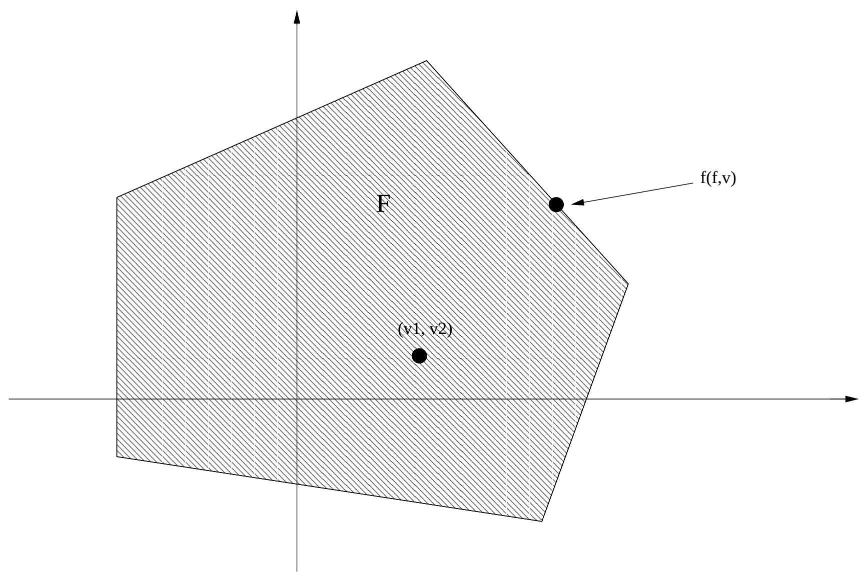

Figure 1: Illustrating strong efficiency and individual rationality

3.1 Axiom 1: Strong Efficiency

Given a feasible set F, we say an allocation $x = (x_1, x_2) \in F$ is *strongly Pareto efficient* or simply *strongly efficient* iff there exists no other point $y = (y_1, y_2) \in F$ such that $y_1 \geq x_1;\ \ y_2 \geq x_2$ with strict inequality satisfied for at least one player. An allocation $x = (x_1, x_2) \in F$ is *weakly Pareto efficient* or *weakly efficient* iff there exists no other point $y = (y_1, y_2) \in F$ such that $y_1 > x_1;\ \ y_2 > x_2$.

The strong efficiency axiom asserts that the solution to any two person bargaining problem should be feasible and strongly Pareto efficient. Formally, $f(F, v) \in F$ and there does not exist any $x = (x_1, x_2) \in F$ such that $x_1 \geq f_1(F, v);\ \ x_2 \geq f_2(F, v)$ with $x_i > f_i(F, v)$ for at least some $i \in \{1, 2\}$. This implies that there should be no other feasible allocation that is better than the solution for one player and not worse than the solution for the other player.

3.2 Axiom 2: Individual Rationality

This axiom states that $f(F, v) \geq v$ which implies that

$$f_1(F, v) \geq v_1;\ \ f_2(F, v) \geq v_2$$

This means that neither player should get less in the bargaining solution than he/she could get in the event of disagreement. Axioms 1 and 2 are illustrated in Figure 1.

3.3 Axiom 3: Scale Covariance

This axiom is stated as follows. For any numbers $\lambda_1, \lambda_2, \mu_1, \mu_2$ with $\lambda_1 > 0$, $\lambda_2 > 0$, define the set

$$G = \{(\lambda_1 x_1 + \mu_1, \lambda_2 x_2 + \mu_2 : (x_1, x_2) \in F\}$$

and the point

$$w = (\lambda_1 v_1 + \mu_1, \lambda_2 v_2 + \mu_2),$$

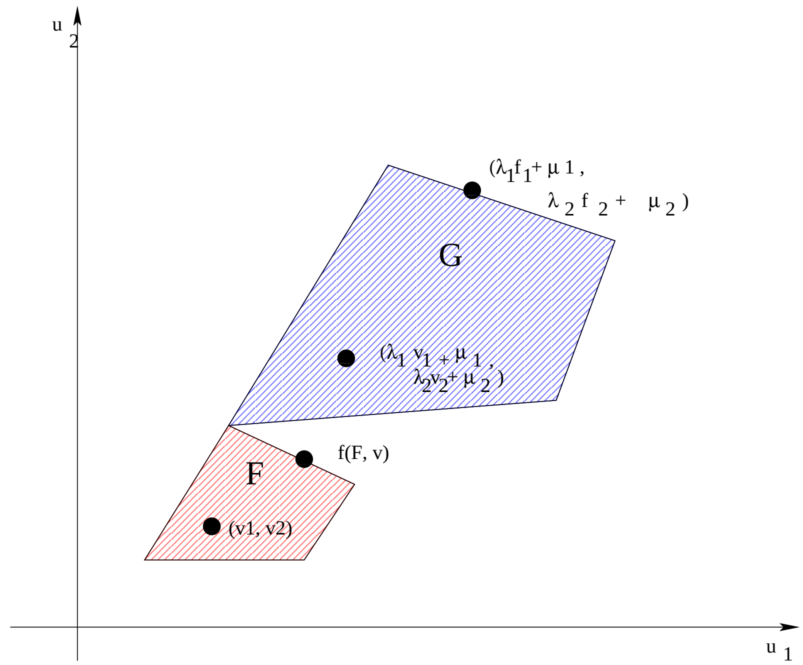

Figure 2: Illustration of scale covariance

Then the solution $f(G, w)$ for the problem (G, w) is given by

$$f(G, w) = (\lambda_1 f_1(F, v) + \mu_1, \lambda_2 f_2(F, v) + \mu_2)$$

In the above, the bargaining problem (G, w) can be derived from (F, v) by applying an increasing affine utility transformations which will not affect any decision theoretic properties of the utility functions. The axiom implies that the solution of (G, w) can be derived from the solution of (F, v) by the same transformation. This axiom is illustrated in Figure 2.

3.4 Axiom 4: Independence of Irrelevant Alternatives

This axiom states that, for any closed convex set G,

$$G \subseteq F \text{ and } f(F, v) \in G \Rightarrow f(G, v) = f(F, v)$$

The axiom asserts that eliminating feasible alternatives (other than the disagreement point) that would not have been chosen should not affect the solution. The eliminated alternatives are referred to as irrelevant alternatives. This axiom is illustrated in Figure 3. If an arbitrator or referee were to select a solution by maximizing some aggregate measure of social gain, that is,

$$f(F, v) = \max_{x \in F} M(x, v)$$

where $M(x, v)$ is a measure of social gain by choosing x instead of v, then Axiom 4 can be shown to be always satisfied.

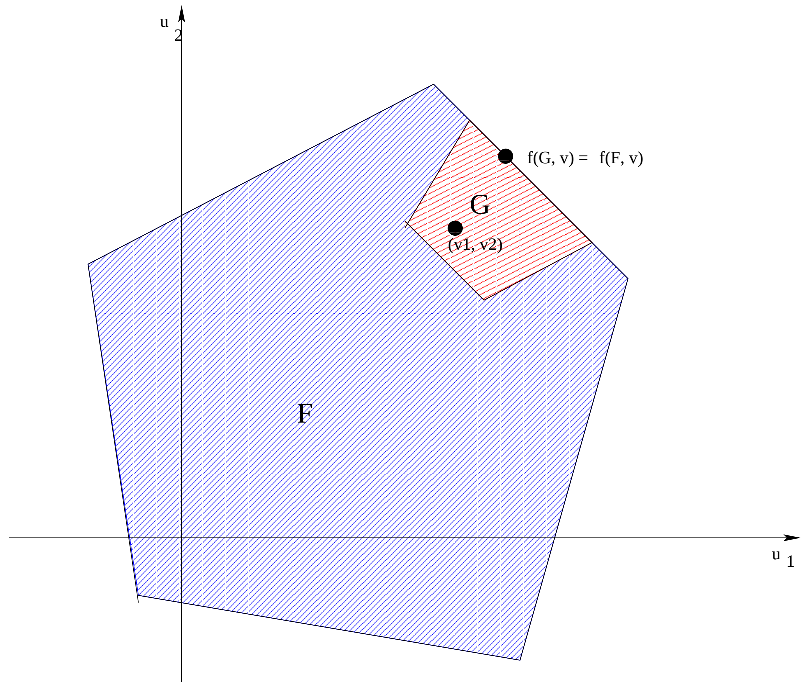

Figure 3: Independence of irrelevant alternatives

3.5 Axiom 5: Symmetry

This axiom asserts that if the positions of players 1 and 2 are completely symmetric in the bargaining problem, then the solution should also treat them symmetrically. Formally,

$$v_1 = v_2 \text{ and } \{(x_2, x_1) : (x_1, x_2) \in F\} = F \Rightarrow f_1(F, v) = f_2(F, v)$$

This axiom is illustrated in Figure 4.

3.6 The Nash Bargaining Solution

Using ingenious arguments, Nash showed the a solution that satisfies all the five axioms exists and moreover it is unique. The following provides the details of the theorem.

Theorem 1 *Given a two person bargaining problem (F, v), there exists is a unique solution function $f(.,.)$ that satisfies Axioms (1) through (5). The solution function satisfies, for every two person bargaining problem (F, v),*

$$f(F,v) \in \underset{\substack{(x_1,x_2) \in F \\ x_1 \geq v_1; x_2 \geq v_2}}{\operatorname{argmax}} (x_1 - v_1)(x_2 - v_2)$$

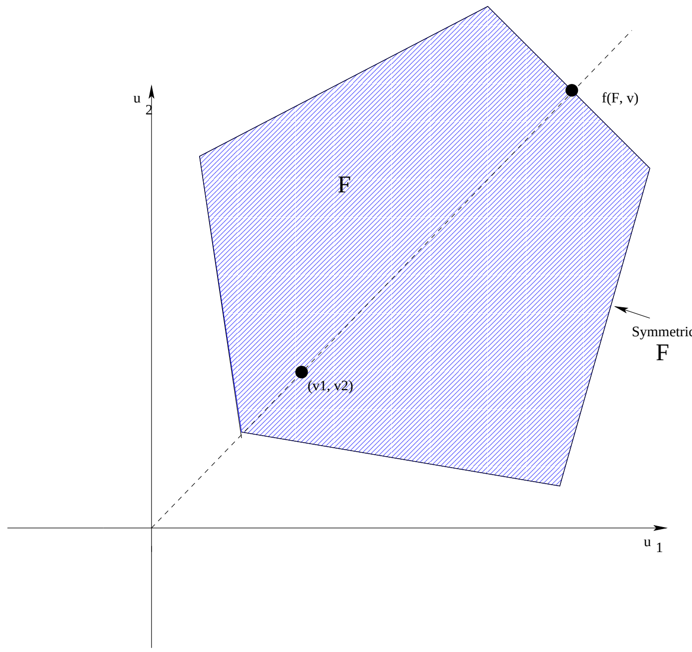

Figure 4: Illustration of symmetry

3.7 An Illustrative Example

Figure 5 shows a closed convex space F which is actually the convex hull enclosing the points (4,0), (1,1), and (0,4). Suppose the default point is (1,1). It can be shown that the Nash bargaining solution is (2,2). This illustrates Pareto efficiency, individual rationality, and symmetry. We shall define a new feasible space G obtained the following scaling: $\lambda_1 = \lambda_2 = \frac{1}{2}; \quad \mu_1 = \mu_2 = 1$. Consider the bargaining problem (G, w) with $w = (1,1) = v$ that is obtained using $w = (\lambda_1 v_1 + \mu_1, \quad \lambda_2 v_2 + \mu_2)$. Using scale covariance, the Nash bargaining solution becomes $(2,2)$. The problems (F, v) and (G, w) also illustrate the independence of irrelevant alternatives axiom. For, we find that $G \subseteq F$; G is closed and convex, and $(2,2) \in G$. Therefore we have $f(G, v) = f(F, v) = (2,2)$. Finally, if H is the feasible space obtained using $\lambda_1 = \lambda_2 = \frac{1}{2}; \quad \mu_1 = \mu_2 = 0$, then the problem $(H, (0.5, 0.5))$ illustrates another instance of scale covariance.

4 Proof of the Nash Bargaining Theorem

First we prove the theorem for a special class of two person bargaining problems called *essential* bargaining problems and subsequently, we generalize this to the entire class of problems. A two person bargaining problem (F, v) is said to be *essential* if there exists at least one allocation $y \in F$ that is strictly better for both the players than the disagreement allocation v, that is $y_1 > v_1$ and $y_2 > v_2$.

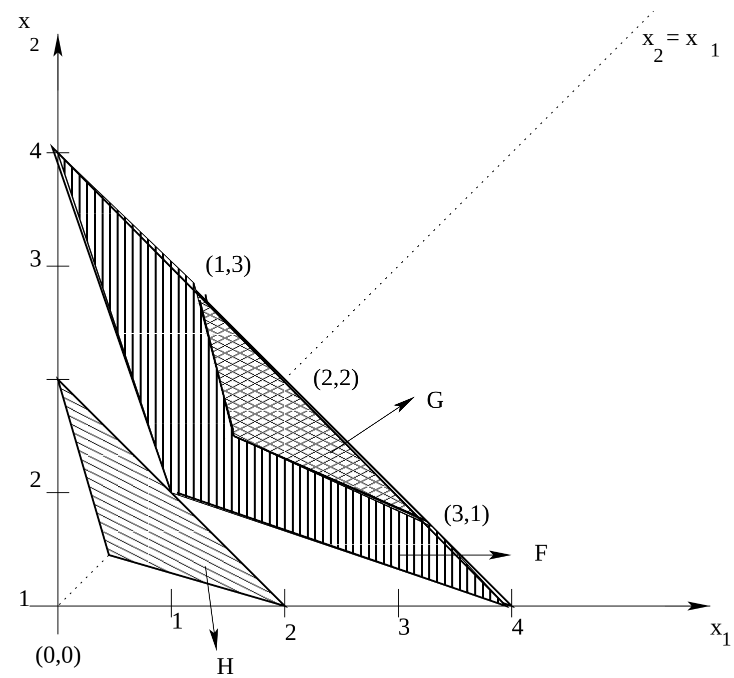

Figure 5: An example to illustrate Nash axioms

4.1 Proof for Essential Bargaining Problems

We are given an essential two person bargaining problem (F, v). Clearly, there exists some $y = (y_1, y_2) \in F$ such that $y_1 > v_1$ and $y_2 > v_2$.

Recall the definition of a quasiconcave function from Chapter 10: A function $f : S \to \mathbb{R}$ where S is non-empty and convex is said to be quasiconcave if

$$f(\lambda x + (1-\lambda)y) \geq \min(f(x), f(y)) \ \ \forall x, y \in S \ \text{ and } \ \forall \lambda \in [0,1]$$

f is strictly quasiconcave if

$$f(\lambda x + (1-\lambda)y) > \min(f(x), f(y)) \ \ \forall x, y \in S \ \text{ and } \ \forall \lambda \in (0,1)$$

It is a standard result that a strictly quasiconcave function will have a unique optimal (maximum) solution.

- Consider the optimization problem:

$$\max_{\substack{(x_1, x_2) \in F \\ x_1 \geq v_1;\, x_2 \geq v_2}} \quad (x_1 - v_1)(x_2 - v_2)$$

  where $(v_1, v_2) \in F$. The function $(x_1 - v_1)(x_2 - v_2)$ is strictly quasiconcave (since we are dealing with an essential bargaining problem) and therefore it has a unique maximizer. Therefore the above optimization problem has a unique optimal solution. Call this solution as (x_1^*, x_2^*).

- Let $F \subset \mathbb{R}^2$ be convex and closed. Suppose $v = (v_1, v_2) \in F$ and

$$F \cap \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq v_1; x_2 \geq v_2\}$$

  is non-empty and bounded. Suppose the function $f(F, v)$ satisfies the five axioms (1) strong efficiency (2) individual rationality (3) scale covariance (4) independence of irrelevant alternatives, and (5) symmetry. Then the Nash bargaining theorem states that $f(F, v) = (x_1^*, x_2^*)$ which happens to be the unique solution of optimization problem above.

The proof of the Nash bargaining theorem proceeds in two parts. In Part 1, we show that the optimal solution (x_1^*, x_2^*) of the optimization problem above satisfies all the five axioms. In Part 2, we show that if $f(F, v)$ satisfies all the five axioms, then $f(F, v) = (x_1^*, x_2^*)$. We use the following notation for the objective function in the rest of this proof:

$$N(x_1, x_2) = (x_1 - v_1)(x_2 - v_2)$$

The objective function is appropriately called the *Nash Product*.

4.2 Proof of Part 1

We have to show that the optimal solution $(x_1^*, x_2)^*$ of the optimization problem above satisfies all the five axioms. We do this one by one.

Strong Efficiency

We have to show that there does not exist $(\hat{x_1}, \hat{x_2}) \in F$ such that $\hat{x_1} \geq x_1^*$ and $\hat{x_2} \geq x_2^*$ with at least one inequality strict. Suppose such a $(\hat{x_1}, \hat{x_2})$ exists. Since there exists a $(y_1, y_2) \in F$ such that $y_1 > v_1$ and $y_2 > v_2$, the maximum value of the Nash product in the optimization problem is strictly greater than zero. Since the objective function $N(x_1, x_2)$ is increasing in x_1 and x_2, we have

$$N(\hat{x_1}, \hat{x_2}) > N(x_1^*, x_2^*)$$

which is not possible since $N(x_1^*, x_2^*)$ is the maximum possible value of $N(x_1, x_2)$ in the optimization problem.

Individual Rationality

This is immediately satisfied being one of the constraints in the optimization problem.

Scale Covariance

For $\lambda_1 > 0, \lambda_2 > 0, \mu_1, \mu_2$, define

$$G = \{\lambda_1 x_1 + \mu_1, \lambda_1 x_2 + \mu_2) : (x_1, x_2) \in F\}$$

Consider the problem

$$\max_{(y_1, y_2) \in G} (y_1 - (\lambda_1 v_1 + \mu_1))(y_2 - (\lambda_2 v_2 + \mu_2))$$

This can be written using $y_1 = \lambda_1 x_1 + \mu_1$ and $y_2 = \lambda_2 x_2 + \mu_2$ as

$$\max_{(x_1, x_2) \in G} (\lambda_1 x_1 + \mu_1 - (\lambda_1 v_1 + \mu_1))(\lambda_2 x_2 + \mu_2 - (\lambda_2 v_2 + \mu_2))$$

The above problem is the same as

$$\max_{(x_1, x_2) \in G} \lambda_1 \lambda_2 (x_1 - v_1)(x_2 - v_2)$$

which attains maximum at (x_1^*, x_2^*). Therefore the problem

$$\max_{(x_1, x_2) \in G} (y_1 - (\lambda_1 v_1 + \mu_1))\,(y_2 - (\lambda_2 v_2 + \mu_2))$$

attains maximum at $(\lambda_1 x_1^* + \mu_1,\ \lambda_2 x_2^* + \mu_2)$.

Independence of Irrelevant Alternatives

We are given $G \subseteq F$ with G closed and convex. Let (x_1^*, x_2^*) be optimal to (F, v) and let (y_1^*, y_2^*) be be optimal to (G, v). It is also given that $(x_1^*, x_2^*) \in G$.

- Since (x_1^*, x_2^*) is optimal to F which is a superset of G, we have

$$N(x_1^*, x_2^*) \geq N(y_1^*, y_2^*)$$

- Since (y_1^*, y_2^*) is optimal to G and $(x_1^*, x_2^*) \in G$, we have

$$N(y_1^*, y_2^*) \geq N(x_1^*, x_2^*)$$

Therefore we have

$$N(x_1^*, x_2^*) = N(y_1^*, y_2^*)$$

Since the optimal solution is unique, we then immediately obtain

$$(x_1^*, x_2^*) = (y_1^*, y_2^*)$$

Symmetry

Suppose we have

$$\{(x_2, x_1) : (x_1, x_2) \in F\} = F$$

and $v_1 = v_2$. Since $v_1 = v_2$, then (x_1^*, x_2^*) maximizes $(x_1 - v_1)(x_2 - v_1)$. Since the optimal solution is unique, we should have $(x_1^*, x_2^*) = (x_2^*, x_1^*)$ which immediately yields $x_1^* = x_2^*$.

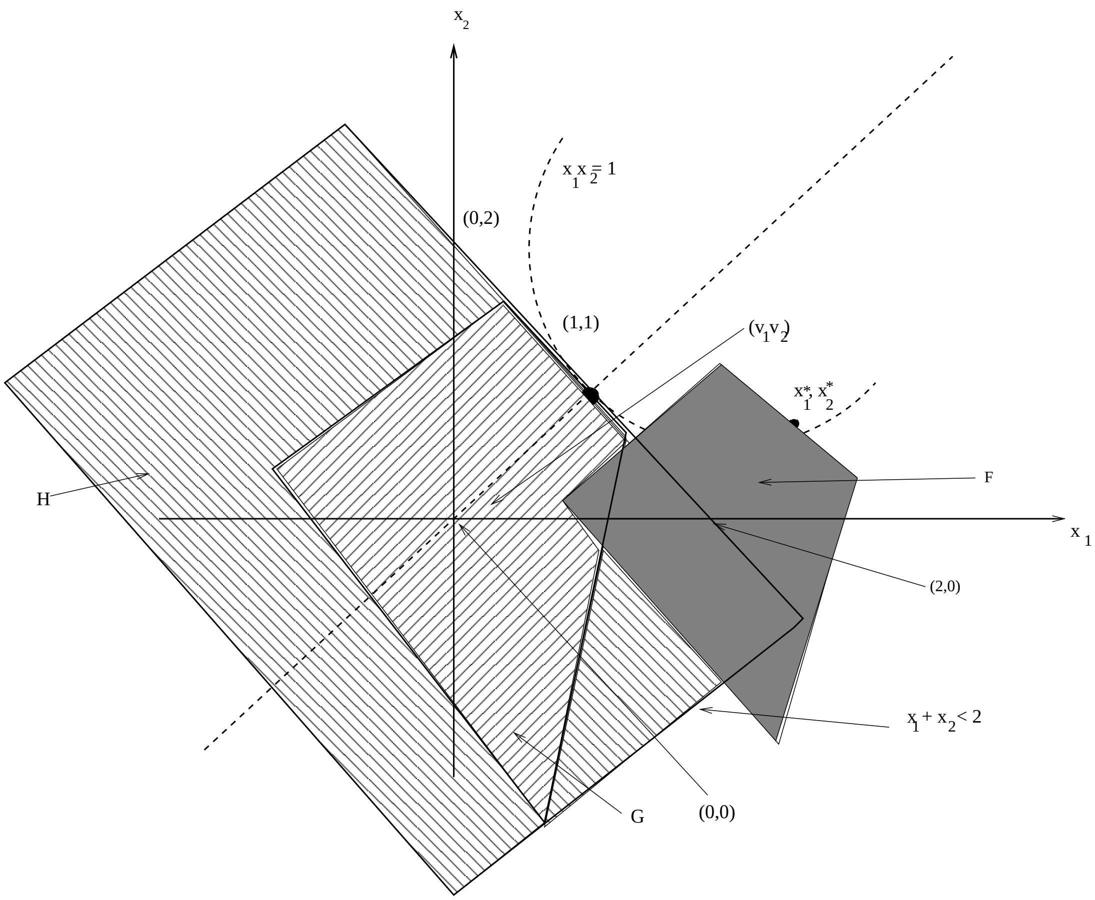

Figure 6: Diagram showing F, G, H for proving the Nash bargaining theorem

4.3 Proof of Part 2

We are given that $f(F, v)$ is a bargaining solution that satisfies all the five axioms. We have to show that $f(F, v)$ is the same as (x_1^*, x_2^*) which is the unique solution of the optimization problem:

$$\max_{\substack{(x_1,x_2)\in F \\ x_1 \geq v_1; x_2 \geq v_2}} (x_1 - v_1)(x_2 - v_2)$$

We have that $x_1^* > v_1$ and $x_2^* > v_2$. Consider the following transformation

$$L(x_1, x_2) = (\lambda_1 x_1 + \mu_1, \lambda_2 x_2 + \mu_2)$$

where

$$\lambda_1 = \frac{1}{x_1^* - v_1}; \ \ \lambda_2 = \frac{1}{x_2^* - v_2}; \ \ \mu_1 = \frac{-v_1}{x_1^* - v_1}; \ \ \mu_2 = \frac{-v_2}{x_2^* - v_2}$$

In other words, we have

$$L(x_1, x_2) = \left(\frac{x_1 - v_1}{x_1^* - v_1}, \ \ \frac{x_2 - v_2}{x_2^* - v_2}\right)$$

Note that $L(v_1, v_2) = (0, 0)$ and $L(x_1^*, x_2^*) = (1, 1)$. Defining $G = \{L(x_1, x_2) : (x_1, x_2) \in F\}$, we have thus transformed the problem $(F, (v_1, v_2))$ to the problem $(G, (0, 0))$. In fact, since $L(x_1^*, x_2^*) = (1, 1)$, it is known that the transformed problem $(G, (0, 0))$ has its solution at $(1, 1)$. See Figure 6. Also observe that the objective function of the transformed problem is $(x_1 - 0)(x_2 - 0) = x_1 x_2$. First we show that

$$x_1 + x_2 \leq 2 \ \ \forall \ (x_1, x_2) \in G$$

To show this, we assume that $x_1 + x_2 > 2$ and arrive at a contradiction. Suppose $\alpha \in (0, 1)$ and

$$t = (1 - \alpha)(1, 1) + \alpha(x_1, x_2) = (1 - \alpha + \alpha x_1, \ \ 1 - \alpha + \alpha x_2)$$

Since G is convex and $(1, 1) \in G$, we therefore have $t \in G$ and we get

$$t_1 t_2 = (1 - \alpha + \alpha x_1)(1 - \alpha + \alpha x_2) = (1 - \alpha)^2 + \alpha^2 x_1 x_2 + (1 - \alpha)\alpha(x_1 + x_2)$$

If $x_1 + x_2 > 2$, we then get

$$t_1 t_2 > (1 - \alpha)^2 + \alpha^2 x_1 x_2 + (1 - \alpha)\alpha = (1 - \alpha) + \alpha^2 x_1 x_2$$

We can always choose α sufficiently small and make $t_1 t_2 > 1$, which is impossible since the optimal (that is, maximal) value of $t_1 t_2$ is 1.

G is bounded and we can always find a rectangle H which is symmetric about the line $x_1 = x_2$ such that $G \subseteq H$ and H is also convex and bounded. Further choose H such that the point $(1, 1) \in G$ is on the boundary of H.

Strong efficiency and symmetry imply that

$$f(H, (0, 0)) = (1, 1)$$

We can now use independence of irrelevant alternatives to get

$$f(G, (0, 0)) = (1, 1)$$

We know G is obtained through scaling. Now use scale covariance to get

$$f(G, (0, 0)) = L(f(f, v))$$

This implies

$$L(f(F, v)) = (1, 1)$$

Since we know that $L(x_1^*, x_2^*) = (1, 1)$, we finally obtain

$$f(F, v) = (x_1^*, \ \ x_2^*)$$

4.4 Proof for Inessential Bargaining Problems

Consider the case when the problem (F, v) is inessential, that is, there does not exist any $(y_1, y_2) \in F$ such that $y_1 > v_1$; $y_2 > v_2$. Since F is convex, the above implies that there exists at least one player i such that

$$y_1 \geq v_1 \ \text{ and } \ y_2 \geq v_2 \Rightarrow y_i = v_i \ \ \forall \ (y_1, y_2) \in F$$

The reason is if we could find $(y_1, y_2), (z_1, z_2) \in F$ such that $(y_1, y_2) \geq (z_1, z_2), y_1 > v_1, z_2 > v_2$, then $\frac{1}{2}(y_1, y_2) + \frac{1}{2}(z_1, z_2)$ would be a point in F that is strictly better than (v_1, v_2) for both the players. In the rest of the discussion, let us assume without loss of generality that

$$y_1 \geq v_1 \text{ and } y_2 \geq v_2 \Rightarrow y_1 = v_1 \;\; \forall \, (y_1, y_2) \in F$$

Suppose x^* is an allocation in F that is best for the player 2 subject to the constraint $x_1 = v_1$ (with this constraint, the maximum value of the Nash product in the optimization problem will be zero). This would imply that the vector (x_1^*, x_2^*) is the unique point that is strongly efficient in F and individually rational relative to v. This would mean that, to satisfy Axioms 1 and 2, we must have $f(F, v) = (x_1^*, x_2^*)$. It can be easily observed that (x_1^*, x_2^*) achieves the maximum value of the Nash product $(x_1 - v_1)(x_2 - v_2)$ which happens to be zero for all individually rational allocations.

4.5 A Note on Essential Bargaining Problems

For essential bargaining problems, the strong efficiency axiom can be replaced by the following weak efficiency axiom:

Axiom $1'$. (Weak Efficiency). $f(F, v) \in F$ and there does not exist any $y \in F$ such that

$$y > f(F, v).$$

Also, it is trivial to see that for essential bargaining problems, the individual rationality assumption is not required. Thus in the case of essential two person bargaining problems, any solution that satisfies Axioms $1', 3, 4, 5$ must coincide with the Nash bargaining solution.

4.6 An Illustrative Example

This example is taken from the book by Myerson [3]. Let $v = (0, 0)$ and

$$F = \{(y_1, y_2) : 0 \leq y_1 \leq 30; \;\; 0 \leq y_2 \leq 30 - y_1\}$$

The bargaining problem (F, v) can be thought of a situation in which the players can share 30 million rupees in any way in which they agree or get zero if they cannot agree, with both players having a linear utility for money (that is, utility varies linearly with money). A player with such a utility function is said to be risk neutral. The Nash bargaining solution $f(F, v)$ is the vector $y = (y_1, y_2)$ such that $y_1 \geq 0$, $y_2 \geq 0$, and y maximizes the Nash product $(y_1 - v_1)(y_2 - v_2)$. This can be easily seen to be $(15, 15)$ which looks perfectly reasonable. This corresponds to each player being allocated 15 million rupees.

Let us modify this example by choosing

$$F = \{(y_1, y_2) : 0 \leq y_1 \leq 30; \;\; 0 \leq y_2 \leq \sqrt{30 - y_1}\}$$

Suppose $v = (0, 0)$ as earlier. This problem is similar to the one above, except that player 1 is risk neutral (that is, has linear utility for money) and player 2 is risk averse (with a utility scale that is proportional to the square root of the money) Note that

$$\frac{d}{dy_1}\left[y_1\sqrt{30 - y_1}\right] = 0$$

yields $y_1 = 20$. The Nash bargaining solution is the allocation $(20, \sqrt{10}) = (20, 3.162)$. This corresponds to a wealth sharing of $(20, 10)$.

To Probe Further

The original discussion of the Nash bargaining problem and its solution is of course found in the classic paper of Nash [1]. The discussion and approach presented in this chapter closely follows that of Myerson [3]. There are two books that deal with the bargaining problem quite extensively. The book by Abhinay Muthoo [4] deals exclusively with the bargaining problem. The book by Osborne and Rubinstein [5] discusses bargaining theory extensively and presents rigorous applications to market situations of different kinds. The book by Straffin [6] provides two real-world applications of the two person bargaining problem: (1) Management - labor union negotiations, and (2) Duopoly Model of two competing companies trying to maximize their revenues. The concept of Nash program is discussed by Nash [2].

There are several appendix sections in this chapter. Appendix 1 discusses two other well known solutions to the bargaining problem, namely egalitarian and utilitarian solutions. Appendix 2 treats a class of games called transferable utility games. Finally Appendix 3 discussed the theory of rational threats which could be used for choosing a disagreement point.

Problems

1. With reference to the axiom on independence of irrelevant alternatives, show the following property: Given a Nash bargaining problem $)F, v)$, if an arbitrator were to select a solution by maximizing some aggregate measure of social gain, that is,

$$f(F, v) = \max_{x \in F} M(x, v)$$

where $M(x, v)$ is a measure of social gain by choosing x instead of v, then Axiom 4 can be shown to be always satisfied.

2. Suppose F is the convex hull enclosing the points A= (1,8); B = (6,7); C = (8,6); D = (9,5); E =(10,3); F = (11, -1); and G = (-1,-1). Suppose the default point (status quo point) is (2,1). Compute the Nash bargaining solution for this situation. Write down a picture and that will be helpful.

3. Consider the following two player game:

	A	B
A	5, 5	2,2
B	2,2	3,3

For this game,

- compute the set of all payoff utility pairs possible (a) under correlated strategies (b) under individually rational correlated strategies.
- What would be the Nash bargaining solution in cases (a) and case (b) assuming the minmax values as the disagreement point.

4. Consider the following situation. It is required to lease a certain quantity of telecom bandwidth from Bangalore to Delhi. There are two service providers in the game $\{1, 2\}$. Provider 1 offers a direct service from Bangalore to New Delhi with a bid of 100 million rupees. Provider 1 also

offers a service from Mumbai to Delhi with a bid of 30 million rupees. On the other hand, Provider 2 offers a a direct service between Bangalore and Delhi with a bid of 120 million dollars and a service from Bangalore to Mumbai with a bid of 50 million rupees. Compute a Nash bargaining solution assuming $(0, 0)$ as the disagreement point. What can you infer from the Nash bargaining solution?

Appendix 2: Other Solutions for the Bargaining Problem: Egalitarian and Utilitarian Solutions

In typical bargaining situations, interpersonal comparisons of utility are made in two different ways.

- **Principle of Equal Gains**: Here a person's argument will be: *you should do this for me because I am doing more for you.* This leads to what is called an *egalitarian* solution.

- **Principle of the Greatest Good**: The argument here goes as follows: *You should do this for me because it helps me more than it hurts you.* This leads to the so called *utilitarian* solution.

For a two-person bargaining problem (F, v), the egalitarian solution is the unique point $(x_1, x_2) \in F$ that is weakly efficient in F and satisfies the equal gains condition:

$$x_1 - v_1 = x_2 - v_2$$

Recall that $((x_1, x_2) \in F$ is said to be weakly efficient if there does not exist any $(y_1, y_2) \in F$ such that $y_1 > x_1$ and $y_2 > x_2)$.

A utilitarian solution is any solution function that selects, for every two person bargaining problem (F, v), an allocation $(x_1, x_2) \in F$ such that

$$x_1 + x_2 = \max_{(y_1, y_2) \in F} (y_1 + y_2)$$

If agents in negotiation or arbitration are guided by the equal gains principle, the natural outcome is the egalitarian solution. If it is guided by the principle of greatest good, a natural outcome is a utilitarian solution.

The egalitarian and the utilitarian solutions violate the axiom of scale covariance. The intuition for this as follows. The scale covariance axiom is based on an assumption that only the individual decision theoretic properties of the utility scales should matter. Also interpersonal comparisons of utility have no decision theoretic significance as long as no player can be asked to decide between being himself or someone else.

λ-Egalitarian Solution

Consider a two person bargaining problem (F, v). Given numbers $\lambda_1, \lambda_2, \mu_1, \mu_2$, with $\lambda_1 > 0$, $\lambda_2 > 0$, let

$$L(y) = (\lambda_1 y_1 + \mu_1, \lambda_2 y_2 + \mu_2) \text{ for } y \in \mathbb{R}^2$$

Given the problem (F, v), define

$$L(F) = \{L(y) : y \in F\}$$

Then the egalitarian solution of $(L(F), L(v))$ is $L(x)$, where x is the unique weakly efficient point in F such that

$$\lambda_1(x_1 - v_1) = \lambda_2(x_2 - v_2)$$

This is called the λ-egalitarian solution of (F, v). If $\lambda = (1, 1)$, this is called the simple egalitarian solution. The egalitarian solution does not satisfy scale covariance because the λ-egalitarian solution is generally different from the simple egalitarian solution.

λ-Utilitarian Solution

A utilitarian solution of $(L(f), L(v))$ is a point $L(z)$ where $z = (z_1, z_2)$ is a point in F such that

$$\lambda_1 z_1 + \lambda_2 z_2 = \max_{(y_1,y_2)\in F}(\lambda_1 y_1 + \lambda_2 y_2)$$

The solution point z is called a λ-utilitarian solution of (F, v). Utilitarian solutions do not satisfy scale covariance because a λ-utilitarian solution is generally not a simple utilitarian solution.

Relationship to the Nash Bargaining Solution

Note that the equal gains principle suggests a family of egalitarian solutions and the greatest good principle suggests a family of utilitarian solutions. These solutions correspond to application of these principles when the payoffs are compared in a λ-weighted utility scale.

As λ_1 increases and λ_2 decreases, the λ-egalitarian solutions trace out the individually rational, weakly efficient frontier moving in the direction of decreasing payoff to player 1. Also as λ_1 increases and λ_2 decreases, the λ-utilitarian solutions trace out the entire weakly efficient frontier, moving in the direction of increasing payoff to player 1.

It turns out that for an essential two person bargaining problem (F, v), there exists a vector $\lambda = (\lambda_1, \lambda_2)$ such that $\lambda > (0, 0)$ and the λ-egalitarian solution of (F, v) is also a λ-utilitarian solution of (F, v). λ_1 and λ_2 that satisfy this property are called *natural scale factors*.

Remarkably, the allocation in F that is both λ-egalitarian and λ-utilitarian in terms of the natural scale factors is the Nash bargaining solution. Thus the Nash bargaining solution can be viewed as a natural synthesis of the equal gains and greatest good principles. The following theorem formalizes this fact.

Theorem

Let (F, v) be an essential two person bargaining problem. Suppose x is an allocation vector such that $x^* \in F$ and $x^* \geq v$. Then x^* is the Nash-bargaining solution for (F, v) iff there exist strictly positive numbers λ_1 and λ_2 such that

$$\lambda_1 x_1^* - \lambda_1 v_1 = \lambda_2 x_2^* - \lambda_2 v_2$$

and

$$\lambda_1 x_1^* - \lambda_2 x_2^* = \max_{y\in F}(\lambda_1 y_1 + \lambda_2 y_2)$$

Proof: Let $H(x, v)$ denote the hyperbola

$$H(x, v) = \{y \in \mathbb{R}^2 : (y_1 - v_1)(y_2 - v_2) = (x_1 - v_1)(x_2 - v_2)\}$$

The allocation x is the Nash bargaining solution of (F, v) if and only if the hyperbola $H(x, v)$ is tangential to F at x. Now, the slope of the hyperbola $H(x, v)$ at x

$$= \frac{-(x_2 - v_2)}{(x_1 - v_1)}$$

This means $H(x, v)$ is tangent at x to the line

$$\{y \in \mathbb{R}^2 : \lambda_1 y_1 + \lambda_2 y_2 = \lambda_1 x_1 + \lambda_2 x_2\}$$

for any two positive numbers λ_1 and λ_2 such that

$$\lambda_1(x_1 - v_1) = \lambda_2(x_2 - v_2)$$

Therefore $x \in F$ is the Nash bargaining solution of (F, v) if any only if F is tangent at x to a line of the form

$$\{y \in \mathbb{R}^2 : \lambda_1 y_1 + \lambda_2 y_2 = \lambda_1 x_1 + \lambda_2 x_2\}$$

for some (λ_1, λ_2) such that

$$\lambda_1(x_1 - v_1) = \lambda(x_2 - v_2)$$

.

An Illustrative Example

Let $v = (0, 0)$ and $F = \{(y_1, y_2) : 0 \leq y_1 \leq 30; 0 \leq y_2 \leq \sqrt{30 - y_1}\}$. The above problem is the same as Example 2. Recall that the Nash bargaining solution is the allocation

$$(20, \sqrt{10}) = (20, 3.162)$$

which corresponds to a monetary allocation of 20 for player 1 and 10 to player 2. Note that the risk averse player is under some disadvantage as per the Nash bargaining solution. The natural scale factors for this problem are

$$\lambda_1 = 1$$
$$\lambda_2 = \sqrt{40} = 6.325$$

Let player 2's utility for a monetary gain of D be 6.325 $\sqrt{D}$ instead of $\sqrt{D}$. This is a decision-theoretically irrelevant change. Let player 1's utility be measured in the same units as money gained ($\lambda_1 = 1$). The representation of this bargaining problem becomes $(G, (0, 0))$ where

$$G = \{(y_1, y_2) : 0 \leq y_1 \leq 30; 0 \leq y_2 \leq 6.325\sqrt{30 - y_1}\}$$

Now the Nash bargaining solution is $(20, 20)$, which still corresponds to a monetary allocation of 20 to player 1 and 10 for player 2. Note that this is both an egalitarian solution and a utilitarian solution.

Appendix 3: Games with Transferable Utility

Let $\Gamma = \langle N, (S_i), (u_i) \rangle$ be a strategic form game. Informally, Γ is said to be a game with transferable utility if, in addition to the strategy options listed in S_i, each player i can

1. give any amount of money to any other player, or
2. simply destroy money

Each unit of net monetary outflow decreases the utility of player i by one unit. This is formalized in the following definition.

Definition: A strategic form game $\Gamma = \langle N, (S_i), (u_i) \rangle$ is called a game with transferable utility if it can be represented by the game $G = \langle N, (\hat{S}_i), (\hat{u}_i) \rangle$ where for $i \in N$,

$$\hat{S}_i = S_i \times \mathbb{R}^n_+$$
$$\hat{u}_i((s_j, t_j)_{j \in N}) = u_i((s_j)_{j \in N}) + [\sum_{j \neq i} (t_j(i) - t_i(j))] - t_i(i)$$

where $t_j = (t_j(k))_{k \in N}$. The monetary transfer $t_j(k)$, for any $k \neq j$, represents the quantity of money given by player j to player k and $t_j(j)$ denotes the amount of money destroyed by j but not given to any other player.

An Example

Let $N = \{1, 2\}$. The payoff function $\hat{u}_i$ would look like the following:

$$\hat{u}_1(s_1, s_2, t_1(1), t_1(2), t_2(1), t_2(2)) = u_1(s_1, s_2) - (t_2(1) - t_1(2)) - t_1(1)$$

$$\hat{u}_2(s_1, s_2, t_1(1), t_1(2), t_2(1), t_2(2)) = u_2(s_1, s_2) - (t_1(2) - t_2(1)) - t_2(2)$$

Note that

$$\hat{u}_1(s_1, s_2, t_1(1), t_1(2), t_2(1), t_2(2)) + \hat{u}_2(s_1, s_2, t_1(1), t_1(2), t_2(1), t_2(2)) = u_1(s_1, s_2) + u_2(s_1, s_2) - t_1(1) - t_2(2)$$

Note in the above definition that $\hat{u}_i$ is linearly dependent on the transfers t_j. Thus risk neutrality is implicitly assumed when we talk of transferable utility in a game. Transferable utility is an assumption which ensures that the given scale factors in a game will also be the natural scale factors for the Nash bargaining solution.

Let (F, v) be a two person bargaining problem derived from a game with transferable utility. Let v_{12} represent the maximum transferable wealth that the players can jointly achieve. Then the feasible set F will be of the form

$$F = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 \leq v_{12}\}$$

This implies that, in the presence of transferable utility, a two person bargaining problem can be fully characterized by three numbers.

1. v_1 = disagreement payoff to player 1
2. v_2 = disagreement payoff to player 2
3. v_{12} = total transferable wealth available to the players if they cooperate.

x^* will be a Nash bargaining solution to this problem iff $\exists \lambda_1 > 0$, $\lambda_2 > 0$ such that

$$\begin{aligned}
\lambda_1 x_1^* - \lambda_1 v_1 &= \lambda_2 x_2^* - \lambda_2 v_2 \\
\lambda_1 x_1^* + \lambda_2 x_2^* &= \max_{(x_1,x_2)\in F} (\lambda_1 x_1 + \lambda_2 x_2)
\end{aligned}$$

To satisfy the above conditions, $\lambda_1 = \lambda_2$, since otherwise

$$\max_{(x_1,x_2)\in F} (\lambda_1 x_1 + \lambda_2 x_2) = +\infty$$

since both x_1 and x_2 are real numbers and $x_1 + x_2 \leq v_{12}$ will allow unbounded values for x_1 and x_2 separately.

Thus the conditions for $f(F, v)$ become

$$\begin{aligned}
f_1(F, v) - v_1 &= f_2(F, v) - v_2 \\
f_1(F, v) + f_2(F, v) &= v_{12}
\end{aligned}$$

Solving these equations, the following general formulae can be obtained for the Nash bargaining solution of a game with transferable utility.

$$\begin{aligned}
f_1(F, v) &= \frac{v_{12} + v_1 - v_2}{2} \\
f_2(F, v) &= \frac{v_{12} + v_2 - v_1}{2}
\end{aligned}$$

Let us say F is derived under the assumption that the players' strategies can be regulated by binding contracts. Then we can say, as a consequence of the transferable utility property,

$$v_{12} = \max_{\sigma\in\Delta(S)} (u_1(\sigma) + u_2(\sigma))$$

Appendix 4: Rational Threats

We have shown that a two player bargaining problem (F, v), in a transferable utility setting, has the Nash bargaining solution

$$\begin{aligned}
f_1(F, v) &= \frac{v_{12} + v_1 - v_2}{2} \\
f_2(F, v) &= \frac{v_{12} + v_2 - v_1}{2}
\end{aligned}$$

- Note that the payoff to player 1 increases as the disagreement payoff to player 2 decreases. That is, hurting player 2 in the event of disagreement may actually help player 1 if a cooperative agreement is reached.
- Thus, reaching a cooperating point depends on a disagreement point and this induces rational players to behave in an antagonistic way in trying to create a more favorable disagreement point.
- This phenomenon is called the *chilling effect*. This is described formally by Nash's theory of rational threats.

Nash's Theory of Rational Threats

Let $\Gamma = \langle \{1,2\}, S_1, S_2, u_1, u_2 \rangle$ be a two player finite strategic form game. Let F be the feasible set derived from Γ. F will take the form

$$F = \{(u_1(\sigma), u_2(\sigma)) : \sigma \in \Delta(S)\}$$

where

$$u_i(\sigma) = \sum_{s \in S} \sigma(s) u_i(s) \text{ for } i = 1, 2$$

in the case of binding contracts without non-transferable utility. In the case of binding contracts with transferable utility, F will take the form

$$F = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 + y_2 \leq v_{12}\}$$

where

$$v_{12} = \max_{\sigma \in \Delta(S)} [u_1(\sigma) + u_2(\sigma)]$$

Before entering into the negotiation or arbitration, each player is required to choose a threat $\tau_i \in \Delta(S_i)$. If the players fail to reach a cooperative agreement, then each player is committed independently to carry out his threat.

If (τ_1, τ_2) is a pair of threats chosen by the two players, the disagreement point in the two person bargaining problem should be

$$(u_1(\tau_1, \tau_2), u_2(\tau_1, \tau_2))$$

Let $w_i(\tau_1, \tau_2)$ be the payoff that player i $(i = 1, 2)$ gets in the Nash bargaining solution with the above disagreement point. That is,

$$w_i(\tau_1, \tau_2) = f_i(F, (u_1(\tau_1, \tau_2), u_2(\tau_1, \tau_2)))$$

The game $\Gamma^* = \langle \{1,2\}, \Delta(S_1), \Delta(S_2), w_1, w_2 \rangle$ is called the *threat game* corresponding to the original game Γ.

Assume that the players expect that they will finally reach a cooperative agreement, that will depend on the disagreement point, according to the Nash bargaining solution. This implies that the players should not be concerned about carrying out their threats but instead should evaluate their threats only in terms of their impact on the final cooperative agreement.

This makes each player i to choose his threat τ_i^* so as to maximize $w_i(\tau_1^*, \tau_2^*)$ given the other player's expected threat. This motivates to call (τ_1^*, τ_2^*) a pair of rational threats iff

$$w_1(\tau_1^*, \tau_2^*) \geq w_1(\sigma_1, \tau_2^*) \ \ \forall \sigma_1 \in \Delta(S_1)$$
$$w_2(\tau_1^*, \tau_2^*) \geq w_2(\tau_1^*, \sigma_2) \ \ \forall \sigma_2 \in \Delta(S_2)$$

Thus (τ_1^*, τ_2^*) is a Nash equilibrium of the threat game

$$\Gamma^* = \langle \{1,2\}, \Delta(S_1), \Delta(S_2), w_1, w_2 \rangle$$

Kakutani's fixed point theorem can be used to prove the existence of rational threats. In the threat game,

- Player 1 wants to choose her threat τ_1^* so as to put the disagreement point $(u_1(\tau_1^*, \tau_2^*), u_2(\tau_1^*, \tau_2^*))$ as favorable to her as possible and as unfavorable to Player 2 as possible.
- Player 2 wants choose his threat τ_2^* so as to put the disagreement point as favorable to him as possible and as unfavorable to Player 1 as possible.

The payoffs in the threat game (in the transferable utility case) are

$$w_1(\tau_1, \tau_2) = \frac{v_{12} + u_1(\tau_1, \tau_2) - u_2(\tau_1, \tau_2)}{2}$$

$$w_2(\tau_1, \tau_2) = \frac{v_{12} + u_2(\tau_1, \tau_2) - u_1(\tau_1, \tau_2)}{2}$$

where

$$v_{12} = \max_{\sigma \in \Delta(S)} [u_1(\sigma) + u_2(\sigma)]$$

Note that v_{12} is a constant. Therefore maximizing $w_1(\tau_1, \tau_2)$ is equivalent to maximizing $u_1(\tau_1, \tau_2) - u_2(\tau_1, \tau_2)$. Similarly maximizing $w_2(\tau_1, \tau_2)$ is equivalent to maximizing $u_2(\tau_1, \tau_2) - u_1(\tau_1, \tau_2)$.

It is straightforward to note that, under the transferable utilities case, τ_1^* and τ_2^* are rational threats for players 1 and 2 iff (τ_1^*, τ_2^*) is an equilibrium of the two person zero sum game

$$\Gamma^{**} = \langle \{1, 2\}, \Delta(S_1), \Delta(S_2), u_1 - u_2, u_2 - u_1 \rangle$$

The game Γ^{**} is called the *difference game* derived from G.

Example to Illustrate Different Ways of Choosing Disagreement Point

In this example from Myerson's book [3], we examine three different ways to determine a disagreement point for deriving the Nash bargaining solution.

1. A Nash equilibrium of game Γ
2. Minimax values
3. Rational threats

Consider the following two player strategic form game

	2	
1	a_2	b_2
a_1	10, 0	-5, 1
b_1	0, -5	0, 10

Note that the maximum total payoff achievable in this game is $v_{12} = 10$.

Choice 1: Nash Equilibrium

Note that (b_1, b_2) is a non-cooperative equilibrium. Let us take the payoffs in the equilibrium as the disagreement point, that is,

$$v = (0, 10)$$

The Nash bargaining solution is

$$f(F, v) = (0, 10)$$

Choice 2: Minimax Values

- Minimax value for player 1 : $v_1 = 0$. This is achieved when player 2 chooses an offensive threat b_2 and player 1 chooses a defensive response b_1.

- Minimax value for player 2: $v_2 = 1$. This is achieved when player 1 chooses a_1 as his optimal offensive threat and player 2 chooses b_2 as an optimal defensive strategy.

Therefore, $v = (0, 1)$. The Nash bargaining solution is $f(F, v) = (4.5, 5.5)$.

Choice 3: Rational Threats

The threat game Γ^* derived from Γ is as follows.

	2	
1	a_2	b_2
a_1	10, 0	2, 8
b_1	7.5, 2.5	0, 10

The unique equilibrium of this threat game = (a_1, b_2) which leads to

$$v = (-5, 1)$$

$$f(F, v) = (2, 8)$$

In all the three cases, player 2 chooses b_2 in any disagreement because a_2 is dominated by b_2 with respect to both

- player 2's defensive objective of maximizing u_2

- player 2's offensive objective of minimizing u_1

Player 1's disagreement behavior is different across the three cases.

- In case 1 (Nash equilibrium case), player 1's behavior in the event of disagreement would be determined by his purely defensive objective of maximizing u_1. So he chooses b_1 allowing player 2 to get his maximum payoff

- In case 2 (minimax case), player 1 is supposed to be able to select between two threats:
  - an offensive threat a_1 for determining v_2
  - a defensive threat b_1 for determining v_1.

- In case 3 (rational threats case), player 1 must choose a single threat that must serve both offensive and defensive purposes simultaneously, so he chooses a_1 because a_1 maximizes the objective

$$\frac{10 + u_1 - u_2}{2}$$

  that is a synthesis of his offensive and defensive criteria.

Which of these three methods is appropriate is an interesting question to ponder. Equilibrium theory of disagreement is appropriate where the players could not commit themselves to any planned strategies in the event of disagreement, until a disagreement actually occurs. The rational threats theory is applicable when each player can, before the negotiation or arbitration process, commit himself to a single planned strategy that he would carry out in the event of disagreement, no matter whose final rejection may have caused the disagreement. It is implicitly assumed that the probability of a disagreement is extremely low. The minimax values theory is appropriate in situations where each player can, before the process starts, commit himself to two planned strategies: (a) defensive (b) offensive. The player would carry out one of these depending on how the disagreement was caused. We can suppose that he would implement his defensive strategy if he himself rejected the last offer in the negotiation or arbitration process and his offensive strategy otherwise.

References

[1] John F. Nash Jr. The bargaining problem. *Econometrica*, 18:155–162, 1950.

[2] John F. Nash Jr. Two person cooperative games. *Econometrica*, 21:128–140, 1953.

[3] Roger B. Myerson. *Game Theory: Analysis of Conflict.* Harvard University Press, Cambridge, Massachusetts, USA, 1997.

[4] Abhinay Muthoo. *Bargaining Theory with Applications.* Cambridge University Press, 1999.

[5] Martin J. Osborne and Ariel Rubinstein. *Bargaining and Markets.* Academic Press, 1990.

[6] Philip D. Straffin Jr. *Game Theory and Strategy.* The Mathematical Association of America, 1993.