Kakutani's Theorem

<u>Therem</u> (Kakutani 1941). Let $M \subset \mathbb{R}^n$ be a compact convex set, Let $F: M \to M$ be an uppel hemi-continuous convex valued correspondence. Then $\exists$ some $x^* \in M$, s.t. $x^* \in F(x^*)$.

<u>Proof</u>. First prove for the case that M is nondegenerate simplex in $\mathbb{R}^n$. then generalize to the case of compact convex subsets of $\mathbb{R}^n$.

Let M be the simplex defined by the $n+1$ points $v_0, v_1, \ldots, v_n$. That is $M = \langle v_0, v_1, \ldots, v_n \rangle$.

Now form the mth barycentric subdivision of M.

We define the continuous function $f^m: M \to M$ as follows: If x is the vertex of any cell of the subdivision let $f^m(x) = y$ for some $y \in F(x)$. For any other x we define $f^m(x)$ cell $\langle x_0, x_1, \ldots, x_n \rangle$ and $x = \sum_{j=0}^{n} \theta_j x_j$ with $\sum_{j=0}^{n} \theta_j = 1$, $\theta_j \geq 0$ then $f^m(x)$ is defined to be $\sum_{j=0}^{n} \theta_j f^m(x_j)$.

Now we apply ~~the~~ Brouwer's theorem to obtain a fixed point of the map f^m, say x^m. If x^m is a vertex of one of the cells in the subdivision then we are done since $x^m = f^m(x^m) \in F(x^m)$. If x^m is not a vertex of one of the cells then let the cell in which it does lie be $\langle x_0^m, x_1^m, \ldots, x_n^m \rangle$, and let $\theta_0^m, \theta_1^m, \ldots$

θ_n^m be the barycentric coordinates of x^m relative to that cell. Thus

$$x^m = \sum_{j=0}^{m} \theta_j^m x_j^m .$$

and

$$(1) \quad x^m = f^m(x^m) = \sum_{j=0}^{n} \theta_j^m y_j^m .$$

where $y_j^m = f^m(x_j^m) \in F(x_j^m)$ for all $j = 0, 1, \ldots, n$.

Now we choose a ~~s~~ subsequence of $m \to \infty$ say $m_k \to \infty$ such that $x^{m_k} \to x^*$, $\theta_j^{m_k} \to \theta_j$ and $y_j^{m_k} \to y_j$. Also, since the cells shrink to points as $m_k \to \infty$ each of the vertices of the cell containing x^{m_k} converges to x^*, i.e., $x_j^{m_k} \to x^*$. Thus from (1)

$$x^* = \sum_{j=0}^{n} \theta_j y_j .$$

Also since F is upper hemi-continuous $y_j \in F(x^*)$. Since $F(x^*)$ is convex and x^* is a convex combination of the y_j's this implies that $x^* \in F(x^*)$ as required.

Now ~~o~~ what happens if M not a simplex? We take some simplex M' containing M and a ~~retractio~~ retraction $\psi: M' \to M$ (ie., a continuous fn taking M' to M that leaves all points of M fixed.) Then $F': M' \to M'$ defined by $F'(x) = F(\psi(x))$ is clearly an upper hemi-continuous correspondence and a fixed point of F' clearly lies in M and so is also a fixed point of F.

This proof is essentially the one originally given by Kakutani.