Proof for MID ≤ 1 in Star-Network Setting

(Has to be read after section III of the paper "On properties of Core Selecting Combinatorial Auctions," by Vineet Abhishek and Bruce Hajek.)

We need to solve the following set of linear equations:

$$\lambda_{j,\theta} \geq 0; \ \sum_{i=1}^{J} \lambda_{i,\theta} + \eta_j \geq \sum_{k=1}^{n_j} \left[\min(\eta_j+\theta, \Delta_{j,k}) - \lambda_{j,\theta} \right]; \quad j = 1, 2, \dots, J,$$

with equality if $\lambda_{j,\theta} > 0$.

We prove that the following set of steps help us find the values of $\lambda_{j,\theta}$:

(i) Solve the set of simultaneous equations with equality.
(ii) In case of negative solutions for any $\lambda_{j,\theta}$, force those $\lambda_{j,\theta}$'s to zero, and solve the remaining set of equations with equality.
(iii) Continue until a non-negative solution is obtained.

Proof: The inequality in the equations can be rewritten as,

$$(1+n_j)\lambda_{j,\theta} + \sum_{\substack{i=1 \\ i\neq j}}^{J} \lambda_{i,\theta} \geq \left\{ \sum_{k=1}^{n_j} \left[\min(\eta_j+\theta, \Delta_{j,k}) \right] - \eta_j \right\} := A_j.$$

With equalities, they become

$$\begin{bmatrix} 1+n_1 & 1 & \cdots & 1 \\ 1 & 1+n_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1+n_J \end{bmatrix} \begin{bmatrix} \lambda_{1,\theta} \\ \lambda_{2,\theta} \\ \vdots \\ \lambda_{J,\theta} \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_J \end{bmatrix}.$$

Call the $J \times J$ matrix to be A. We need to find the inverse of A to solve $\lambda_{j,\theta}$. We first find the determinant of A.

$$|A| = (1+n_1)(\text{Cofactor of } A_{11}) - (\text{Cofactor of } A_{12}) + \dots + (-1)^{J+1}(\text{Cofactor of } A_{1J}).$$

$$\text{Cofactor of } A_{11} = \begin{vmatrix} 1+n_2 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1+n_J \end{vmatrix}$$

We will prove by induction that $|A| = \left(\prod_{i=1}^{J} n_i\right)\left(1 + \sum_{i=1}^{J} \frac{1}{n_i}\right)$.

For $J=1$, $|1+n_1| = 1+n_1 = n_1\left(1+\frac{1}{n_1}\right)$.

For $J=2$, $\begin{vmatrix} 1+n_1 & 1 \\ 1 & 1+n_2 \end{vmatrix} = 1+n_1+n_2+n_1n_2-1 = n_1+n_2+n_1n_2 = n_1n_2\left(1+\frac{1}{n_1}+\frac{1}{n_2}\right)$

Assume this to be true for $J=k$. Then, for $J=k+1$, we have,

$$|A| = (1+n_1)(\text{cofactor of } A_{11}) - (\text{cofactor of } A_{12}) + \dots + (-1)^k (\text{cofactor of } A_{1(k+1)})$$

$$= (1+n_1)\left(\prod_{i=2}^{k+1} n_i\right)\left(1+\sum_{i=2}^{k+1}\frac{1}{n_i}\right) - (\text{cofactor of } A_{12}) + \dots + (-1)^k(\text{cofactor of } A_{1(k+1)})$$

To find cofactor of A_{1j}, let us do the following:

$$\text{cofactor of } A_{1j} = \begin{vmatrix} 1 & 1 & \cdots & 1 & \cdots & 1 \\ 1+n_2 & 1 & \cdots & 1 & \cdots & 1 \\ 1 & 1+n_3 & \cdots & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 & \cdots & 1+n_{k+1} \end{vmatrix}$$

{∵ Observe that 1^{st} row and $(j-1)^{th}$ column is full of 1's.}

$$= \begin{vmatrix} 1 & 1 & \cdots & 1 & \cdots & 1 \\ n_2 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & n_3 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & n_{k+1} \end{vmatrix}$$

$\left\{ ∵ R_2 \to R_2 - R_1, \ \vdots, \ R_k \to R_k - R_1 \right\}$

$$= \left(\prod_{i=2}^{k+1} n_i\right) \cdot \frac{(-1)^j}{n_j}$$

The last step comes because the cofactor of $1,(j-1)^{th}$ element is non-zero and all the other cofactors turn out to be zero.

$$\therefore |A| = \left(\prod_{i=1}^{k+1} n_i\right)\left(1+\sum_{i=2}^{k+1}\frac{1}{n_i}\right) + \prod_{i=2}^{k+1} n_i$$ {∵ All the other terms get cancelled}

$$= \left(\prod_{i=1}^{k+1} n_i\right)\left(1+\sum_{i=1}^{k+1}\frac{1}{n_i}\right)$$

Thus by induction, $|A| = \left(\prod_{i=1}^{J} n_i\right)\left(1+\sum_{i=1}^{J}\frac{1}{n_i}\right)$.

To find the value of A^{-1}, we can find cofactors of each element by the same method mentioned above. The elements of A^{-1} are given by,

$$(A^{-1})_{ij} = \frac{-1}{|A|} \cdot \frac{\left(\prod_{k=1}^{J} n_k\right)}{n_i n_j}, \quad \forall\, i \neq j,$$

$$= \frac{1}{|A|} \frac{\left(\prod_{k=1}^{J} n_k\right)}{n_i} \left(1 + \sum_{\substack{k=1 \\ k \neq i}}^{J} \frac{1}{n_k}\right), \quad \forall\, i = j.$$

Thus $\lambda_{j,\theta}$ values are given by,

$$\boxed{\lambda_{j,\theta} = \frac{A_j}{|A|} \left(\frac{\prod_{i=1}^{J} n_i}{n_j}\right) + \sum_{\substack{k=1 \\ k \neq j}}^{J} \frac{(A_j - A_k)}{|A|} \left(\frac{\prod_{i=1}^{J} n_i}{n_j n_k}\right).}$$

If all $\lambda_{j,\theta} \geq 0$, then the system of equations is satisfied by these values of $\lambda_{j,\theta}$ and the proof is complete. So let us consider some $\lambda_{j,\theta} < 0$.

Without Loss of Generality, let $A_1 \leq A_2 \leq \dots \leq A_J$.

~~This means that if $\lambda_{j,\theta} < 0$, then so are $\lambda_{j-1,\theta}, \lambda_{j-2,\theta}, \dots, \lambda_{1,\theta}$.~~
~~(Note that if~~

We have
$$\lambda_{j,\theta} = \frac{(A_j - A_{j-1})}{|A|} \left(\frac{\prod_{i=1}^{J} n_i}{n_j n_{j-1}}\right) + \frac{n_{j-1}}{|A|} \left[A_j \left(\frac{\prod_{i=1}^{J} n_i}{n_j n_{j-1}}\right) + \sum_{\substack{k=1 \\ k \neq j, j-1}}^{J} (A_j - A_k) \left(\frac{\prod_{i=1}^{J} n_i}{n_j n_{j-1} n_k}\right)\right]$$
and
$$\lambda_{j-1,\theta} = \frac{(A_{j-1} - A_j)}{|A|} \left(\frac{\prod_{i=1}^{J} n_i}{n_j n_{j-1}}\right) + \frac{n_j}{|A|} \left[A_j \left(\frac{\prod_{i=1}^{J} n_i}{n_j n_{j-1}}\right) + \sum_{\substack{k=1 \\ k \neq j, j-1}}^{J} (A_j - A_k) \left(\frac{\prod_{i=1}^{J} n_i}{n_j n_{j-1} n_k}\right)\right]$$

If $\lambda_{j,\theta} < 0$, then so is $\lambda_{j-1,\theta}$. This is clear as $A_j \geq A_{j-1}$. But this also means that if $\lambda_{j-1,\theta} \leq 0$, then $\lambda_{j-2,\theta} < 0$, and so on. Thus

$$\lambda_{j,\theta} < 0 \Rightarrow \lambda_{k,\theta} < 0 \quad \forall\, k = 1, 2, \dots, j-1.$$

Thus let us say that $\lambda_{1,\theta}, \lambda_{2,\theta}, \dots, \lambda_{K_1,\theta}$ turned out to be negative and were forced to zero in the first step. We solve the remaining set of equations, force the negative solutions to zero, until finally ending up having a non-zero solution vector. Let $\lambda_{1,\theta}, \lambda_{2,\theta}, \dots, \lambda_{K,\theta}$ be forced to zero by then. It is clear that the last $(J-K)$ equations are satisfied with equality. We need to prove that inequality holds in the first K equations, (ie.,) to prove,

$$\sum_{j=K+1}^{J} \lambda_{j,\theta} \geq A_K.$$

It is enough to prove this, since $A_K \geq A_{K-1} \geq \dots \geq A_1$.

We know that
$$\lambda_{j,\theta} = \frac{A_j}{|A_k|}\left(\frac{\prod_{i=k+1}^{J} n_i}{n_j}\right) + \sum_{\substack{k=k+1 \\ k \neq j}}^{J} \frac{(A_j - A_k)}{|A_k|}\left(\frac{\prod_{i=k+1}^{J} n_i}{n_j\, n_k}\right).$$

where $|A_k| = \left(\prod_{i=k+1}^{J} n_i\right)\left(1 + \sum_{i=k+1}^{J} \frac{1}{n_i}\right)$.

$$\boxed{\therefore \sum_{j=k+1}^{J} \lambda_{j,\theta} = \frac{\sum_{j=k+1}^{J} A_j \left(\frac{\prod_{i=k+1}^{J} n_i}{n_j}\right)}{|A_k|} \longrightarrow (1)}$$

$\therefore$ We need to prove
$$\frac{\sum_{j=k+1}^{J} A_j \left(\frac{\prod_{i=k+1}^{J} n_i}{n_j}\right)}{|A_k|} \geq A_K$$

That is to prove, $$0 \geq A_K \left(\prod_{i=k+1}^{J} n_i\right) + \sum_{k=k+1}^{J} (A_K - A_k)\left(\frac{\prod_{i=k+1}^{J} n_i}{n_k}\right)$$

That is to prove, $0 \geq \lambda_{K,\theta}$.

But $\lambda_{K,\theta} < 0$ is what we already know.

Hence the result. ∎

It still remains to prove that MID ≤ 1, (ie.,) to prove
$$\frac{\partial P_0}{\partial \theta} \leq 1.$$

But $P_0 = \min\left(v_0 + \sum_{j=1}^{J} \lambda_{j,\theta},\ b_0\right)$

$$= v_0 + \min\left(\theta, \sum_{j=1}^{J} \lambda_{j,\theta}\right).$$

So we need to prove $\frac{\partial}{\partial \theta}\left[\sum_{j=1}^{J} \lambda_{j,\theta}\right] \leq 1$, when $\left(\sum_{j=1}^{J} \lambda_{j,\theta}\right) \leq \theta$.

From (1), $$\sum_{j=1}^{J} \lambda_{j,\theta} = \frac{\sum_{j=k+1}^{J} A_j \left(\frac{\prod_{i=k+1}^{J} n_i}{n_j}\right)}{|A_k|} \quad \left\{\because \lambda_{1,\theta} = \lambda_{2,\theta} = \dots = \lambda_{k,\theta} = 0\right\}$$

But $A_j = \sum_{k=1}^{n_j} \left[\min(\eta_j + \theta, \Delta_{j,k})\right] - \eta_j = c_j + \eta_j(a_j - 1) + \theta a_j$,

where $a_j = \#\,\Delta_{j,k}$'s that are at most $(\eta_j + \theta)$. Observe that $0 \leq a_j \leq n_j$.

Substituting the values of A_j's, we get,

$$\sum_{j=k+1}^{J} \lambda_{j,\theta} = \frac{\sum_j [c_j + \eta_j (a_j - 1)] \left(\frac{\prod_{i=k+1}^{J} n_i}{n_j} \right) + \theta \frac{a_j}{n_j} \left(\prod_{i=k+1}^{J} n_i \right)}{|A_k|}$$

The term $\left(\sum_{j=k+1}^{J} \lambda_{j,\theta} \right)$ is of the form $a + b\theta$. If $\theta \geq \sum_{j=k+1}^{J} \lambda_{j,\theta}$, it implies that $\theta \geq a + b\theta$. If $b > 1$, then $a < 0$. We are done if we prove that $\sum_{j=k+1}^{J} [c_j + \eta_j (a_j - 1)] \geq 0$.

The reverse can occur only if at least one of the a_j's is zero. But if $a_j = 0$, ~~then the RHS of~~ and $c_j < \eta_j$ (so that $c_j + \eta_j \cdot (a_j - 1) < 0$), we have j^{th} equation in the system to be,

$$(1 + n_j) \lambda_{j,\theta} + \sum_{\substack{i=k+1 \\ i \neq j}}^{J} \lambda_{i,\theta} \geq c_j - \eta_j .$$

The RHS of the equation is negative, (i.e.,) $A_j < 0$. Then we have,

$$\lambda_{j,\theta} = \frac{A_j}{|A_k|} \left(\frac{\prod_{i=k+1}^{J} n_i}{n_j} \right) + \sum_{\substack{k=k+1 \\ k \neq j}}^{J} \frac{(A_j - A_k)}{|A_k|} \left(\frac{\prod_{i=k+1}^{J} n_i}{n_j n_k} \right)$$

Each of these terms is negative if $j = k+1$. If not, then $A_{k+1} \leq A_j < 0$, because of which $\lambda_{(k+1),\theta} < 0$. A contradiction!

Thus $\sum_{j=k+1}^{J} [c_j + \eta_j (a_j - 1)] \geq 0$, and thus $MID \leq 1$.

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