Game Theory

Lecture Notes By

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COOPERATIVE GAME THEORY
Correlated Strategies and Correlated Equilibrium

Note: *This is a only a draft version, so there could be flaws. If you find any errors, please do send email to* `hari@csa.iisc.ernet.in`. *A more thorough version would be available soon in this space.*

We commence our study of cooperative game theory with a discussion of games and solution concepts which enable us to capture cooperation among players. In this chapter, we first study games with contracts followed by games with communication. We introduce the key concept of correlated strategies and correlated equilibrium.

The motivation to look at games with cooperation is the fact that in many games (for example, the Prisoner's Dilemma), Nash equilibria yield non-optimal payoffs compared to certain non-equilibrium outcomes. Let us consider the following modified version of the prisoner's dilemma problem whose payoff matrix is shown in Figure 1.

	2	
1	x_2	y_2
x_1	2,2	0,6
y_1	6,0	1,1

Figure 1: Payoff matrix of a modified version of prisoner's dilemma game

Note in the above that the unique equilibrium (which happens to be a strictly dominant strategy equilibrium) is (y_1, y_2) which yields a payoff profile (1,1). The non-equilibrium outcome (x_1, x_2) yields higher payoffs (2,2). In situations like these, the players may like to transform the game to extend the set of equilibria to include better outcomes. There could be several ways of achieving this transformation:

- The players communicate among themselves to coordinate their moves
- The players formulate contractual agreements

1	2		
	x_2	y_2	a_2
x_1	2,2	0,6	0,6
y_1	6,0	1,1	1,1
a_1	6,0	1,1	2,2

Figure 2: Payoff matrix with contract 1

- The players try to create long-term relationships
- The players decide to play the game repeatedly

1 Games with Contracts

In a game with contracts, a player who signs a contract is required to play according to a designated correlated strategy. Contracts transform games with less desirable equilibria to games with more desirable equilibria. We shall understand the meaning of a contract and correlated strategy through the example below.

Example: Modified Prisoner's Dilemma with a Contract

In game shown in Figure 1, let us say the two players sign the following contract (call it contract 1).

1. If both players sign this contract, then player 1 (player 2) chooses to play the strategy x_1 (x_2).
2. If the contract is signed by only player 1, player 1 would choose y_1
3. If the contract is signed by only player 2, player 2 would choose y_2

Call the action of signing the contract by player i ($i = 1, 2$) as a_i. We can now expand the strategy sets as $S - 1 = \{x_1, y_1, a_1\}$ and $S_2 = \{x_1, y_1, a_1\}$. The transformed game has the payoff matrix shown in Figure 2. The transformed game now has a new equilibrium (a_1, a_2) which is a weakly dominant strategy equilibrium and yields payoff the payoff (2,2). The old equilibrium (y_1, y_2) continues to be an equilibrium but it is not a dominant strategy equilibrium anymore.

Example: Modified Prisoner's Dilemma with an Additional Contract

Even better payoffs could be achieved if a second contract (call it contract 2) is introduced in addition to contract 1 above. This additional contract commits the players to a correlated strategy (also called a jointly randomized strategy). This contract is as follows.

- If both players sign this new contract, then a coin will be tossed. In the event of a "heads" they will be implement (x_1, y_2) and in the event of a "tails", they will implement (y_1, x_2).
- If player 1 alone signs this new contract, then player 1 chooses y_1.
- If player 2 alone signs this new contract, then player 2 chooses y_2.

1	2			
	x_2	y_2	a_2	b_2
x_1	2,2	0,6	0,6	0,6
y_1	6,0	1,1	1,1	1,1
a_1	6,0	1,1	2,2	1,1
b_1	6,0	1,1	1,1	3,3

Figure 3: Modified prisoner's dilemma with contract 1 and contract 2

If b_1 and b_2 represent the actions of players 1 and 2 corresponding to signing of this new contract, the extended payoff matrix would be as shown in Figure 3. This new game has the following equilibria:

- (y_1, y_2) with payoff $(1, 1)$.
- (a_1, a_2) with payoff (2,2)
- (b_1, b_2) with payoff (3,3)
- $((0, 0, \frac{2}{3}, \frac{1}{3}), (0, 0, \frac{2}{3}, \frac{1}{3}))$ where the mixed strategy $(0, 0, \frac{2}{3}, \frac{1}{3})$ for player 1 means a_1 with probability $\frac{2}{3}$ and b_1 with probability $\frac{1}{3}$. This equilibrium leads to a payoff of $\frac{5}{3}$ for both player 1 and player 2.

It turns out that none of the above equilibria are dominant strategy equilibria.

2 Correlated Strategies

Let $\Gamma = \langle N, (S_i), (u_i) \rangle$ be a strategic form game. A *correlated strategy* for a set of players $C \subseteq N$ is any probability distribution over the set of possible combinations of pure strategies that these players can choose. In other words, a correlated strategy, τ_C for a given coalition C belongs to $\Delta(S_C)$ where

$$S_C = \Delta(\times_{i \in C} \ S_i)$$

N is called the *grand coalition* and the symbol τ_N denotes a correlated strategy of the grand coalition.

Example: Correlated Strategies

Let $N = \{1, 2, 3\}; S_1 = \{x_1, y_1\}; S_2 = \{x_2, y_2\}; S_3 = \{x_3, y_3, z_3\}$. If $C = \{2, 3\}$, then

$$S_C = S_2 \times S_3 = \{(x_2, x_3), (x_2, y_3), (x_2, z_3), (y_2, x_3), (y_2, y_3), (y_2, z_3)\}$$

A correlated strategy for the set C is a probability distribution on S_C. For example, $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12})$ would correspond to (x_2, x_3) with probability $\frac{1}{4}$, (x_2, y_3) with probability $\frac{1}{4}$ etc. Note the difference between a correlated strategy and a mixed strategy profile. The former corresponds to $\Delta(\times(S_i))$ and the latter to $\times(\Delta(S_i))$.

A correlated strategy $\tau_C \in \Delta(\times_{i \in C} S_i)$ can be implemented as follows. A reliable mediator or a random number generator picks randomly a profile of pure strategies in S_C according to distribution τ_C. The mediator asks each player to play the strategy chosen in this pure strategy profile.

2.1 Contract Signing Game

A vector of correlated strategies of all possible coalitions is called a *contract*. More formally, consider the vector $\tau = (\tau_C)_{C \subseteq N}$. Note that

$$\tau \in \times_{C \subseteq N} \ \Delta(\times_{i \subseteq C} S_i).$$

The vector τ is a *contract*. Note that τ_C for $C \subseteq N$ gives the correlated strategy that would be implemented by players in C if C were the set of players to sign the contract. Clearly, a contract defines (in fact induces) an extended game and this extended game is called the *contract signing game*.

Example: Contract in terms of Correlated Strategies

Note that contract 1 is described by by $(\tau_1, \tau_2, \tau_{\{1,2\}})$ where $\tau_1 = (x_1 : 0, y_1 : 1)$; $\tau_2 = (x_2 : 0, y_2 : 1)$; $\tau_{\{1,2\}} = ((x_1, x_2) : 1;\ (x_1, y_2) : 0;\ (y_1, x_2) : 0;\ (y_1, y_2) : 0)$. The payoff matrix in Figure 2 defines the contract signing game induced by contract 1. Contract 2 is given by $(\tau_1, \tau_2, \tau_{\{1,2\}})$ where $\tau_1 = (x_1 : 0, y_1 : 1)$; $\tau_2 = (x_2 : 0, y_2 : 1)$; $\tau_{\{1,2\}} = ((x_1, x_2) : 0;\ (x_1, y_2) : \frac{1}{2};\ (y_1, x_2) : \frac{1}{2};\ (y_1, y_2 : 0)$. The payoff matrix in Figure 3 defines the contract signing game induced by contract 1 and contract 2.

2.2 Expected Payoff under a Correlated Strategy

Let $\alpha \in \Delta(S)$ be any correlated strategy for all players. Let $U_i(\alpha)$ denote the expected payoff to player i when α is implemented. It can be easily seen that

$$U_i(\alpha) = \sum_{s \in S} \alpha(s) u_i(s)$$

Let $U(\alpha) = (U_1(\alpha), \ldots, U_n(\alpha))$ denote the *expected payoff allocation* to players when they implement α.

Given any allocation in the set $\{U(\alpha) : \alpha \in \Delta(S)\}$, there exists a contract such that if the players all signed this contract, then they would get this expected payoff allocation. The set of possible expected payoff allocations $\{U(\alpha) : \alpha \in \Delta(S)\}$ can be shown to be a *closed and convex* subset of $\mathbb{R}^n$.

2.3 Concept of Individual Rationality

Not all contracts would be signed by everyone in an equilibrium of the implicit contract signing game. For example, in the modified prisoner's dilemma game of Figure 1, player 1 will not agree to sign a contract that would commit the players to implement (x_1, y_2) since it gives a him a payoff 0. Player 1 can always guarantee himself a payoff of 1 by signing nothing and simply choosing y_1. Which contracts would a player be interested in signing at all? This leads to the concept of a security level for players.

For any player i, his *security level* is the player's minimax value:

$$v_i = \min_{\tau_{N-i} \in \Delta(S_{N-i})} \max_{s_i \in S_i} U_i(s_i, \tau_{N-i})$$

where

$$U_i(s_i, \tau_{N-i}) = \sum_{s_{N-i} \in S_{N-i}} \tau_{N-i}(s_{N-i}) u_i(s_i, s_{N-i}).$$

$$\begin{aligned} S_{N-i} &= S_{N\setminus\{i\}} \\ &= \times_{j\in N\setminus\{i\}} \ S_i \\ s_{N-i} &\in S_{N-i} \\ \tau_{N-i} &= \tau_{N\setminus\{i\}} \in \Delta(S_{N-i}) \end{aligned}$$

v_i is the minimum expected payoff that player i is guaranteed to get against any correlated strategy that the other players could use against player i. A minimax strategy against player i is any correlated strategy τ_{N-i} in $\Delta(S_{N-i})$ such that the payoff of player i is the minimax value v_i. For example, in the modified prisoner's dilemma problem (Figure 1), $v_1 = v_2 = 1$.

By the theory of two player zero-sum games, we know that the minimax value v_i also satisfies

$$v_i = \max_{\tau_i \in \Delta(S_i)} \ \min_{s_{N-i}\in S_{N-i}} U_i(\tau_i, s_{N-i})$$

where

$$U_i(\tau_i, s_{N-i}) = \sum_{s_i\in S_i} \tau_i(s_i) u_i(s_i, s_{N-i})$$

Thus player i has a randomized strategy that achieves the above maximum and gets him an expected payoff that is no less than his minimax value v_i, regardless of what other players do.

2.4 Individually Rational Correlated Strategy

It is reasonable for player i to sign a contract to play a correlated strategy α only if

$$U_i(\alpha) \geq v_i$$

The above is called the *individual rationality* or *participation* constraint for player i. A correlated strategy

$$\alpha \in \Delta(\times_{i\in N} S_i)$$

for all the players in N is said to be *individually rational* if

$$U_i(\alpha) \geq v_i \ \ \forall i \in N$$

2.5 Equilibria of the Contract Signing Game

Suppose the players make their decisions about which contract to sign independently. Then, given any individually rational correlated strategy α, there exists a contract τ with $\tau_N = \alpha$ such that all players signing this contract is an equilibrium of the implicit contract signing game. We show this as follows.

Let $\alpha \in \Delta(S)$ be an individually rational correlated strategy. That is

$$U_i(\alpha) \geq v_i \ \ \forall i \in N$$

Consider the contract

$$\tau = (\tau_C)_{C\subseteq N}$$

such that $\tau_N = \alpha$ and τ_{N-i} is a minimax strategy against player i. τ_C for all other coalitions (that is, coalitions other than $N, N-1, \ldots, N-n$ could be chosen arbitrarily. Let $(a_1, \ldots, a_n)$ be the profile of contract signing strategies for this contract. Note that the profile $(a_1, \ldots, a_n)$ corresponds to the situation when all the players sign the contract. Now,

$$U_i(a_1, \ldots, a_n) = U_i(\alpha) \text{ since } \tau_N = \alpha$$

Therefore, we get for all $i \in N$,

$$U_i(a_1, \ldots, a_n) \geq v_i$$

For any $s_i \in S_i$ such that $s_i \neq a_i$, note that the profile (s_i, a_{-i}) corresponds to the situation when player i plays s_i and the rest of the players sign the contract. We have,

$$u_i(s_i, a_{-i}) = U_i(s_i, \tau_{N-i})$$

$$\leq v_i$$

since τ_{N-i} is a minimax strategy against player i. Therefore,

$$u_i(a_i, a_{-i}) \geq u_i(s_i, a_{-i}) \ \ \forall s_i \in S_i \ \ \forall i \in N$$

Thus $(a_1, a_2, \ldots, a_n)$ is a Nash equilibrium of the contract signing game.

Thus any individually rational correlated strategy $\alpha \in \Delta(S)$ will lead to a contract such that all players signing the contract is a Nash equilibrium of the contract signing game.

Now consider any Nash equilibrium (s_i^*, s_{-i}^*) of the contract signing game induced by an individually rational correlated strategy α. It can be shown that in this Nash equilibrium, the payoff for player i would be $\geq v_i$. To prove this, assume to the contrary. That is, the payoff for player i in this equilibrium is $< v_i$. Now player i can decide not to sign the contract and instead play the strategy that guarantees him the minimax value v_i. This provides the contradiction. Thus

$$\{U(\alpha) : \alpha \in \Delta(S) \text{ and } \ U_i(\alpha) \geq v_i \ \forall i \in N\}$$

is exactly the set of payoff allocations that can be achieved in equilibria of the contract signing game when every player has the option to sign nothing and choose an action in S_i. This set is also the set of expected payoff allocations corresponding to individually rational correlated strategies. This set is also *closed and convex*.

Example: Payoff Allocations in the Modified Prisoner's Dilemma

Consider the modified prisoner's dilemma game with payoff matrix as in Figure 1. The expected payoff in a correlated strategy is a convex combination of the payoffs in different strategy profiles and hence the set of all payoff allocations for the above game is the convex set with extreme points (0,6), (6,0), and (1,1). See Figure 4. Note that this set is also closed and that the point (2,2) is in the interior of this convex set. The set of possible expected payoff allocations satisfying individual rationality is the triangle with corners at (1,1), (5,1), and (1,5) as shown. This is because $v_1 = 1$ and $v_2 = 1$ for this example. Note that this set is also closed and convex.

3 Games with Communication

We have so far seen how contracts can transform a game with less desirable equilibria to a game with more desirable equilibria. However, in many situations, players may not be able to commit themselves to binding contracts. The reasons for this could be many [1]:

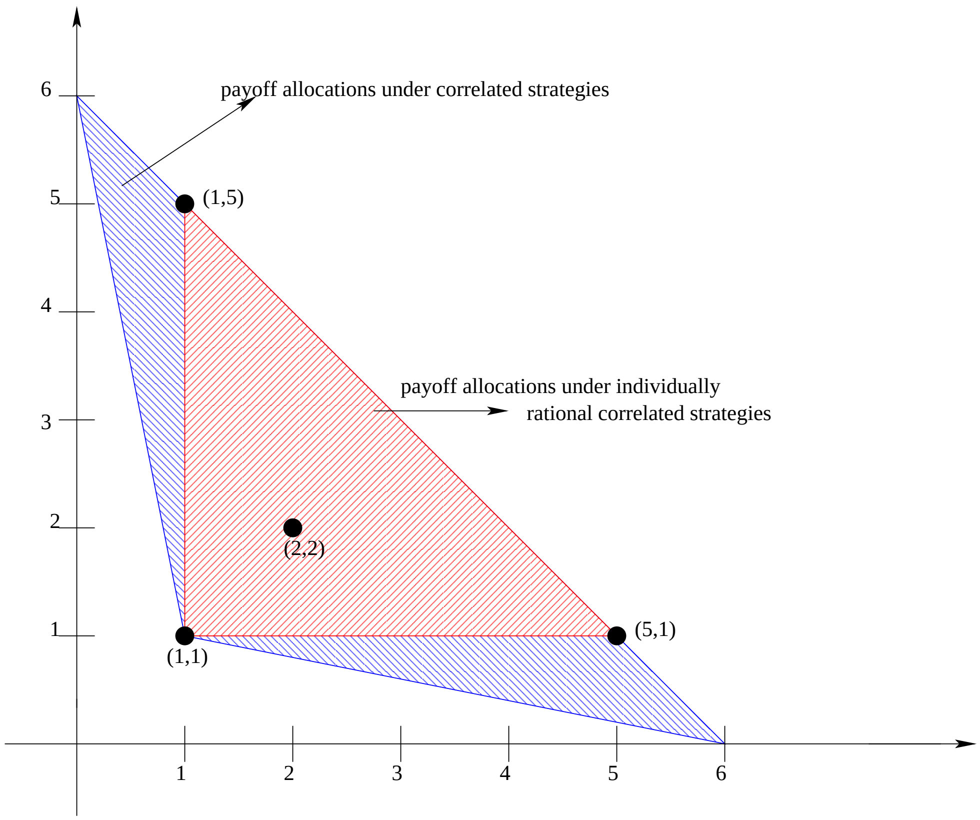

Figure 4: Payoff allocation vectors under correlated strategies for the modified prisoner's dilemma problem

- Player's strategies may not be observable to the enforcers of contracts.
- There may not be adequate sanctions to guarantee compliance with contracts.
- Player's strategies might actually involve inalienable rights.

In these above situations also, it may still be possible for the players to communicate and coordinate and achieve a self-enforcing equilibrium with desirable payoff structures. Such games correspond to *games with communication*. A game with communication is one which, in addition to the strategy options explicitly specified, the players have a range of implicit options to communicate with each other. A game with communication need not have any contracts. Such a game may still achieve interesting results in spite of contracts being absent.

Example: A Game with Communication

Consider a two player game having the payoff matrix shown in Figure 5. Clearly, the above game has three Nash equilibria.

- (x_1, x_2) with payoff allocation (5,1)
- (y_1, y_2) with allocation (1,5)

	2	
1	x_2	y_2
x_1	5,1	0,0
y_1	4,4	1,5

Figure 5: A two player game to illustrate games with communication

- Mixed strategy Nash equilibrium $((\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}))$ which yields the outcome (2.5, 2.5).

Note that (y_1, x_2) is not a Nash equilibrium though it is a very desirable outcome. It can be realized through a binding contract. Assume that contracts cannot be used. We can ask the question whether we can achieve the outcome (2.5, 2.5) without contracts. It turns out that correlated strategies exist that achieve an even better allocation than (2.5, 2.5).

3.1 Correlated Strategy 1

Let us say players choose to toss a coin and select the outcome (x_1, x_2) with probability $\frac{1}{2}$ and the outcome (y_1, y_2) with probability $\frac{1}{2}$. This refers to the following correlated strategy

$$\alpha = ((x_1, x_2); \frac{1}{2}; (x_1, y_2) : 0; (y_1, x_2) : 0; (y_1, y_2) : \frac{1}{2})$$

Note that $U_1(\alpha) = U_2(\alpha) = 3$. To implement this correlated strategy, players can toss a coin and choose the outcome (x_1, x_2) in the event of a *heads* and choose the outcome (y_1, y_2) in the event of a *tails*. The above correlated strategy is implemented without a binding contract since tossing a coin does not refer to any binding force on the players. However, communication and coordination are indeed required. The outcome suggested by this correlated strategy is *self-enforcing* since neither player will gain by unilaterally deviating from this.

The above correlated strategy can also be implemented with the help of a trusted mediator who who helps the players to communicate and share information: the mediator recommends, randomly, with probability 0.5 each the profiles (x_1, x_2) and (y_1, y_2). Assume that each player learns only the strategy recommended to that player by the mediator.

- Player 1, if recommended x_1 by the mediator thinks that player 2 is recommended x_2. Believing that player 2 obeys the recommendation x_2, player 1 finds it a best response to choose x_1 and thus accepts the mediator's recommendation.
- Player 1, if recommended y_1 by the mediator thinks that player 2 is recommended y_2. Believing that player 2 obeys the recommendation y_2, player 1 finds it a best response to choose y_1 and again accepts the mediator's recommendation.
- Player 2, if recommended x_2 by the mediator thinks that player 1 is recommended x_1. Believing that player 1 obeys the recommendation x_1, player 2 finds it a best response to choose x_2 and thus accepts the mediator's recommendation.
- Player 2, if recommended y_2 by the mediator thinks that player 1 is recommended y_1. Believing that player 1 obeys the recommendation y_1, player 2 finds it a best response to choose y_2 and again accepts the mediator's recommendation.

Thus mediation can also be used to implement the above correlated strategy.

3.2 Correlated Strategy 2

We now explore a different correlated strategy that can be realized with the help of a mediator. Consider the correlated strategy

$$\alpha = ((x_1, x_2); \frac{1}{3}; (x_1, y_2) : 0; (y_1, x_2) : \frac{1}{3}; (y_1, y_2) : \frac{1}{3})$$

Note that $U_1(\alpha) = U_2(\alpha) = \frac{10}{3}$. To implement this correlated strategy, the mediator recommends, randomly, with probability $\frac{1}{3}$, each of the profiles $(x_1, x_2), (y_1, y_2), (y_1, x_2)$. Again assume that each player learns only the strategy recommended to that player by the mediator.

- Suppose the mediator recommends x_1 to player 1. Then player 1 knows that player 2 is recommended x_2. When player 2 plays x_2 (as recommended by the mediator), it is a best response for player 1 to play x_1, so he would be happy to play x_1 and thus accept the recommendation of the mediator.

- Suppose the mediator recommends y_1 to player 1. Then player 1 knows that the mediator would recommend the mixed strategy $x_2 : 0.5; y_2 : 0.5$ to player 2. When player 2 plays the above mixed strategy, then player 1 gets a payoff of 2.5 if he plays x_1 and gets a payoff of 2.5 even if he plays y_1. Thus player 1 can be indifferent between x_1 and y_1 and will not mind accepting the recommendation of the mediator to play y_1.

The above shows that player 1 would be happy to listen to the mediator if player 1 expected player 2 also to listen to the mediator.

Similarly, it can be shown that player 2 would be happy to do as recommended by the mediator under the belief that player 1 would obey the mediator.

The above shows that the two players can reach a self-enforcing understanding to obey the mediator if the mediator recommends the correlated strategy

$$((x_1, x_2) : \frac{1}{3}; (x_1, y_2) : 0; (y_1, x_1) : \frac{1}{3}, (y_1, x_2) : \frac{1}{3}))$$

In other words, even though the mediator's recommendation is not binding on the two players, the two players find it in their best interest to follow this. Thus there is a Nash equilibrium of the transformed game with mediated communication without contracts. This is the idea behind the notion of *Correlated Equilibrium* which is discussed next.

4 Correlated Equilibrium

Consider the following setup of a game with communication. Let $\Gamma = \langle N, S_i), (u_i) \rangle$ be any finite strategic form game. Assume that there is a mediator who recommends a particular strategy to each player. Based on the recommendation, the player either obeys it or chooses any other strategy from his strategy set S_i. Let $\delta_i : S_i \to S_i$ describe player i's choice of a strategy based on the mediator's recommendation. That is, $\delta_i(s_i)$ gives the strategy that the player i chooses to play when the mediator recommends s_i to him. $\delta_i(s_i) = s_i$ means that the player i obeys the mediator when the mediator recommends s_i. Let $\alpha \in \Delta(S)$ be the correlated strategy recommended by the mediator. Assume that α is common knowledge. The correlated strategy α would induce an equilibrium for all players to obey the mediator's recommendation if and only if

$$\sum_{s \in S} \alpha(s_i, s_{-i}) u_i(s_i, s_{-i}) = U_i(\alpha) \geq \sum_{s \in S} \alpha(s_i, s_{-i}) u_i(\delta_i(s_i), s_{-i}) \ \ \forall \delta_i : S_i \to S_i \qquad \forall i \in N \ldots (1)$$

Such a correlated strategy α of players is called a *correlated equilibrium*.

4.1 Computing Correlated Equilibria

Note that a correlated equilibrium is any correlated strategy for the players which could be self-enforcingly implemented with the help of a mediator who can make non-binding recommendations to each player. It can be shown that the inequalities (1) are equivalent to the following set of inequalities.

$$\sum_{s_{-i} \in S_{-i}} \alpha(s)[u_i(s_i, s_{-i}) - u_i(s_i', s_{-i})] \geq 0 \quad \forall s_i \in S_i \ \forall s_i' \in S_i \quad \forall i \in N \ \ldots (2)$$

The equivalence can be shown by fixing s_i and unfolding the original inequalities. Equation (2) asserts that no player i could expect to increase his expected payoff by using some disobedient action s_i' when the mediator recommends s_i. The constraints (1) or equivalently (2) are called *strategic incentive constraints*. They are the constraints to be satisfied by a mediator's correlated strategy for ensuring that all players could rationally obey the recommendations. It can also be noted that

$$\begin{aligned} \alpha(s) &\geq 0 \quad \forall s \in S \\ \sum_{s \in S} \alpha(s) &= 1 \end{aligned}$$

It can be shown that the set of all correlated equilibria in a finite game is a *compact and convex* set. This is a very rich structure.

Let us look at the following linear program:

$$\max \sum_{i \in N} U_i(\alpha)$$

subject to

$$\begin{aligned} \sum_{s_{-i} \in S_{-i}} \alpha(s) \left[u_i(s_i, s_{-i}) - u_i(s_i', s_{-i})\right] &\geq 0 \qquad \forall i \in N \ \forall s_i \in S_i \ \forall s_i' \in S_i \\ \alpha(s) &\geq 0 \quad \ \forall s \in S \\ \sum_{s \in S} \alpha(s) &= 1 \end{aligned}$$

Any feasible solution of this linear program will give a correlated equilibrium. An optimal solution of this linear program will give a correlated equilibrium that maximizes the social welfare.

Example: Computation of Correlated Equilibria

Consider the game shown in Figure 5. The linear program here is: maximize

$$6 \ \alpha(x_1, x_2) + 0 \ \alpha(x_1, y_2) + 8 \ \alpha(y_1, x_2) + 6 \ \alpha(y_1, y_2)$$

subject to

$$
\begin{aligned}
(5-4)\ \alpha\ (x_1,x_2)+(0-1)\ \alpha\ (x_1,y_2) &\geq 0\\
(4-5)\ \alpha\ (y_1,x_2)+(1-0)\ \alpha\ (y_1,y_2) &\geq 0\\
(1-0)\ \alpha\ (x_1,x_2)+(4-5)\ \alpha\ (y_1,x_2) &\geq 0\\
(0-1)\ \alpha\ (x_1,y_2)+(5-4)\ \alpha\ (y_1,y_2) &\geq 0\\
\alpha\ (x_1,x_2)+\ \alpha\ (x_1,y_2)+\alpha\ (y_1,x_2)+\alpha\ (y_1,y_2) &= 1\\
\alpha\ (x_1,x_2)\geq 0,\ \alpha\ (x_1,y_2) &\geq 0;\\
\alpha\ (y_1,x_2)\geq 0,\ \alpha\ (y_1,y_2) &\geq 0
\end{aligned}
$$

This yields the solution

$$\alpha\ (x_1,x_2)=\alpha\ (y_1,x_2)=\alpha\ (y_1,y_2)=\frac{1}{3};\ \ \alpha(x_1,y_2)=0$$

This shows that the above correlated strategy yields the maximum total payoff to the two players. The above also shows that the sum of expected payoffs cannot exceed $6\frac{2}{3}$ under non-binding mediated communication scenario. Figure 6 shows the sets of all payoff allocations for this example under correlated strategies, under individually rational correlated strategies, and under correlated equilibria.

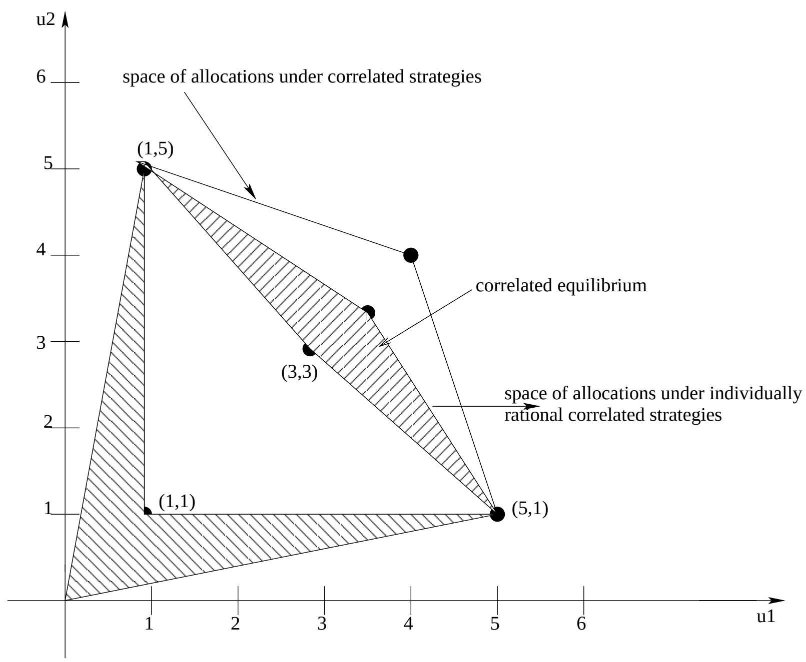

Figure 6: Payoff allocation vectors under correlated strategies and correlated equilibria

5 Conclusion

In this chapter, we have discussed three different sets of payoff allocations. (1) Set of all payoff allocations under correlated strategies (2) Set of all payoff allocations under individually rational correlated strategies which is the same as the set of allocations under Nash equilibria of the contract signing games induced by individually rational strategies (3) Set of all payoff allocations under correlated equilibria. All the three sets are convex and closed. We can now ask the following question: Can we select a small number of (perhaps a single) desirable or best outcomes among these. For two player games, this issue was settled by the Nash bargaining theorem which we discuss in the next chapter. For multiplayer games (including two player games), a variety of solution concepts have been suggested: The Core, Shapley Value, Bargaining Sets, Nucleolus, Kernel, etc. We will be studying those in the other following chapters.

To Probe Further

The material discussed in this chapter draws upon mainly from the the book by Myerson [1]. The concept of correlated equilibrium was first introduced by Robert Aumann [2].

Problems

1. Given a strategic form game, show that the following sets are closed and convex.
   - The space of all utility vectors achievable under correlated strategies
   - The space of all utility vectors achievable under individually rational correlated strategies
   - The space of all utility vectors achievable under correlated equilibria
2. Compute all correlated equilibria of

	2	
1	x_2	y_2
x_1	2, 2	0, 6
y_1	6, 0	1, 1

	2	
1	x_2	y_2
x_1	2, 2	0, 0
y_1	0, 0	1, 1

3. Show the equivalence of the two sets of inequalities presented in the definition of correlated equilibrium.
4. Consider the following two player game:

	A	B
A	5, 5	1,1
B	1,1	3,3

   For this game,
   - compute the set of all payoff utility pairs possible (a) under correlated strategies (b) under individually rational correlated strategies.
   - What would be the Nash bargaining solution in case (a) and case (b) assuming the minmax values as the disagreement point.
   - Also compute all correlated equilibria that maximize the sum of utilities of the two players.

References

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