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An Optimal Mechanism for Sponsored Search Auctions and Comparison with Other Mechanisms

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In this paper, we first describe a framework to model the sponsored search auction on the web as a mechanism design problem. Using this framework, we describe two well known mechanisms for sponsored search auction - *Generalized Second Price (GSP)* and *Vickrey-Clarke-Groves (VCG)*. We then derive a new mechanism for sponsored search auction which we call *Optimal (OPT)* mechanism. The OPT mechanism maximizes the search engine's expected revenue while achieving Bayesian incentive compatibility and individual rationality of the advertisers. We then undertake a detailed comparative study of the mechanisms GSP, VCG, and OPT. Our investigation shows that the expected revenue earned by the search engine is the same for all the three mechanisms provided the advertisers are symmetric and the number of sponsored slots is strictly less than the number of advertisers; this is a consequence of a generalization of the classical revenue equivalence theorem that we derive. We also derive exact expressions for the expected revenue of the three mechanisms under more general conditions. We also compare the three mechanisms in terms of incentive compatibility, individual rationality, and computational complexity.

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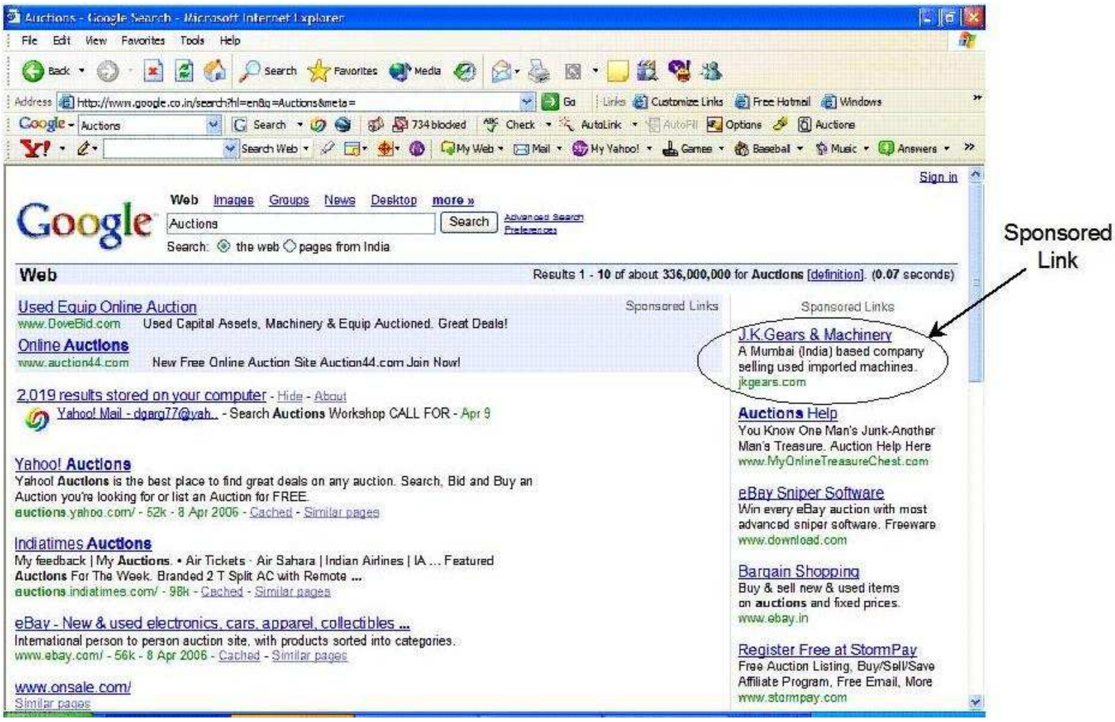
1. Introduction

The rapid growth of the Internet and the World Wide Web is transforming the way information is being accessed and used. Newer and innovative models for distributing, sharing, linking, and marketing the information are appearing. As with any major medium, a major way of financially supporting this growth has been advertising (popularly known as *Internet Advertising* or *Web Advertising*). The advertisers-supported web site is one of the successful business models in the emerging web landscape. The rise of Internet advertising has witnessed a range of advertising formats. The major advertising formats on the web are Banner Ads or Display Ads, Rich Media Ads, Email Attachment Ads, Classified, and Search Ads. A detailed description of the various advertising formats can be found in Seda (2004), Hoffman and Novak (2000), Adams (2003), and Zeff (1999). The Interactive Advertising Bureau is another rich source of the information about various advertising formats (URL: `http://www.iab.net/`). In today's web advertising industry, *Search Ads* constitute the highest revenue generating model among all Internet advertising formats. In this format, advertisers pay on-line companies to list and/or link their company site domain names to a specific search word or phrase. In this format, the text links appear at the top or side of the search results for specific keywords. The more the advertiser pays, the higher the position it gets. When a user clicks on the sponsored link, he is sent to the advertiser's web page. The advertiser then pays the search engine for sending the user to its web page. Such pricing models are known as *Pay-Per-Click (PPC)* Models. The PPC models were originally introduced by `Overture` in 1997 and today they have almost become a standard pricing model for search engine companies, such as `Google, MSN`, and `Yahoo!`. The PPC models for the search engines basically rely on some or other form of the auction models. These auctions are popularly known as *Sponsored Search Auctions*.

Figure 1 Result of a search performed on Google



1.1. Sponsored Search Auctions

When an Internet user (which we will sometimes refer to as the user, searcher, or customer) enters a keyword (i.e. a search term) into a search engine, he gets back a page with results, containing both the links most relevant to the query and the sponsored links, i.e., paid advertisements. When a user clicks on a sponsored link, he is sent to the respective advertiser's web page. The advertiser then pays the search engine for sending the user to its web page. Figure 1 depicts the result of a search performed on Google using the keyword '*auctions*'. There are two different stacks - the left stack contains the links that are most relevant to the query term and the right stack contains the sponsored links. Sometimes, a few sponsored links are placed on top of the search search results as shown in the Figure 1. Typically, a number of merchants (advertisers) are interested in advertising alongside the search results of a keyword. However, the number of slots available to display the sponsored links is limited. Therefore, against every search performed by the user, the search engine faces the problem of matching the advertisers to the slots. In addition, the search engine also needs to decide on a price to be charged to each advertiser. Note that each advertiser has different desirability for different slots on the search result page. The visibility of an Ad shown at the top of the page is much better than an Ad shown at the bottom and, therefore, it is more likely to be clicked by the user. Therefore, an advertiser naturally prefers a slot with higher visibility. Hence, search engines need a system for allocating the slots to advertisers and deciding on a price to be charged to each advertiser. Due to increasing demands for advertising space, most search engines are currently using auction mechanisms for this purpose. In a typical sponsored search auction, advertisers are invited to submit bids on keywords, i.e. the maximum amount they are willing to pay for an Internet user clicking on the advertisement. This is typically referred by the term *Cost-Per-Click (CPC).* Based on the bids submitted by the advertisers for a particular keyword,

Table 1 Historical revenue performance of the Internet advertising industry (revenues in billions of US dollars)

Year	Q1	Q2	Q3	Q4	Annual Revenue	Year/Year Growth	Market Share of Sponsored Search	Market Share of Display Ads
2005	2.802	2.985	3.1	3.6	12.487	+30%	-	-
2004	2.230	2.369	2.333	2.694	9.626	+33%	40%	19%
2003	1.632	1.660	1.793	2.182	7.267	+21%	35%	21%
2002	1.520	1.458	1.451	1.580	6.010	-16%	15%	29%
2001	1.872	1.848	1.773	1.641	7.134	-12%	04%	36%
2000	1.922	2.091	1.951	2.123	8.087	+75%	01%	48%
1999	0.693	0.934	1.217	1.777	4.621	+141%	-	56%
1998	0.351	0.423	0.491	0.656	1.920	+112%	-	56%
1997	0.130	0.214	0.227	0.336	0.907	+239%	-	55%

Source: Interactive Advertising Bureau. URL: `http://www.iab.net/resources/ad_revenue.asp` (accessed on March 20, 2006)

the search engine (which we will sometimes refer to as the auctioneer or the seller) picks a subset of advertisements along with the order in which to display. The actual price charged also depends on the bids submitted by the advertisers. There are many terms currently used in practice to refer to these auctions models, e.g. *search auctions, Internet search auctions, sponsored search auctions, paid search auctions, paid placement auctions, AdWord auctions, slot auctions, etc.*

In a relatively short time (not more than 10 years), advertising on the Internet has become a common activity embraced by advertisers and marketers across all industry sectors; Table 1 gives a quick estimate of the size of the market dominated by Internet advertising and the pace with which it is growing. The columns *Q1* through *Q4* represent the revenue generated from Internet advertising in each quarter of the years 1997–2005. The *Annual Revenue* and *Year/Year* columns give the annual revenue generated and year-by-year growth of the Internet advertising industry. The last two columns are important in the sense they give an estimate of the market share of two major formats of the Internet advertising - sponsored search and display Ads. As pointed out by Edelman et al. (2006), it is believed that `Google`'s total revenue in 2004 was equal to \$ 3.189 billion. Over 98% of the revenue came from Internet advertising. Similarly, `Yahoo!`'s total revenue in 2004 was equal to \$ 3.574 billion and over 50% of it came from Internet advertising. Thus, Table 1 shows that sponsored search is a key factor in deciding the revenue performance of any search engine company. In this paper, we are interested in studying appropriate mechanisms for sponsored search auction and investigate their performance.

1.2. Related Literature

The motivation for our work comes from several recent research articles. The work of Edelman et al. (2006) investigates the Generalized Second Price (GSP) mechanism for sponsored search auction under *static settings*. The work assumes that the value derived out of a single user-click by an advertiser is publicly known to all the rival advertisers, and then they analyze the underlying static one-shot game of complete information. Our approach generalizes their analysis to the more realistic case of incomplete information through a detailed analysis of the induced Bayesian game.

Another strand of work which is closely related to ours is due to Lahaie (2006). The objective of this paper is to clarify the incentive, efficiency, and revenue properties of the two popular slot auctions - first price and second price, under settings of incomplete and complete information. The work does not attempt to derive any optimal mechanism.

Another line of work that is closely related to ours is due to Feng (2005) where the author studies the allocation mechanisms under a setting in which the advertisers have a consistent ranking of advertising positions but different rates of decrease in absolute valuation. The model and underlying assumptions of this paper are quite different than ours. Among other interesting work in this area, is the work of Feng et al. (2003, 2005), where they examine the paid-placement ranking strategies of the two dominant firms in this industry, and compare their revenue under different scenarios via computational simulation.

In a recent paper, Varian (2006) analyzes the equilibria of an assignment game that arises in the context of Ad auctions. These equilibria are closely related to the equilibria of assignment game studied by Shapley and Shubik (1972), Demange et al. (1986), and Roth and Sotomayor (1990). The author characterizes the symmetric Nash equilibria of such assignment games and uses it to derive an upper bound and a lower bound on the revenue generated by the search engine. Further, this revenue is also compared with the revenue in the VCG mechanism.

In another related work by Aggarwal et al. (2006), the authors design a simple truthful auction for a general class of ranking functions that includes direct ranking and revenue ranking. More specifically, the authors study the case where the merchants are assigned arbitrary weights which do not depend on the bids, and then ranked in decreasing order of their weighted bids. They call such an auction as laddered auction, since the price for a merchant builds on the price of each merchant ranked below it. They show that this auction is truthful.

We would also like to mention some interesting papers in this area which have some indirect connections to our work. Bhargava and Feng (2002) have formulated the search engine design problem as a tradeoff between placement revenue and user-based revenue. Borgs et al. (2005) study a multi-unit (corresponds to a sequence of searches each with a single slot) auction with multiple agents, each of whom has a private valuation and budget. Aggarwal and Hartline (2005) consider a special version of Ad auction as the private value knapsack problem. Mehta et al. (2005) address the online version of the sponsored search auctions problem. Balcan et al. (2005) use techniques from sample-complexity in machine learning theory to reduce the design of revenue maximizing incentive-compatible mechanisms to algorithmic pricing questions relevant to sponsored search.

1.3. Contributions and Outline of the Paper

In this paper, we first develop a framework to model the sponsored search auction problem as a mechanism design problem. Using this framework, we describe three well known auction mechanisms - *Generalized First Price (GFP), Generalized Second Price (GSP)*, and *Vickrey-Clarke-Groves (VCG)*. We then pursue the objective of designing a mechanism that is superior to the above three mechanisms. For this, we impose the following well known requirements, which we believe are practical requirements for sponsored search auction, for any mechanism in this setting - *revenue maximization, individual rationality,* and *Bayesian incentive compatibility* or *dominant strategy incentive compatibility.* Motivated by this, we propose a new mechanism which we call the *Optimal (OPT)* mechanism. This mechanism maximizes the search engine's expected revenue subject to achieving Bayesian incentive compatibility and individual rationality. Next, we compare the OPT mechanism with the GSP and the VCG mechanisms along different dimensions such as *incentive compatibility, expected revenue earned by the search engine, individual rationality*, and *computational complexity.* The following are the findings and contributions of our study.

1. The expected revenue earned by the search engine is the same for all the mechanisms GSP, VCG, and OPT, provided the advertisers are symmetric and the number of slots is strictly less than the number of advertisers. This is a direct consequence of a revenue equivalence theorem for sponsored search auctions which we state and prove.
2. We derive expressions for the expected revenue generated by the search engine in equilibrium under all the three mechanisms, under general conditions. To do this, we compute an equilibrium for the GSP mechanism.

3. We show that the GSP and the VCG mechanisms are individually rational in the specific context of sponsored search.
4. We evaluate the computational complexity of all the three mechanisms. Under reasonable assumptions, the worst case complexity of the OPT mechanism is the same as that of the VCG mechanism and this complexity is higher than that of the GSP mechanism.

The rest of the paper is organized as follows. In Section 2, we model the sponsored search auction as a mechanism design problem and use this as the basic framework in the subsequent sections to study the different mechanisms for sponsored search auction. In this framework, we describe the allocation and payment rules for the Generalized First Price (GFP), the Generalized Second Price (GSP), and the Vickrey-Clarke-Groves (VCG) mechanisms. In Section 3, we propose a new mechanism for sponsored search auctions, namely *Optimal (OPT)* mechanism. Subsection 3.3 discusses a special case of the OPT mechanism where advertisers are symmetric. In Section 4, we undertake a detailed comparison of the GSP, VCG, and OPT mechanisms. First, we investigate the incentive compatibility of the three auction mechanisms, namely GSP, VCG, and OPT. In particular, we show that the GSP mechanism is not incentive compatible. In Section 4.2, we present an interesting result which we call as the revenue equivalence theorem for sponsored search auctions. We use this theorem to show the revenue equivalence of the GSP, the VCG, and the OPT mechanisms under some special conditions. Section 4.3 is devoted to computing the expected revenue generated by the three mechanisms under fairly general conditions. In Section 4.4, we investigate the individual rationality of the three mechanisms. In Section 4.5, we investigate the computational complexity of the three mechanisms. Section 5 summarizes the contributions of this paper and proposes a few direction for further research in this area.

2. Sponsored Search Auction as a Mechanism Design Problem in Linear Environment

Consider a search engine that has received a query from an Internet user and it immediately faces the problem of conducting an auction for selling its advertising space among the available advertisers for this particular query word. Let us assume that

1. There are n advertisers interested in this particular keyword and $N = \{1, 2, \ldots, n\}$ represents the set of these advertisers. Also, there are m slots available with search engine to display the Ads and $M = \{1, 2, \ldots, m\}$ represents the set of these advertising slots.
2. α_{ij} is the probability that a user will click on the i^{th} advertiser's Ad if it is displayed in the j^{th} position (slot), where the first position refers to the top most position. We assume that the following condition is satisfied.

$$1 \geq \alpha_{i1} \geq \alpha_{i2} \geq \ldots \geq \alpha_{im} \geq 0 \, \forall i \in N \tag{1}$$

   Note here that we are assuming that click probability α_{ij} does not depend on which other advertiser has been allocated to what other position. We refer to this assumption as *absence of allocative externality* among the advertisers.
3. Each advertiser precisely knows the value derived out of each click performed by the user on his Ad but does not know the value derived out of a single user-click by the other advertisers. Note that this value should be independent of the position of the Ad and should only depend on whether or not a user clicks on the Ad. Formally, this is modeled by supposing that advertiser i observes a parameter, or signal θ_i that represents his value for each user click. The parameter θ_i is referred to as advertiser i's *type*. The set of possible types of advertiser i is denoted by Θ_i.
4. Each advertiser perceives any other advertiser's valuation as a draw from some probability distribution. Similarly, he knows that the other advertisers regard his own valuation as a draw from some probability distribution. More precisely, for advertiser i, $i = 1, 2, \ldots, n$, there is some

probability distribution $\Phi_i(\cdot)$ from which he draws his valuation θ_i. Let $\phi_i(\cdot)$ be the corresponding PDF. We assume that the θ_i takes values from a closed interval $\left[\underline{\theta_i}, \overline{\theta_i}\right]$ of the real line. That is, $\Theta_i = \left[\underline{\theta_i}, \overline{\theta_i}\right]$. We also assume that any advertiser's valuation is statistically independent from any other advertiser's valuation. That is, $\Phi_i(\cdot), i = 1, 2, \ldots, n$ are mutually independent. This is the classical *independent private values assumption.* Note that the probability distribution $\Phi_i(\cdot)$ can be viewed as the distribution of a random variable that gives the profit earned by advertiser i when a random customer clicks on the advertiser's Ad.

5. Each advertiser i is rational and intelligent in the sense of Myerson (1997). This fact is modeled by assuming that the advertisers always try to maximize a Bernoulli utility function $u_i : X \times \Theta_i \rightarrow \mathbb{R}$, where X is the set of outcomes which will be defined shortly.
6. The probability distribution functions $\Phi_i(\cdot)$, the type sets $\Theta_1, \ldots, \Theta_n$, and the utility functions $u_i(\cdot)$ are assumed to be common knowledge among the advertisers. Note that utility function $u_i(\cdot)$ of advertiser i depends on both the outcome x and the type θ_i. The type θ_i is not a common knowledge; but by saying that $u_i(\cdot)$ is common knowledge we mean that for any given type θ_i, the auctioneer (that is, search engine in this case) and every other advertiser can evaluate the utility function of advertiser i.

In view of the above modeling assumptions, the sponsored search auction problem can now be restated as follows. For any query word, each interested advertiser i bids an amount $b_i \geq 0$, which depends on his actual type θ_i. Now each time the search engine receives this query word, it first retrieves the information from its database of all the advertisers who are interested in displaying their Ads against the search result of this query and their corresponding bid vector $b = (b_1, \ldots, b_n)$. The search engine then decides the winning advertisers and the order in which their Ads will be displayed against the search results and the amount that will be paid by each advertiser if the user clicks on his Ad. These are called as *allocation* and *payment rules*, respectively. A sponsored search auction can be viewed as an *indirect mechanism* $\mathcal{M} = ((B_i)_{i \in N}, g(\cdot))$, where $B_i \subset \mathbb{R}^+$ is the set of bids that an advertiser i can report to the search engine and $g(\cdot)$ is an allocation and payment rule. Note, if we assume that for each advertiser i, the set of bids B_i is the same as the type set Θ_i, then the indirect mechanism $\mathcal{M} = ((B_i)_{i \in N}, g(\cdot))$ becomes a direct revelation mechanism $\mathcal{D} = ((\Theta_i)_{i \in N}, f(\cdot))$, where $f(\cdot)$ becomes the allocation and payment rule. In the rest of this paper, we will assume that $B_i = \Theta_i \ \forall\, i = 1, \ldots, n$. Thus, we regard a sponsored search auction as a direct revelation mechanism. The various components of a typical sponsored search mechanism design problem are listed below.

Outcome Set X

An outcome in the case of sponsored search auction may be represented by a vector $x = (y_{ij}, p_i)_{i \in N, j \in M}$, where y_{ij} is the probability that advertiser i is allocated the slot j and p_i denotes the price-per-click charged from the advertiser i. The set of feasible alternatives is then

$$X = \left\{ (y_{ij}, p_i)_{i \in N, j \in M} \,\middle|\, y_{ij} \in [0,1] \, \forall i \in N, \, \forall j \in M, \sum_{i=1}^{n} y_{ij} \leq 1 \, \forall j \in M, \sum_{j=1}^{m} y_{ij} \leq 1 \, \forall i \in N, \; p_i \geq 0 \, \forall i \in N \right\}$$

Note that the randomized outcomes are also included in the above outcome set. This implies that randomized mechanisms are also part of the design space.

Utility Function of Advertisers $u_i(\cdot)$

The Bernoulli utility function of advertiser i, for $x = (y_{ij}, p_i)_{i \in N, j \in M}$, is given by

$$u_i(x, \theta_i) = \left(\sum_{j=1}^{m} y_{ij} \alpha_{ij} \right) (\theta_i - p_i)$$

Allocation and Payment Rule $f(\cdot)$

The general structure of the allocation and payment rule for this case is

$$f(b) = (y_{ij}(b), p_i(b))_{i \in N, j \in M}$$

where $b = (b_1, \dots, b_n)$ is a bid vector of the advertisers. The functions $y_{ij}(\cdot)$ form the allocation rule and the functions $p_i(\cdot)$ form the payment rule.

Linear Environment

Through a slight modification in the definition of allocation rule, payment rule, and utility functions, we can show that sponsored search auction is indeed a mechanism in linear environment. To transform the underlying environment to a linear one, we redefine the allocation and payment rule as below.

$$f(b) = (y(b), t_i(b))_{i \in N, j \in M}$$

where $y(b) = (y_{ij}(b))_{i \in N, j \in M}$ and $t_i(b) = \left(\sum_{j=1}^{m} y_{ij}(b)\alpha_{ij}\right) p_i(b)$. The quantity $t_i(b)$ can be viewed as the average payment made by the advertiser i to the search engine against every search query received by the search engine and when the bid vector of the advertisers is $b = (b_1, \dots, b_n)$.

Now, we can rewrite the utility functions in following manner

$$u_i(f(b), \theta_i) = \theta_i v_i(y(b)) - t_i(b)$$

where $v_i(y(b)) = \left(\sum_{j=1}^{m} y_{ij}(b)\alpha_{ij}\right)$. The quantity $v_i(y(b))$ can be interpreted as the probability that advertiser i will receive a user click whenever there is a search query received by the search engine and when the bid vector of the advertisers is $b = (b_1, \dots, b_n)$. Now, it is easy to verify that the underlying environment is *linear.*

2.1. Generalized First Price (GFP) Mechanism

In 1997, `Overture` introduced the first auction mechanism ever used for sponsored search. The term Generalized First Price Auction is coined by Edelman et al. (2006).

2.1.1. Allocation Rule The m advertising slots are allocated to advertisers in *descending order of their bids.* If two advertisers place the same bid, then the tie can be broken by an appropriate rule. In order to define the allocation rule $y_{ij}(\cdot)$ for the GFP mechanism, we define $b^{(k)}$ to be the k^{th} highest element in $(b_1, \dots, b_n)$ and $(b_{-i})^{(k)}$ to be the k^{th} highest element in $(b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_n)$. In view of these definitions, we can say that for all $i \in N$ and all $j \in M$,

$$y_{ij}(b) = \begin{cases} 1 & : \text{ if } b_i = b^{(j)} \\ 0 & : \text{ otherwise} \end{cases} \tag{2}$$

2.1.2. Payment Rule Every time a user clicks on a sponsored link, an advertiser's account is automatically billed *the amount of the advertiser's bid.* That is, if $b = (b_1, b_2, \dots, b_n)$ is the profile of bids received from the n advertisers then, for all $i \in N$,

$$p_i(b) = \begin{cases} b_i & : \text{ if advertiser } i\text{'s Ad is displayed} \\ 0 & : \text{ otherwise} \end{cases} \tag{3}$$

2.2. Generalized Second Price (GSP) Mechanism

The primary motivation for this auction mechanism was the instability of the GFP mechanism. The difficulties with the GFP mechanism are discussed by Edelman et al. (2006). In particular, it has been shown that under the GFP mechanism, truth-telling is not an equilibrium bidding strategy for the advertisers and this fact leads to instability in the system which in turn leads to inefficient investments on behalf of the advertisers. The GFP mechanism also creates volatile prices which in turn causes allocative inefficiencies. There are three different versions of the GSP mechanism (depending on the allocation rule).

2.2.1. Allocation Rule

1. `Yahoo!/Overture`**'s Allocation Rule:** This rule is the same as the allocation rule of GFP mechanism.
2. **Greedy Allocation Rule:** The primary motivation for this rule is *allocative efficiency* (which we will discuss later). In this rule, the first slot is allocated to the advertiser $i \in N$ for whom the quantity $\alpha_{i1}b_i$ is the maximum. If there is a tie then it is broken by an appropriate rule. The winning advertiser is removed from the set N and an advertiser among the remaining ones is chosen for whom $\alpha_{i2}b_i$ is maximum and the second slot to allocated to this advertiser. In similar fashion, the rest of the slots are allocated.
3. `Google`**'s Allocation Rule:** In practice, `Google` uses a stylized version of the *greedy* allocation rule. In `Google`'s actual version of GSP mechanism, for each advertiser `Google` computes its estimated *Click-Through-Rate (CTR)*, that is the ratio of the number of clicks received by the Ad to the number of times the Ad was displayed against the search results-popularly known as number of *impressions*. Now the advertisers are ranked in decreasing order of the *ranking scores*, where the ranking score of an advertiser is defined as the product of the advertiser's bid and estimated *CTR*.

In order to understand the relationship among these three allocation rules, we need to first understand the relationship between click probability and CTR.

2.2.2. Relationship between Click Probability and CTR Recall the following definitions that we presented earlier:

$$\begin{aligned}
\alpha_{ij} &= \text{Probability that a user will click on the } i^{th} \text{ advertiser's Ad if it is displayed in } j^{th} \text{ position} \\
\text{CTR}_i &= \text{Probability that a user will click on the } i^{th} \text{ advertiser's Ad if it is displayed} \\
y_{ij} &= \text{Probability that advertiser } i\text{'s Ad is displayed in position } j
\end{aligned}$$

It is easy to verify that

$$\begin{aligned}
\text{CTR}_i &= \sum_{j=1}^{m} y_{ij}\alpha_{ij} \ \forall\, i \in N \\
\Rightarrow\ \text{CTR}_i &\le \sum_{j=1}^{m} \alpha_{ij} \ \forall\, i \in N
\end{aligned}$$

In practice, the click probabilities (α_{ij}) and CTR are learned by means of available data. Immorlica et al. (2005) have proposed different ways in which one can learn these quantities.

2.2.3. Relationship among Different Allocation Rules Assume that $b = (b_1, b_2, \ldots, b_n)$ is the profile of bids received from the n advertisers. Consider the following optimization problem

Maximize
$$\sum_{i=1}^{n} b_i v_i(y(b)) = \sum_{i=1}^{n}\sum_{j=1}^{m} (b_i\alpha_{ij}) y_{ij}(b)$$

subject to

$$\sum_{i=1}^{n} y_{ij}(b) \leq 1 \; \forall j \in M$$
$$\sum_{j=1}^{m} y_{ij}(b) \leq 1 \; \forall i \in N$$
$$y_{ij}(b) \geq 0 \; \forall i \in N, \; \forall j \in M$$

It is easy to see that for a given click probabilities α_{ij}, where these probabilities satisfy the condition (1), the greedy allocation rule basically provides a solution of the above optimization problem. Such an allocation would be an *efficient* allocation. The Yahoo!/Overture's allocation rule and Google's allocation rule become special cases of the greedy allocation rule under certain conditions that are summarized in following propositions.

PROPOSITION 1. *If*
1. *click probabilities satisfy the assumption of absence of allocative externality among the advertisers, that is,* $1 \geq \alpha_{i1} \geq \alpha_{i2} \geq \ldots \geq \alpha_{im} \geq 0 \; \forall i \in N$
2. *click probabilities depend only on the positions of the Ads and are independent of the identities of the advertisers, that is,* $\alpha_{1j} = \alpha_{2j} = \ldots = \alpha_{nj} = \alpha_j \; \forall j \in M$

then for any bid vector $b = (b_1, \ldots, b_n)$*, both the greedy allocation rule and the* ***Yahoo!/Overture****'s allocation rule result in the same allocation.*

PROPOSITION 2. *If*
1. *click probabilities satisfy the assumption of absence of allocative externality among the advertisers, that is,* $1 \geq \alpha_{i1} \geq \alpha_{i2} \geq \ldots \geq \alpha_{im} \geq 0 \; \forall i \in N$
2. *click probabilities depend only on the identities of the advertisers and are independent of the positions of the Ads, that is,* $\alpha_{i1} = \alpha_{i2} = \ldots = \alpha_{im} = \alpha_i = CTR_i \; \forall i \in N$

then for any bid vector $b = (b_1, \ldots, b_n)$*, both the greedy allocation rule and the* ***Google****'s allocation rule result in the same allocation.*

In rest of the paper, we will stick to the following assumptions:
1. Click probabilities depend only on the positions of the Ads and are independent of the identities of the advertisers. That is, $\alpha_{1j} = \alpha_{2j} = \ldots = \alpha_{nj} = \alpha_j \; \forall j \in M$
2. The allocation rule in GSP mechanism is the same as the greedy allocation rule, which would be the same as Yahoo!/Overture's allocation rule because of the previous assumption.

2.2.4. Payment Rule In the GSP auction mechanism, every time a user clicks on a sponsored link, an advertiser's account is automatically billed *the amount of the advertiser's bid who is just below him in the ranking of the displayed Ads plus a minimum increment (typically* \$0.01*)*. The advertiser whose Ad appears at the bottom-most position is charged the amount of the highest bid among the disqualified bids plus the minimum increment. If there is no such bid then he is charged nothing. If $b = (b_1, b_2, \ldots, b_n)$ is the profile of bids received from the n advertisers, then because of the assumptions we have made regarding the allocation rule in GSP mechanism, the price per click that is charged to an advertiser i is given by

$$p_i(b) = \begin{cases} \sum_{j=1}^{m} \left(b^{(j+1)} y_{ij}(b) \right) & : \text{ if either } m < n \text{ or } n \leq m \text{ but } b_i \neq b^{(n)} \\ 0 & : \text{ otherwise} \end{cases}$$

where $b^{(j+1)}$ is the $(j+1)^{th}$ highest bid which is the same as the bid of an advertiser whose Ad is allocated to position $(j+1)$. We have ignored the small increment \$0.01 because all the future analysis and results are insensitive to this amount.

2.3. Vickrey-Clarke-Groves (VCG) Mechanism

2.3.1. Allocation Rule By definition, the VCG mechanism is allocatively efficient. Therefore, in the case of sponsored search auction, the allocation rule $y^*(\cdot)$ in the VCG mechanism is

$$y^*(\cdot) = \underset{y(\cdot)}{\arg\max} \sum_{i=1}^{n} b_i v_i(y(b)) = \underset{y_{ij}(\cdot)}{\arg\max} \sum_{i=1}^{n}\sum_{j=1}^{m}(b_i\alpha_{ij})y_{ij}(b) \tag{4}$$

In the previous section, we have already seen that the greedy allocation rule is a solution to (4). Moreover, under the assumption that click probabilities are independent of advertisers' identities, the allocation $y^*(\cdot)$ allocates the slots to the advertisers in the decreasing order of their bids. That is, if $b = (b_1, b_2, \ldots, b_n)$ is the profile of bids received from the n advertisers then $y^*(\cdot)$ must satisfy the following condition

$$y^*_{ij}(b) = \begin{cases} 1 & : \quad b_i = b^{(j)} \\ 0 & : \quad \text{otherwise} \end{cases} \tag{5}$$

We state below an interesting observation regarding GFP and GSP mechanisms which is based on the above observations.

PROPOSITION 3. *If click probabilities depend only on the positions of the Ads and are independent of the identities of the advertisers, then*
1. *The GFP mechanism is allocatively efficient.*
2. *The GSP mechanism is allocatively efficient if it uses greedy allocation rule which is the same as* `Yahoo!/Overture`*'s allocation rule.*
3. *The allocation rule for the VCG mechanism, which is an efficient allocation, is given by (5). Moreover, this allocation rule is precisely the same as the GFP allocation rule and the* `Yahoo!/Overture`*'s allocation rule.*

2.3.2. Payment Rule As per the definition of the VCG mechanism given in Mas-Colell et al. (1995), the expected payment $t_i(b)$ made by an advertiser i, when the profile of the bids submitted by the advertisers is $b = (b_1, \ldots, b_n)$, must be computed using the following Groves payment formula:

$$t_i(b) = \left[\sum_{j\neq i} b_j v_j(y^*(b))\right] + h_i(b_{-i}) \tag{6}$$

where $h_i(b_{-i})$ is an arbitrary function of b_{-i}. A special case of the above Groves payment scheme is the Clarke's payment scheme in which the payment rule is given by the following formula:

$$t_i(b) = \left[\sum_{j\neq i} b_j v_j(y^*(b))\right] - \left[\sum_{j\neq i} b_j v_j(y^*_{-i}(b))\right] \tag{7}$$

where $y^*_{-i}(\cdot)$ is an efficient allocation of the slots among the advertisers when advertiser i is removed from the scene. Substituting the value of $y^*(\cdot)$ from Equation (5)and making use of the fact that $v_i(y^*(b)) = \sum_{j=i}^{m} y^*_{ij}(b)\alpha_j$, Equation (7) can be written as follows

Case 1 ($m < n$):

$$p^{(j)}(b) = \frac{1}{\alpha_j} t^{(j)}(b) = \begin{cases} \frac{1}{\alpha_j}\left[\sum_{k=j}^{m-1} \beta_k b^{(k+1)}\right] + \frac{\alpha_m}{\alpha_j} b^{(m+1)} & : \quad \text{if } 1 \leq j \leq (m-1) \\ b^{(m+1)} & : \quad \text{if } j = m \\ 0 & : \quad \text{if } m < j \leq n \end{cases} \tag{8}$$

where

—$t^{(j)}(b)$ is the expected payment made by the advertiser whose Ad is displayed in j^{th} position, for every search query received by the search engine and when the bid profile of the advertisers is $b=(b_1,\ldots,b_n)$,
—$p^{(j)}(b)$ is the payment made by the advertiser, whose Ad is displayed in j^{th} position, for every click made by the user and when the bid profile of the advertisers is $b=(b_1,\ldots,b_n)$,
—and $\beta_j=(\alpha_j-\alpha_{j+1})$
—$b^{(j)}$ has its usual interpretation.

Case 2 ($n\le m$):

$$p^{(j)}(b)=\frac{1}{\alpha_j}t^{(j)}(b)=\begin{cases}\frac{1}{\alpha_j}\sum_{k=j}^{n-1}\beta_k b^{(k+1)} & : \text{ if } 1\le j\le(n-1)\\ 0 & : \text{ if } j=n\end{cases} \tag{9}$$

Thus, we can say that Equation (5) describes the allocation rule for the VCG mechanism and Equations (8) and (9) describe the payment rule for the VCG mechanism.

3. Optimal (OPT) Mechanism

We believe that a mechanism for sponsored search auction must satisfy three basic properties - *revenue maximization, individual rationality,* and *dominant strategy incentive compatibility* or *Bayesian incentive compatibility.* Myerson (1981) first studied such an auction mechanism in the context of selling a single indivisible good. Myerson called such an auction mechanism as *optimal auction.* Following the same terminology, we would prefer to call a similar mechanism for the sponsored search auction as *optimal mechanism* for sponsored search auction (OPT mechanism for short). In this section, our goal is to compute the allocation and payment rule $f(\cdot)$ that results in an optimal mechanism for the sponsored search auction. This calls for extending Myerson's optimal auction to the case of the sponsored search auction. We follow a line of attack which is similar to that of Myerson (1981). Recall that we formulated the sponsored search auction as a direct revelation mechanism $\mathcal{D}=((\Theta_i)_{i\in N},f(\cdot))$ in linear environment, where the Bernoulli utility function of an advertiser i is given by

$$\begin{aligned}u_i(f(b),\theta_i) &= \left(\sum_{j=1}^{m}y_{ij}(b)\alpha_j\right)(\theta_i-p_i(b))\\ &= v_i(y(b))(\theta_i-p_i(b))\\ &= \theta_i v_i(y(b))-t_i(b)\end{aligned}$$

where $v_i(y(b))=\left(\sum_{j=1}^{m}y_{ij}(b)\alpha_j\right)$ is known as *value function* of the advertiser i.

3.1. Allocation Rule

It is convenient to define

- $\bar{t}_i(b_i)=E_{\theta_{-i}}[t_i(b_i,\theta_{-i})]$ is the expected payment made by advertiser i when he bids an amount b_i and all the advertisers $j\neq i$ bid their true types.
- $\overline{v}_i(b_i)=E_{\theta_{-i}}\left[v_i(y(b_i,\theta_{-i}))\right]$ is the probability that advertiser i will receive a user click if he bids an amount b_i and all the advertisers $j\neq i$ bid their true types.
- $U_i(\theta_i)=\theta_i\overline{v}_i(\theta_i)-\bar{t}_i(\theta_i)$ gives advertiser i's expected utility from the mechanism conditional on his type being θ_i when he and all other advertisers bid their true types.

The problem of designing an optimal mechanism for the sponsored search auction can now be written as one of choosing functions $y_{ij}(\cdot)$ and $U_i(\cdot)$ to solve:

$$\text{Maximize } \sum_{i=1}^{n} \int_{\underline{\theta_i}}^{\overline{\theta_i}} \left(\theta_i \overline{v}_i(\theta_i) - U_i(\theta_i)\right) \phi_i(\theta_i) d\theta_i$$

subject to

(i) $\overline{v}_i(\cdot)$ is non-decreasing $\forall i \in N$

(ii) $y_{ij}(\theta) \in [0,1], \sum_{j=1}^{m} y_{ij}(\theta) \leq 1, \sum_{i=1}^{n} y_{ij}(\theta) \leq 1 \;\; \forall i \in N, \forall j \in M, \forall \theta \in \Theta$

(iii) $U_i(\theta_i) = U_i(\underline{\theta_i}) + \int_{\underline{\theta_i}}^{\theta_i} \overline{v}_i(s) ds \;\; \forall i \in N, \forall \theta_i \in \Theta_i$

(iv) $U_i(\theta_i) \geq 0 \; \forall i \in N, \forall \theta_i \in \Theta_i$

In the above formulation, the objective function is the total expected payment received by the search engine from all the advertisers. Note that constraints (iv) are the advertisers' interim individual rationality constraints while constraint (ii) is the feasibility constraint. Constraints (i) and (iii) are the necessary and sufficient conditions for the allocation and payment rule $f(\cdot) = (y_{ij}(\cdot), t_i(\cdot))_{i \in N, j \in M}$ to be Bayesian incentive compatible. These constraints are taken from Myerson (1981). We have a critical observation to make here. Note that in the above optimization problem, we have replaced the bid b_i by the actual type θ_i. This is because we are imposing the Bayesian incentive compatibility constraints on the allocation and payment rule and, hence, every advertiser will bid his true type. Thus, while dealing with the OPT mechanism, we can safely interchange θ_i and b_i for any $i \in N$. Define, as in Myerson (1981),

$$J_i(\theta_i) = \theta_i - \frac{1 - \Phi_i(\theta_i)}{\phi_i(\theta_i)}$$

Then, following the same line of arguments as in Myerson (1981), we can show that if the constraint (i) is ignored then $y_{ij}(\cdot)$ is a solution to the above problem iff no slot is allocated to any advertiser having negative value $J_i(\theta_i)$, and the rest of the advertisers' Ads are displayed in the same order as the values of $J_i(\theta_i)$. That is,

$$y_{ij}(\theta) = \begin{cases} 0 & \forall j = 1,2,\ldots,m \quad : \text{ if } J_i(\theta_i) < 0 \\ 1 & \forall j = 1,2,\ldots,m < n \quad : \text{ if } J_i(\theta_i) = J^{(j)} \\ 1 & \forall j = 1,2,\ldots,n \leq m \quad : \text{ if } J_i(\theta_i) = J^{(j)} \\ 0 & \quad : \text{ otherwise} \end{cases} \tag{10}$$

where $J^{(j)}$ is the j^{th} highest values among $J_i(\theta_i)$s.

Now, recall the definition of $\overline{v}_i(\cdot)$. It is easy to write down the following expression:

$$\overline{v}_i(\theta_i) = E_{\theta_{-i}}\left[v_i(y(\theta_i, \theta_{-i}))\right] = E_{\theta_{-i}}\left[\sum_{j=1}^{m} y_{ij}(\theta_i, \theta_{-i}) \alpha_j\right] \tag{11}$$

Now if we assume that $J_i(\cdot)$ is non-decreasing in θ_i, it is easy to see that the above solution $y_{ij}(\cdot)$, given by (10), will be non-decreasing in θ_i, which in turn implies, by looking at expression (11), that $\overline{v}_i(\cdot)$ is non-decreasing in θ_i. Thus, the solution to this relaxed problem actually satisfies constraint (i) under the assumption that $J_i(\cdot)$ is non-decreasing. Assuming that $J_i(\cdot)$ is non-decreasing, the solution given by (10) appears to be the solution of the optimal mechanism design problem for sponsored search auction. The condition that $J_i(\cdot)$ is non-decreasing in θ_i is met by most distribution functions such as Uniform and Exponential. In the rest of this paper, we will stick to the assumption that for every advertiser i, $J_i(\cdot)$ is non-decreasing in θ_i. We have interesting observations to make here.

PROPOSITION 4. *If the advertisers have non-identical distribution functions $\Phi_i(\cdot)$ then the advertiser who has the k^{th} largest value of $J_i(b_i)$ is* not *necessarily the advertiser who has bid the k^{th} highest amount. Thus the OPT mechanism need not be allocatively efficient and therefore, need not be ex post efficient.*

PROPOSITION 5. *If the advertisers are symmetric in following sense*
- $\Theta_1 = \ldots = \Theta_n = \Theta$,
- $\Phi_1(\cdot) = \ldots = \Phi_n(\cdot) = \Phi(\cdot)$,

and for every advertiser i, we have $J_i(\cdot) > 0$ and $J_i(\cdot)$ is non-decreasing, then
- $J_i(\cdot) = \ldots = J_n(\cdot) = J(\cdot)$
- *The rank of an advertiser in the decreasing order sequence of $J_1(b_1), \ldots, J_n(b_n)$ is precisely the same as the rank of the same advertiser in the decreasing order sequence of $b_1, \ldots, b_n$.*
- *For a given bid vector b, the OPT mechanism results in the same allocation as suggested by the GFP, the GSP, and the VCG mechanisms.*
- *The OPT mechanism is allocatively efficient.*

3.2. Payment Rule

Following Myerson (1981) line of attack , the optimal expected payment rule $t_i(\cdot)$ must be chosen in such a way that it satisfies

$$\bar{t}_i(\theta_i) = E_{\theta_{-i}}[t_i(\theta_i, \theta_{-i})] = \theta_i \overline{v}_i(\theta_i) - U_i(\theta_i) = \theta_i \overline{v}_i(\theta_i) - \int_{\underline{\theta_i}}^{\theta_i} \overline{v}_i(s) ds \tag{12}$$

Looking at the above formula, we can say that if the payment rule $t_i(\cdot)$ satisfies the following formula (13) then it would also satisfy the formula (12).

$$t_i(\theta_i, \theta_{-i}) = \theta_i v_i(y(\theta_i, \theta_{-i})) - \int_{\underline{\theta_i}}^{\theta_i} v_i(s, \theta_{-i}) ds \;\; \forall\, \theta \in \Theta \tag{13}$$

The above formula can be rewritten in a more intuitive way for which, we need to define the following quantities for any vector θ_{-i}.

$$\begin{aligned}
z_{i1}(\theta_{-i}) &= \inf \left\{ \theta_i | J_i(\theta_i) > 0 \text{ and } J_i(\theta_i) \geq J_{-i}^{(1)} \right\} \\
z_{i2}(\theta_{-i}) &= \inf \left\{ \theta_i | J_i(\theta_i) > 0 \text{ and } J_{-i}^{(1)} > J_i(\theta_i) \geq J_{-i}^{(2)} \right\} \\
\vdots &= \vdots \\
z_{i\gamma}(\theta_{-i}) &= \inf \left\{ \theta_i | J_i(\theta_i) > 0 \text{ and } J_{-i}^{(\gamma-1)} > J_i(\theta_i) \right\}
\end{aligned}$$

where $\gamma = m$ if $m < n$, otherwise $\gamma = n$. The quantity $J_{-i}^{(k)}$ is the k^{th} highest value among $J_1(\theta_1), \ldots, J_{i-1}(\theta_{i-1}), J_{i+1}(\theta_{i+1}), \ldots, J_n(\theta_n)$. The quantity $z_{ik}(\theta_{-i})$ is the infimum of all the bids for advertisers i which can win him the k^{th} slot against the bid vector θ_{-i} from the other advertisers. In view of the above definitions, we can write

$$v_i(y(\theta_i, \theta_{-i})) = \begin{cases} \alpha_1 & : \text{ if } \theta_i \geq z_{i1}(\theta_{-i}) \\ \alpha_2 & : \text{ if } z_{i1}(\theta_{-i}) > \theta_i \geq z_{i2}(\theta_{-i}) \\ \vdots & : \vdots \\ 0 & : \text{ if } z_{i\gamma}(\theta_{-i}) > \theta_i \end{cases}$$

This gives us the following expression for $\int_{\underline{\theta_i}}^{\theta_i} v_i(s,\theta_{-i})ds$. In these expressions, r is the position of the advertiser i's Ad.

$$\int_{\underline{\theta_i}}^{\theta_i} v_i(y(s,\theta_{-i}))ds = \begin{cases} \alpha_r(\theta_i - z_{ir}(\theta_{-i})) + \sum\limits_{j=(r+1)}^{\gamma} \alpha_j \left(z_{i(j-1)}(\theta_{-i}) - z_{ij}(\theta_{-i})\right) & : \text{ if } 1 \le r \le (\gamma-1) \\ \alpha_\gamma(\theta_i - z_{i\gamma}(\theta_{-i})) & : \text{ if } r = \gamma \\ 0 & : \text{ otherwise} \end{cases}$$

Substituting the above value for $\int_{\underline{\theta_i}}^{\theta_i} v_i(y(s,\theta_{-i}))ds$ in formula (13), we get

$$p_i(\theta_i,\theta_{-i}) = \frac{1}{\alpha_r} t_i(\theta_i,\theta_{-i}) = \begin{cases} \frac{\alpha_\gamma}{\alpha_r} z_{i\gamma}(\theta_{-i}) + \frac{1}{\alpha_r}\sum\limits_{j=r}^{\gamma-1} \beta_j z_{ij}(\theta_{-i}) : & \text{if } 1 \le r \le (\gamma-1) \\ z_{i\gamma}(\theta_{-i}) & : \text{ if } r=\gamma \\ 0 & : \text{ otherwise} \end{cases} \tag{14}$$

The above relations say that an advertiser i must pay only when his Ad receives a click, and he pays an amount equal to $p_i(\theta)$. Note that in above expressions, we have expressed the payment rule $p_i(\cdot)$ as a function of the actual type profile θ of the advertisers rather than the bid vector b. This is because in OPT mechanism, each advertiser bids his true type and we have $b_i = \theta_i \ \forall\, i = 1,\ldots,n$. Thus, we can say that Equation (10) describes the allocation rule for the OPT mechanism and Equation (14) describe the the payment rule for the OPT mechanism.

In what follows, we discuss an important special cases of the OPT mechanism when the advertisers are symmetric.

3.3. OPT Mechanism and Symmetric Advertisers

Let us assume that advertisers are symmetric in the following sense:
- $\Theta_1 = \ldots = \Theta_n = \Theta = [L,U]$
- $\Phi_1(\cdot) = \ldots = \Phi_n(\cdot) = \Phi(\cdot)$

Also, we assume that
- $J(\cdot)$ is non-decreasing over the interval $[L,U]$
- $J(x) > 0 \ \forall\, x \in [L,U]$

Note that if $J(L) > 0$ then we must have $L > 0$.

Proposition 5 shows that if the advertisers are symmetric, then the allocation rule under the OPT mechanism is the same as the GFP, the GSP, and the VCG mechanisms. Coming to the payment rule, it is easy to verify that if advertiser i is allocated the slot r for the bid vector (θ_i,θ_{-i}) then we should have

Case 1 $(m < n)$**:**

$$z_{ij}(\theta_{-i}) = \begin{cases} \theta^{(j)} & : \text{ if } 1 \le j \le (r-1) \\ \theta^{(j+1)} & : \text{ if } r \le j \le m \end{cases} \tag{15}$$

Case 2 $(n \le m)$**:**

$$z_{ij}(\theta_{-i}) = \begin{cases} \theta^{(j)} & : \text{ if } 1 \le j \le (r-1) \\ \theta^{(j+1)} & : \text{ if } r \le j \le (n-1) \\ L & : \text{ if } j = n \end{cases} \tag{16}$$

If we substitute Equations (15) and (16) into Equation (14) then we get the following payment rule for the OPT mechanism when the advertisers are symmetric.

Case 1 $(m<n)$

$$p_i(\theta_i,\theta_{-i}) = \frac{1}{\alpha_r} t_i(\theta_i,\theta_{-i}) = \begin{cases} \frac{\alpha_m}{\alpha_r}\theta^{(m+1)} + \frac{1}{\alpha_r}\sum\limits_{j=r}^{m-1}\beta_j\theta^{(j+1)} & : \text{ if } 1 \le r \le (m-1) \\ \theta^{(m+1)} & : \text{ if } r = m \\ 0 & : \text{ otherwise} \end{cases} \tag{17}$$

Case 2 $(n \le m)$

$$p_i(\theta_i,\theta_{-i}) = \frac{1}{\alpha_r} t_i(\theta_i,\theta_{-i}) = \begin{cases} \frac{\alpha_n}{\alpha_r}L + \frac{1}{\alpha_r}\sum\limits_{j=r}^{n-1}\beta_j\theta^{(j+1)} & : \text{ if } 1 \le r \le (n-1) \\ L & : \text{ if } r = n \\ 0 & : \text{ otherwise} \end{cases} \tag{18}$$

Compare the above equations with the payment rule of the VCG mechanism given by Equations (8) and (9). This comparison leads to the following proposition.

PROPOSITION 6. *If the advertisers are symmetric in following sense*
- $\Theta_1 = \ldots = \Theta_n = \Theta = [L,U]$
- $\Phi_1(\cdot) = \ldots = \Phi_n(\cdot) = \Phi(\cdot)$

and for every advertiser i, we have $J_i(\cdot) > 0$ and $J_i(\cdot)$ is non-decreasing over the interval $[L,U]$, then
- *the payment rule for Case 1 coincides with the corresponding payment rule in the VCG mechanism,*
- *and the payment rule for the Case 2 differs from the corresponding payment rule of the VCG mechanism just by a constant amount L.*

Note that L cannot be zero because of the assumption that $J(L) > 0$.

4. Comparison of GSP, VCG, and OPT Mechanisms

We now compare the mechanisms GSP, VCG, and OPT along four dimensions:
1. Incentive compatibility
2. Expected revenue earned by the search engine
3. Individual rationality
4. Computational complexity

For the purpose of comparison we will make the following assumptions which include the symmetry of advertisers:
- $\Theta_1 = \ldots = \Theta_n = \Theta = [L,U]$
- $\Phi_1(\cdot) = \ldots = \Phi_n(\cdot) = \Phi(\cdot)$
- $J(\cdot)$ is non-decreasing over the interval $[L,U]$
- $J(x) > 0 \; \forall \, x \in [L,U]$

4.1. Incentive Compatibility

Note that by design itself, the OPT mechanism is Bayesian incentive compatible and the VCG mechanism is dominant strategy incentive compatible. In this section, we show that the GSP mechanism is not Bayesian incentive compatible. Our proof follows a line of attack similar to the one used by McAfee and McMillan McAfee and McMillan (1987) to compute the equilibrium bidding strategy of the buyers during the auction of a single indivisible good.

Consider an advertiser i, whose actual type is θ_i. He conjectures that the other advertisers are following a bidding strategy $s(\cdot)$: that is, he predicts that any other advertiser j will bid an amount $s(\theta_j)$ if his type is θ_j (although advertiser i does not know this type). Assume that

1. $L \leq s(\theta_j) \leq \theta_j \ \forall \theta_j \in [L,U]$
2. $s(\cdot)$ is a monotonically increasing function in θ_j

What is the advertiser i's best bid? Advertiser i chooses his bid b_i to maximize his expected utility, which in this case is given by

Case 1 ($m < n$):

$$\pi_i(\theta_i, b_i) = \int_L^{\xi} \left(\sum_{j=1}^{m} \left[\alpha_j \, j \binom{n-1}{j} [\overline{\Phi}(\xi)]^{j-1} \, \Phi(x)^{n-j-1} \right] \right) (\theta_i - s(x))\phi(x)dx \tag{19}$$

Case 2 ($n \leq m$):

$$\pi_i(\theta_i, b_i) = \alpha_n \theta_i [\overline{\Phi}(\xi)]^{n-1} + \int_L^{\xi} \left(\sum_{j=1}^{n-1} \left[\alpha_j \, j \binom{n-1}{j} [\overline{\Phi}(\xi)]^{j-1} \Phi(x)^{n-j-1} \right] \right) (\theta_i - s(x))\phi(x)dx \tag{20}$$

where

- $\xi = s^{-1}(b_i)$
- $\overline{\Phi}(\cdot) = 1 - \Phi(\cdot)$
- The quantity $\int_L^{\xi} \alpha_j \, j \binom{n-1}{j} [\overline{\Phi}(\xi)]^{j-1} [\Phi(x)]^{n-j-1} \phi(x)dx$ gives the probability that advertiser i will be allocated to slot j if he bids b_i and all the other advertisers bid according to the strategy $s(\cdot)$.

Thus, advertiser i chooses bid b_i such that

$$\frac{\partial \pi(\theta_i, b_i)}{\partial b_i} = 0 \tag{21}$$

Note that due to the Envelope Theorem, we can write

$$\frac{d\pi_i(\theta_i, b_i)}{d\theta_i} = \frac{\partial \pi_i(\theta_i, b_i)}{\partial b_i} \frac{db_i}{d\theta_i} + \frac{\partial \pi_i(\theta_i, b_i)}{\partial \theta_i} \tag{22}$$

Thus, by substituting Equation (21) in Equation(22), we get the following condition which an optimally chosen bid b_i must satisfy

$$\frac{d\pi_i(\theta_i, b_i)}{d\theta_i} = \frac{\partial \pi_i(\theta_i, b_i)}{\partial \theta_i} \tag{23}$$

By differentiating (19) and (20), we get

$$\frac{d\pi_i(\theta_i, b_i)}{d\theta_i} = \sum_{j=1}^{\gamma} \left[\alpha_j \binom{n-1}{j-1} [\overline{\Phi}(\xi)]^{j-1} \, \Phi(\xi)^{n-j} \right] \tag{24}$$

where $\gamma = m$ if $m < n$, otherwise $\gamma = n$.

So far, we have examined advertiser i's best response to an arbitrary bidding strategy $s(\cdot)$ being used by his rivals. Now we impose the Nash requirement: the rivals' use of the bidding strategy $s(\cdot)$ must be consistent with the rivals themselves acting rationally. Together with an assumption of symmetry (any two advertisers with the same type will submit the same bid), this implies that advertiser i's bid b_i, satisfying (23), must be the bid implied by the decision rule $s(\cdot)$ - in other words, at a Nash equilibrium, $b_i = s(\theta_i)$ or equivalently $\xi = \theta_i$. When we substitute this Nash condition into (24), we obtain the following equations

$$\frac{d\pi_i(\theta_i)}{d\theta_i} = \sum_{j=1}^{\gamma} \left[\alpha_j \binom{n-1}{j-1} [\overline{\Phi}(\theta_i)]^{j-1} \, [\Phi(\theta_i)]^{n-j} \right] \tag{25}$$

We solve the above differential equations for π_i simply by integrating in conjunction with the boundary condition $s(L) = L$. This results in the following expressions for π_i.

$$\pi_i(\theta_i) = \begin{cases} \int_L^{\theta_i} \sum_{j=1}^{m} \left[\alpha_j \binom{n-1}{j-1} [\overline{\Phi}(x)]^{j-1} [\Phi(x)]^{n-j}\right] dx & : \text{ if } m < n \\ \alpha_n L + \int_L^{\theta_i} \sum_{j=1}^{n} \left[\alpha_j \binom{n-1}{j-1} [\overline{\Phi}(x)]^{j-1} [\Phi(x)]^{n-j}\right] dx & : \text{ if } n \le m \end{cases} \tag{26}$$

We now use the definition of π_i (Equations (19) and (20)) and Nash condition $s_i(\theta_i) = b_i$ or equivalently $\xi = \theta_i$ to obtain the following relations:

Case 1 ($m < n$):

$$\int_L^{\theta_i} \sum_{j=1}^{m} \left[\alpha_j \binom{n-1}{j-1} [\overline{\Phi}(x)]^{j-1} [\Phi(x)]^{n-j}\right] dx =$$
$$\int_L^{\theta_i} \left(\sum_{j=1}^{m} \left[\alpha_j\, j \binom{n-1}{j} [\overline{\Phi}(\theta_i)]^{j-1} [\Phi(x)]^{n-j-1}\right]\right) (\theta_i - s(x))\phi(x)dx$$

Case 2 ($n \le m$):

$$\alpha_n L + \int_L^{\theta_i} \sum_{j=1}^{n} \left[\alpha_j \binom{n-1}{j-1} [\overline{\Phi}(x)]^{j-1} [\Phi(x)]^{n-j}\right] dx = \alpha_n \theta_i [\overline{\Phi}(\theta_i)]^{n-1} +$$
$$\int_L^{\theta_i} \left(\sum_{j=1}^{n-1} \left[\alpha_j\, j \binom{n-1}{j} [\overline{\Phi}(\theta_i)]^{j-1} [\Phi(x)]^{n-j-1}\right]\right) (\theta_i - s(x))\phi(x)dx$$

Differentiating the above equations with respect to θ_i, we get each advertiser's bidding strategy $s(\cdot)$ as a solution of the following integral equations

$$s(\theta_i) = \begin{cases} \theta_i - \frac{1}{g(\theta_i, m)} \int_L^{\theta_i} f(x, \theta_i, m) s'(x) dx & : \text{ if } m < n \\ \theta_i - \frac{1}{g(\theta_i, (n-1))} \int_L^{\theta_i} f(x, \theta_i, (n-1)) s'(x) dx & : \text{ if } n \le m \end{cases} \tag{27}$$

where

$$f(x, \theta_i, k) = \sum_{j=1}^{k} \alpha_j (j-1) \binom{n-1}{j-1} [\overline{\Phi}(\theta_i)]^{j-2} [\Phi(x)]^{n-j}$$
$$g(\theta_i, k) = \sum_{j=1}^{k-1} \left[\beta_j\, j \binom{n-1}{j} [\overline{\Phi}(\theta_i)]^{j-1} [\Phi(\theta_i)]^{n-j-1}\right] + k\alpha_k \binom{n-1}{k} [\overline{\Phi}(\theta_i)]^{k-1} [\Phi(\theta_i)]^{n-k-1}$$

It is easy to see from the above equations that truth-telling is not an equilibrium strategy of the advertisers and, therefore, the allocation and payment rule for GSP mechanism is not Bayesian incentive compatible. Observe that if $m = 1$ and $1 < n$, then this is precisely the scenario of auctioning a single indivisible good with n bidders. For this scenario, the allocation and payment rules under GSP coincides precisely with the allocation and payment rules of classical *Second Price (Vickrey) Auction*.

4.2. Revenue Equivalence Theorem for Sponsored Search Auctions

Here we show that under some reasonable set of assumptions, the mechanisms we have discussed for sponsored search will fetch the same expected revenue to the search engine. We call this the *revenue equivalence theorem* for sponsored search auctions. The classical revenue equivalence theorem is a key result in the literature of single object auction and different versions of this theorem are presented in Mas-Colell et al. (1995), McAfee and McMillan (1987), and Milgrom and Weber (1982).

Theorem 1 **(A Revenue Equivalence Theorem for Sponsored Search Auctions)**. *Consider a sponsored search auction setting, in which*
1. *The advertisers are risk neutral,*
2. *The advertisers are symmetric, i.e.*
    - $\Theta_1 = \ldots = \Theta_n = \Theta = [L, U]$
    - $\Phi_1(\cdot) = \ldots = \Phi_n(\cdot) = \Phi(\cdot)$,
3. *For each advertiser i, we have $\phi_i(\cdot) > 0$; and*
4. *The advertisers draw their types independently.*

Consider two different auction mechanisms, each having a symmetric and increasing Bayesian Nash equilibrium, such that
1. *For each possible realization of $(\theta_1, \ldots, \theta_n)$, every advertiser i has an identical probability of getting slot j in the two mechanisms; and*
2. *Every advertiser i has the same expected utility level in the two mechanisms when his type θ_i is at its lowest possible level, i.e. L.*

Then these equilibria of the two mechanisms generate the same expected revenue for the search engine against every search query.

Proof: By the revelation principle, we know that any given indirect mechanism can be converted into a Bayesian incentive compatible direct revelation mechanism that results in the same outcome as the original mechanism for every type profile θ of the advertisers. This implies that the expected revenue earned by the search engine under both of these mechanisms will be the same. Therefore, we can establish the above theorem by showing that if two Bayesian incentive compatible direct revelation mechanisms have the same allocation rule $(y_{ij}(\theta))_{i\in N, j\in M}$ and the same value of $(U_i(L))_{i\in N}$ then they generate the same expected revenue for the search engine.

To show this, we derive an expression for the search engine's expected revenue from an arbitrary Bayesian incentive compatible direct revelation mechanism. Note, first, that the search engine's expected revenue from an arbitrary Bayesian incentive compatible direct revelation mechanism, under the assumption of risk neutral, symmetric, and independent advertisers, is equal to

$$R = n \int_{\theta_i = L}^{U} \bar{t}_i(\theta_i)\phi(\theta_i)d\theta_i = n \int_{\theta_i = L}^{U} \left(\theta_i \overline{v}_i(\theta_i) - U_i(\theta_i)\right)\phi(\theta_i)d\theta_i \tag{28}$$

We have already seen that due to the result of Myerson (1981) about characterization of Bayesian incentive compatible mechanisms, a direct revelation mechanism is Bayesian incentive compatible iff

(i) $\overline{v}_i(\cdot)$ is non-decreasing $\forall i \in N$

(ii) $U_i(\theta_i) = U_i(\underline{\theta_i}) + \int_{\underline{\theta_i}}^{\theta_i} \overline{v}_i(s)ds \;\; \forall i \in N, \forall \theta_i \in \Theta_i$

Therefore, substituting for $U_i(\theta_i)$ Equation (28), we get

$$R = n \int_{\theta_i = L}^{U} \left(\overline{v}_i(\theta_i)\theta_i - U_i(\underline{\theta_i}) - \int_{s=L}^{\theta_i} \overline{v}_i(s)ds \right) \phi_i(\theta_i)d\theta_i$$

Integrating by parts implies that

$$
\begin{aligned}
R &= \int_L^U \cdots \int_L^U \left[\sum_{i=1}^{n} v_i(y(\theta_i,\theta_{-i}))J_i(\theta_i)\right]\left[\prod_{i=1}^{n}\phi_i(\theta_i)\right] d\theta_n \ldots d\theta_1 - \sum_{i=1}^{n} U_i(\underline{\theta_i}) \\
&= \int_L^U \cdots \int_L^U \left[\sum_{i=1}^{n}\left(\sum_{j=1}^{m} y_{ij}(\theta_i,\theta_{-i})\right) J_i(\theta_i)\right]\left[\prod_{i=1}^{n}\phi_i(\theta_i)\right] d\theta_n \ldots d\theta_1 - \sum_{i=1}^{n} U_i(\underline{\theta_i}) \qquad (29)
\end{aligned}
$$

where $J_i(\theta_i)=\theta_i-\frac{1-\Phi_i(\theta_i)}{\phi_i(\theta_i)}$. By inspection of (29), we see that any two Bayesian incentive compatible direct revelation mechanisms that generate the same allocation functions $(y_{ij}(\cdot))_{i\in N,j\in M}$ and the same values of $(U_1(L),\ldots,U_n(L))$ generate the same expected revenue for the search engine.

Q.E.D.

PROPOSITION 7 **(Revenue Equivalence of GSP, VCG, and OPT Mechanisms)**. *Consider a sponsored search auction setting, in which*

1. *The advertisers are risk neutral,*
2. *The advertisers are symmetric, i.e.*
   - $\Theta_1=\ldots=\Theta_n=\Theta=[L,U]$
   - $\Phi_1(\cdot)=\ldots=\Phi_n(\cdot)=\Phi(\cdot)$,
3. *For each advertiser i, we have $\phi_i(\cdot)>0$; and*
4. *The advertisers draw their types independently,*
5. *For each advertiser i, we have $J_i(\cdot)>0$ and $J_i(\cdot)$ is non-decreasing function.*

If R_{GSP}, R_{VCG} and R_{OPT} be the expected revenue earned by the search engine, against every search query received by the search engine, under the GSP, the VCG, and the OPT mechanisms, respectively, then

$$
\begin{array}{lcl}
R_{GSP}=R_{VCG}=R_{OPT} & : & \text{if } m<n \\
R_{VCG}\le R_{GSP}\le R_{OPT} & : & \text{if } n\le m
\end{array}
$$

Proof: Recall Proposition 5 which says that under the assumptions which are stated above, the VCG and the OPT mechanisms result in the same allocation for any given bid vector $b=(b_1,\ldots,b_n)$. Also, recall that the VCG and the OPT mechanisms are incentive compatible which implies that the advertisers bid their true types under both of these two mechanisms. Therefore, we can conclude that under the assumptions stated above, the VCG and the OPT mechanisms result in the same allocation for any given type profile $\theta=(\theta_1,\ldots,\theta_n)$. Note that this result holds irrespective of whether $m<n$ or $n\le m$. Now coming to the GSP mechanism, the Equation (27) shows that the GSP mechanism has a symmetric and increasing Bayesian Nash equilibrium. Therefore, if $\theta=(\theta_1,\ldots,\theta_n)$ is the type profile of the advertisers then the bid profile would be $(s(\theta_1),\ldots,s(\theta_n))$, where $s(\cdot)$ is given by Equation (27). Because $s(\cdot)$ is increasing, the ordering of the bids $s(\theta_1),\ldots,s(\theta_n)$ is the same as the ordering of the types $\theta_1,\ldots,\theta_n$. Therefore, the GSP mechanism will also result in the same allocation as the VCG and the OPT. Once again this result holds irrespective of whether $m<n$ or $n\le m$. Thus, we have shown that irrespective of whether $m<n$ or $n\le m$, for any given type profile $\theta=(\theta_1,\ldots,\theta_n)$ of the advertisers, advertiser i has an identical probability of getting slot j in all the three mechanisms namely GSP, VCG, and OPT. This confirms the first condition required for the revenue equivalence theorem.

In order to show the second condition, we need to consider three scenarios separately. It is easy to see that if an advertiser i has $\theta_i=L$, then under each one of these three mechanisms, the outcome of the mechanism will conform to one of the following three scenarios.

1. **The advertiser i does not get any slot:** Note that this scenario occurs only when $m < n$. In such a situation, irrespective of the auction mechanism, the advertiser i neither pays any amount to the search engine nor gets any click in return. Therefore, the utility of the advertiser i under this scenario is zero for all the three mechanisms.

2. **The advertiser i gets the last slot:** This scenario may arise in both the cases - $m < n$ and $n \leq m$. We analyze these cases separately.

(a) $m < n$: It is straightforward to verify that all the loosing bids will be equal to L under all the three mechanisms. This is because the VCG and the OPT mechanisms are incentive compatible. Hence, no bid can be smaller than L for these mechanisms. Similarly, for the GSP mechanism, by virtue of Equation (27), we have $s(L) = L$ and moreover the function $s(\cdot)$ is increasing. This again implies that all the losing bids will be equal to L for the GSP mechanism as well. By invoking the respective payment rules for these three mechanisms, we can verify that under this case, the advertiser i needs to pay an amount L to the search engine for every click received from a user under each one of the three mechanisms. Thus, the advertiser i pays an amount L for each user click and gets a benefit of L under each mechanism. Therefore, the net utility of the advertiser under this case is zero for each one of the three mechanisms.

(b) $n \leq m$**:** Here, no advertiser loses. Therefore, by invoking the respective payment rules for the three mechanisms, we can say that under this case, the advertiser i needs to pay an amount equal to $0, 0$, and L under the GSP, the VCG, and the OPT mechanisms, respectively. This implies that the advertiser's utility for every user click is L, L, and 0 for the GSP, the VCG, and the OPT mechanisms, respectively.

3. **The advertiser i gets a slot other than the last slot:** Note that this scenario can arise under both the cases - $m < n$ and $n \leq m$. Let us analyze each case.

(a) $m < n$: It is straightforward to verify that under all the three mechanisms, the bid of an advertiser must be equal to L if either the advertiser gets a slot that is below the advertiser i or the advertiser does not get any slot. Now by invoking the respective payment rules, we can claim that in this case, the advertiser i needs to pay an amount L to the search engine for every click received from a user under each of the three mechanisms. Thus, we see that for this case, the advertiser i pays an amount L for each user click and gets a benefit of L under each mechanism. Therefore, the net utility of the advertiser under this case is zero for each mechanism.

(b) $n \leq m$**:** Similar to the previous case, it is easy to verify in this case that under all the three mechanisms, the bid of an advertiser must be equal to L if the advertiser gets a slot that is below the advertiser i. By invoking the respective payment rules for the three mechanisms, we can say that the advertiser i needs to pay an amount equal to $L, L(1 - \frac{\alpha_n}{\alpha_j})$, and L under the GSP, the VCG, and the OPT mechanisms, respectively. Here α_j is the click probability of the slot at which advertiser i's Ad is displayed. This implies that the utility of advertiser i for every user click is $0, L\frac{\alpha_n}{\alpha_j}$, and 0 for the GSP, the VCG, and the OPT mechanisms, respectively.

The above discussion implies that advertiser i has zero expected utility level in all the three mechanisms when his type θ_i is at its lowest possible level and when $m < n$. Thus, we can now invoke the revenue equivalence theorem and get the first part of the desired result, that is

$$R_{\text{GSP}} = R_{\text{VCG}} = R_{\text{OPT}} \text{ if } m < n$$

In order to get the second part, observe that in Equation (29), if the allocation rule is the same then the expected revenue of the search engine depends solely on the values of $U_i(\underline{\theta_i})$. In the above discussion we have shown that for any advertiser i we have,

$$U_i^{\text{OPT}}(\underline{\theta_i}) \leq U_i^{\text{GSP}}(\underline{\theta_i}) \leq U_i^{\text{VCG}}(\underline{\theta_i})$$

The above inequality can be used in conjunction with Equation (29) to conclude the second part of the desired result, that is

$$R_{\text{GSP}} \le R_{\text{VCG}} \le R_{\text{OPT}} \text{ if } n \le m$$

Q.E.D.

In what follows, we actually derive the exact expressions for the expected revenue earned by the search engine under these three different mechanisms.

4.3. Expected Revenue under GSP, VCG, and OPT

We first compute the equilibrium bidding strategies of the advertisers under each mechanism. Next we compute the expected revenue earned by the search engine under each mechanism assuming that the advertisers will respond with corresponding equilibrium bidding strategies. We have already seen that

- Truth revelation constitutes a dominant strategy equilibrium under the VCG mechanism
- Truth revelation constitutes a Bayesian Nash equilibrium under the OPT mechanism
- Truth revelation does not constitute an equilibrium under the GSP mechanism.

We follow the assumptions made in Section 4 in the rest of the discussion.

4.3.1. Expected Revenue under the VCG Mechanism Under the assumption of the symmetric advertisers, we compute the expected revenue, R_{VCG}, earned by the search engine in the following way.

$$R_{\text{VCG}} = E_\theta \left[\sum_{j=1}^{\min(m,n)} \alpha_j p^{(j)}(\theta) \right], \tag{30}$$

where $p^{(j)}(\theta)$ is the payment made by the advertiser, whose Ad is displayed in the i^{th} position, to the search engine against every click made by the user and when the bid profile of the advertisers is $\theta = (\theta_1, \dots, \theta_n)$. Since truth-telling is a dominant strategy equilibrium for the advertisers in the VCG mechanism, the reported type (or bid) profile of the advertisers is indeed their actual type profile. We consider two cases separately:

Case 1 ($m < n$)**:** Substituting Equation (8) in Equation (30), we get the following relation

$$R_{\text{VCG}} = E_\theta \left[m\alpha_m \theta^{(m+1)} + \sum_{j=1}^{m-1} j\beta_j \theta^{(j+1)} \right] \tag{31}$$

Now we need to compute the expectation of each term separately. For this, notice that the advertisers are assumed to be symmetric and they choose their bids independently, therefore, the probability that $(j+1)^{th}$ highest bid lies in an interval $[x, x+dx]$ can be given by

$$n\binom{n-1}{j}[1-\Phi(x)]^j[\Phi(x)]^{n-j-1}\phi(x)dx$$

where $j = 0, \dots, n-1$ and $x \in [L, U]$. Therefore, the expected value of the $(j+1)^{th}$ highest bid is given by

$$E_\theta\left[\theta^{(j+1)}\right] = \int_L^U x\, n\binom{n-1}{j}[1-\Phi(x)]^j[\Phi(x)]^{n-j-1}\phi(x)dx \tag{32}$$

Substituting Equation (32) in Equation (31), we get the following relation for expected revenue earned by the search engine under this case

$$R_{\text{VCG}} = \int_{L}^{U} \left[m\alpha_m \binom{n-1}{m} [\overline{\Phi}(x)]^m [\Phi(x)]^{n-m-1} + \sum_{j=1}^{m-1} j\beta_j \binom{n-1}{j} [\overline{\Phi}(x)]^j [\Phi(x)]^{n-j-1} \right] xn\phi(x)dx \tag{33}$$

where $\overline{\Phi}(\cdot) = 1 - \Phi(\cdot)$.

Case 2 ($n \leq m$)**:** Substituting Equation (9) in Equation (30), we get the following relation

$$R_{\text{VCG}} = E_\theta \left[\sum_{j=1}^{n-1} j\beta_j \theta^{(j+1)} \right] \tag{34}$$

Following the same approach as for the case 1, we get the following relation for expected revenue earned by the search engine:

$$R_{\text{VCG}} = \int_{L}^{U} \left[\sum_{j=1}^{n-1} j\beta_j \binom{n-1}{j} [\overline{\Phi}(x)]^j [\Phi(x)]^{n-j-1} \right] xn\phi(x)dx \tag{35}$$

4.3.2. Expected Revenue under the OPT Mechanism Because of the symmetric advertisers assumption and the fact that truth-telling is a Bayesian Nash equilibrium for the advertisers under the OPT mechanism, the expected revenue, R_{OPT}, earned by the search engine under the OPT mechanism can be computed by Equation (30) that were discussed earlier in the context of R_{VCG}. Once again, We consider two cases separately.

Case 1 ($m < n$)**:** Substituting Equation (17) in Equation (30), we get the following relation

$$R_{\text{OPT}} = E_\theta \left[m\alpha_m \theta^{(m+1)} + \sum_{j=1}^{m-1} j\beta_j \theta^{(j+1)} \right] \tag{36}$$

Following the same approach as for the case 1 of R_{VCG}, we get the following relation for expected revenue earned by the search engine under this case

$$R_{\text{OPT}} = \int_{L}^{U} \left[m\alpha_m \binom{n-1}{m} [\overline{\Phi}(x)]^m [\Phi(x)]^{n-m-1} + \sum_{j=1}^{m-1} j\beta_j \binom{n-1}{j} [\overline{\Phi}(x)]^j [\Phi(x)]^{n-j-1} \right] xn\phi(x)dx \tag{37}$$

It is easy to verify that $R_{\text{OPT}} = R_{\text{VCG}}$ for the case when $m < n$. This matches with the previous result about revenue equivalence of the OPT and the VCG mechanisms stated in the form of Proposition 7.

Case 2 ($n \leq m$)**:** Substituting Equation (18) in Equation (30), we get the following relation

$$R_{\text{OPT}} = E_\theta \left[n\alpha_n L + \sum_{j=1}^{n-1} j\beta_j \theta^{(j+1)} \right] \tag{38}$$

Following the same approach as for the case 1, we get the following relation for expected revenue earned by the search engine under this case

$$R_{\text{OPT}} = n\alpha_n L + \int_L^U \left[\sum_{j=1}^{n-1} j\beta_j \binom{n-1}{j} [\overline{\Phi}(x)]^j [\Phi(x)]^{n-j-1} \right] xn\phi(x)dx \tag{39}$$

It is easy to verify that $R_{\text{VCG}} \leq R_{\text{OPT}}$ for the case when $n \leq m$. The equality holds if and only if $L = 0$. This matches with the previous result about revenue equivalence of the OPT and the VCG mechanisms stated in the form of Proposition 7.

4.3.3. Expected Revenue under the GSP Mechanism Recall that truth-telling need not be a Bayesian Nash equilibrium for the advertisers under the GSP mechanism. Therefore, the methods for computing the expected revenue of the search engine under this auction mechanism can be modified in following manner.

Method 1:

$$R_{\text{GSP}} = n \int_{\theta_i = L}^{U} \bar{t}_i(s(\theta_i))\phi(\theta_i)d\theta_i \tag{40}$$

where $s(\cdot)$ is the symmetric equilibrium bidding strategy of the advertiser i and is given by Equation (27).

Method 2:

$$R_{\text{GSP}} = E_\theta \left[\sum_{j=1}^{\min(m,n)} \alpha_j p^{(j)}(s(\theta_1), \ldots, s(\theta_n)) \right] \tag{41}$$

where $s(\cdot)$ is the symmetric equilibrium bidding strategy of the advertiser i and is given by Equation (27).

Note that computing the exact expression for expected revenue R_{GSP} is a difficult problem because computing the exact expression for $s(\cdot)$ by solving the Equation (27) is a hard problem. We, therefore, take a different approach here and instead of computing the exact expression for R_{GSP}, we appeal to the Proposition 7 which says that

$$\begin{array}{lcl} R_{\text{GSP}} = R_{\text{VCG}} = R_{\text{OPT}} & : & \text{if } m < n \\ R_{\text{VCG}} \leq R_{\text{GSP}} \leq R_{\text{OPT}} & : & \text{if } n \leq m \end{array}$$

Note that we have already computed R_{VCG} and R_{OPT} for both the cases - $m < n$ and $n \leq m$. Therefore, we can get the exact expression for R_{VCG} when $m < n$, and an upper and a lower bound when $n \leq m$ by making use of the Equations (33), (37), (35), and (39).

4.4. Individual Rationality

We know that the OPT mechanism satisfies interim individual rationality by definition. In order to check whether or not the GSP mechanism satisfies it, we need to make the observation that under the GSP mechanism, an advertiser would never pay more than what he has has bid for each user click on his Ad. Therefore, as long each advertiser i uses a bidding strategy $s_i(\theta_i)$ such that $s_i(\theta_i) \leq \theta_i \,\forall\, \theta_i \in \Theta_i$, it will immediately imply that $U_i(\theta_i) \geq 0 \,\forall\, \theta_i \in \Theta_i$. This would satisfy the interim individual rationality constraints. It is easy to verify that under the symmetry assumption, no equilibrium of the GSP mechanism will ever have $s_i(\theta_i) > \theta_i$ for any advertiser i and for any $\theta_i \in \Theta_i$. This proves that the GSP mechanism always satisfies interim individual rationality.

The VCG mechanism is also interim individually rational. This can be verified by observing that in the VCG mechanism, the payment made by an advertiser against each user click is always less than or equal to his bid amount and the bid amount of each advertiser is always his true valuation. To show that the payment made by an advertiser per user click is less than or equal to his bid amount, we start with the payment rule of the VCG mechanism which is given by Equations (8) and (9). We consider each case separately.

Case 1 $(m < n)$**:** Notice that under this case,

—If an advertiser is not allocated any slot then by virtue of Equation (8), he pays nothing, which ensures interim IR

—If an advertiser i, with his bid θ_i, is allocated the last position, i.e. the m^{th} position, then as per Equation (8), he pays an amount $\theta^{(m+1)}$ for each user click. It is easy to see that $\theta^{(m+1)} \leq \theta_i$ because in the VCG mechanism, the advertisers are allocated the slots in decreasing order of their bids and advertiser i has received the m^{th} slot. This again ensures interim IR.

—If an advertiser i, with his bid θ_i, is allocated the position r, where $1 \leq r \leq (m-1)$, then according to Equation (8), he will be paying an amount

$$p_i(\theta_i, \theta_{-i}) = \frac{1}{\alpha_r}\left[\sum_{j=r}^{m-1} \beta_j \theta^{(j+1)}\right] + \frac{\alpha_m}{\alpha_r}\,\theta^{(m+1)}$$

for every user click. Notice that because in the VCG mechanism, the advertisers are allocated the slots in decreasing order of their bids and the advertiser i has already received the r^{th} slot, we must have

$$p_i(\theta_i, \theta_{-i}) \leq \frac{1}{\alpha_r}\left[\sum_{j=r}^{m-1} \beta_j \theta_i\right] + \frac{\alpha_m}{\alpha_r}\,\theta_i = \theta_i\left[\sum_{j=r}^{m-1} \frac{\alpha_j - \alpha_{j+1}}{\alpha_r} + \frac{\alpha_m}{\alpha_r}\right] = \theta_i$$

This ensures interim IR even for this case.

Case 2 $(n \leq m)$**:** For this case we make use of Equation (9) and go about applying similar arguments which we used in the previous case and show that the VCG mechanism is interim IR even under this case as well. Therefore, we can say that the VCG mechanism is interim individually rational.

4.5. Computational Complexity

Note that in all the discussed auction schemes, after receiving the query word, the search engine needs to retrieve from its database the bids of the advertisers who are interested in displaying their Ads. After getting these bid values, say $b_1, \ldots, b_n$, the search engine needs to sort them in decreasing order if it is either the GSP or the VCG mechanism. As is well known, the worst case complexity of sorting n numbers is $O(n \log n)$. The sorted bids $b^{(1)}, \ldots, b^{(n)}$ can now be used for computing the allocation and the payment of each advertiser. It is easy to verify that the allocation operation has a worst case complexity of $O(\min(m,n))$ for both the GSP and the VCG mechanisms. The payment operation has a worst case complexity of $O(\min(m,n))$ for the GSP mechanism and $O((\min(m,n))^2)$ for the VCG mechanism. Thus, the worst case computational complexity of the GSP is $O(n \log n + \min(m,n))$, which is the same as $O(n \log n)$, and the worst case complexity of the VCG mechanism is $O(n \log n + (\min(m,n))^2)$.

The practical implementation of the OPT mechanism has its own challenges. Recall that the design of the OPT mechanism intrinsically assumes that the search engine precisely knows the type distribution $\Phi_i(\cdot)$ of each advertiser i. However, in practice this may not be true. The search engine typically has no information about an advertiser except his bid value and the history of click streams. However, the search engine can always learn the type distribution $\Phi_i(\cdot)$ of each advertiser i from these given data. Assuming that the search engine knows the type distributions $\Phi_i(\cdot)$ for each advertiser i and that $\phi_i(\cdot)$ is a positive function for each i and $J_i(\cdot)$ is a non-decreasing function for each i, our objective here is to compute the computational complexity of the OPT mechanism. Note that after receiving the bid values (which is same as actual types), say $\theta_1, \ldots, \theta_n$, from its database, the search engine needs to compute $J_1(\theta_1), \ldots, J_n(\theta_n)$. This is an $O(n)$ operation. Next, the search engine needs to sort $J_1(\theta_1), \ldots, J_n(\theta_n)$ in decreasing order which is an $O(n \log n)$ operation. The search engine can use these sorted values to compute the assignment of the advertisers which is an $O(\min(m, n))$ operation. Thus the complexity of the allocation operation under OPT mechanism is $O(n + n \log n + \min(m, n))$, which is the same as $O(n \log n)$. As for the payment determination, note that the search engine needs to compute the quantities $z_{ij}(\theta_{-i})$ for each advertiser i. Assuming that functions $J_i(\theta_i)$ are invertible (a popular example is the uniform distribution), computing the quantity $z_{ij}(\theta_{-i})$ is a constant time operation. To illustrate this, suppose the advertiser i is allocated the r^{th} position, then we have

$$z_{ij} = \begin{cases} J_i^{-1}(J^{(j)}) & : \text{ if } j = 1, \ldots, r-1 \\ J_i^{-1}(J^{(j+1)}) & : \text{ if } j = r, \ldots, \min(m, n) \end{cases}$$

where $J^{(j)}$ is the value of the quantity $J_k(\theta_k)$ for an advertiser k whose Ad is allocated to the j^{th} position. In view of the assumption of invertibility of the functions $J_i(\cdot)$, we can say that computing the quantities $z_{ij}(\theta_{-i})$ is an $O((\min(m, n))^2)$ operation. After computing these quantities the payment for the advertisers can be computed in $O((\min(m, n))^2)$ time. Thus the complexity of the payment operation under the OPT mechanism is $O((\min(m, n))^2)$. Therefore, the computational complexity of the OPT mechanism, under the assumption that the function $J_i(\cdot)$ is invertible for every i, is $O(n \log n + (\min(m, n))^2)$ which is the same as the computational complexity of the VCG mechanism.

5. Summary

In this paper, we formulated the sponsored search auction as a mechanism design problem in linear environment and then showed that three well known mechanisms, GFP, GSP, and VCG can be conveniently described in this framework. Next, we proposed a new mechanism, called the OPT mechanism. We compared the OPT mechanism with the GSP and VCG mechanisms from the point of view of incentive compatibility, expected revenue earned by the search engine, and individual rationality. We derived a symmetric equilibrium bidding strategy of the advertisers for the GSP mechanism and this was instrumental in showing that the GSP is not a Bayesian incentive compatible mechanism. We extended the classical revenue equivalence theorem to the setting of sponsored search auction and used it to show the revenue equivalence of the three mechanisms. Finally, we also computed expressions for the expected revenue earned by the search engine under the GSP, the VCG, and the OPT mechanisms.

We can summarize the results of the comparative study for three different sponsored search auction mechanisms in the form of Table 2.

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Table 2 Properties of various sponsored search auction mechanisms

Auction	**AE**	**DSIC**	**BIC**	**Ex Post IR**	**Complexity**
GSP	√	×	×	√	$O(n \log n)$
VCG	√	√	√	√	$O(n \log n + (\min(m,n))^2)$
OPT	√	×	√	√	$O(n \log n + (\min(m,n))^2)$

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