Game Theory

Lecture Notes By

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Chapter 3: Strategic Form Games

Note: *This is a only a draft version, so there could be flaws. If you find any errors, please do send email to* `hari@csa.iisc.ernet.in`. *A more thorough version would be available soon in this space.*

Strategies

The notion of strategy is a key notion in game theory. A *strategy* is a complete contingent plan which specifies what a player will do at each of his information sets where he is called upon to play.

- Recall that the set of information sets of a player represents the set of all possible distinguishable circumstances in which the player may be required to play.
- A player's strategy completely specifies the action he is going to choose in each of his information sets, if it is reached during play of the game.

Let IS_i denote the collection of information sets of player i in a game. Let A be the set of possible actions in the game. Let J be an information set of player i, i.e., $J \in IS_i$. Recall that $C(J) =$ set of actions possible at information set J.

A strategy of player i can be defined as a mapping $s_i : IS_i \to A$ such that $s_i(J) \in C(J) \ \forall J \in IS_i$. Note that a strategy of a player is a *complete contingent plan* that specifies an action for every information set of the player. The player can make a table with two columns, one for information sets and another for corresponding actions and a representative can take over and play the game using the table lookup.

Example: Matching Pennies (Version B) with Observation

Information sets of player 1: $IS_1 = \{\{v_1\}\}$.
Information sets of player 2 :$IS_2 = \{\{v_2\}, \{v_3\}\}$ Strategies of Player 1:

$$s_{11} : \quad \{v_1\} \to H$$
$$s_{12} : \quad \{v_1\} \to T$$

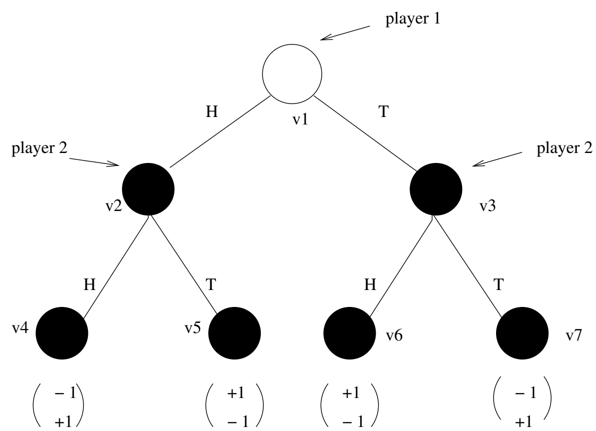

Figure 1: Game tree for matching pennies with observation

Strategies of Player 2:

$$
\begin{array}{ll}
s_{21}: & \{v_2\} \to H \quad \{v_3\} \to H \\
s_{22}: & \{v_2\} \to H \quad \{v_3\} \to T \\
s_{23}: & \{v_2\} \to T \quad \{v_3\} \to H \\
s_{24}: & \{v_2\} \to T \quad \{v_3\} \to T
\end{array}
$$

We can now represent the payoffs in the following way.

1	2			
	s_{21}	s_{22}	s_{23}	s_{24}
s_{11}	$-1, +1$	$-1, +1$	$+1, -1$	$+1, -1$
s_{12}	$+1, -1$	$-1, +1$	$+1, -1$	$-1, +1$

$$
\begin{array}{rcl}
N & = & \{1, 2\} \\
S_1 & = & \{s_{11}, s_{12}\} \\
S_2 & = & \{s_{21}, s_{22}, s_{23}, s_{24}\} \\
u_1 & : & S_1 \times S_2 \to R \\
u_2 & : & S_1 \times S_2 \to R
\end{array}
$$

defined as in the table above $(N, S_1, S_2, u_1(\cdot), u_2(\cdot))$ defines the strategic form or the normal form of the game. This is also abbreviated as $(N, (S_i)_{i \in N}, (u_i)_{i \in N})$, written often as $(N, (S_i), (u_i))$. This is a two player zero sum game.

Example: Matching Pennies Without Observation (Version C)

$$
\begin{array}{rcl}
IS_1 & = & \{\{v_1\}\} \\
IS_2 & = & \{\{v_2, v_3\}\}
\end{array}
$$

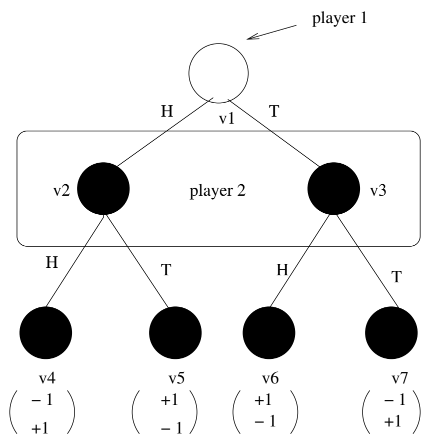

Figure 2: Game tree for matching pennies without observation

Player 1 has two strategies and player 2 has two strategies.

$$
\begin{aligned}
s_{11} &: \{v_1\} \to H \\
s_{12} &: \{v_1\} \to T \\
s_{21} &: \{v_2, v_3\} \to H \\
s_{22} &: \{v_2, v_3\} \to T
\end{aligned}
$$

The payoff matrix here is

1	2	
	s_{21}	s_{22}
s_{11}	$-1, +1$	$+1, -1$
s_{12}	$+1, -1$	$-1, +1$

We now have a strategic form game $(N, (S_i), (u_i))$ with

$$
\begin{aligned}
N &= \{1, 2\} \\
S_1 &= \{s_{11}, s_{12}\} \\
S_2 &= \{s_{21}, s_{22}\}
\end{aligned}
$$

u_1, u_2 are as defined in the above table. This is again a two player zero sum game.

Example: Matching Pennies with Simultaneous Moves (Version A)

This has the same normal form representation as that of version C.

We are now ready to define a strategic form game or a normal form game.

Definition: A strategic form game G is a tuple $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ where $N = \{1, 2, \ldots, n\}$ is a finite set of players; $S_1, S_2, \ldots, S_n$ are the strategy sets of the players; and $u_i : S_1 \times S_2 \times \cdots \times S_n \to R$ for $i = 1, 2, \ldots, n$ are von Neumann - Morgenstern utility functions We often denote $S = S_1 \times S_2 \times \cdots \times S_n$.

- S is called the set of strategy profiles of the players
- The strategies here are also called "actions" or "pure strategies"
- Note that the utility of a player depends not only on his own strategy but on an entire strategy profile.
- Utility functions are also called *payoff functions*.
- Every profile of strategies induces an *outcome* in the game.
- The idea behind the normal form representation is that a player's decision problem is to simply choose a strategy that he thinks will counter best the strategies adopted by the other players.
- Each player is faced with the problem and therefore the players can be thought of as simultaneously choosing their strategies from the respective sets $S_1, S_2 \ldots, S_n$.
- One can view the play of a strategic game as follows: each player simultaneously writes down a chosen strategy on a piece of paper and hands it over to a referee who then computes the outcome and the utilities.
- Every extensive form game has a unique normal form representation. The uniqueness is up to any renaming or renumbering of strategies.
- However, a given normal form game may correspond to multiple extensive form games.

Example: The extensive form game in Figure 3 has the same normal form representation as version B of matching pennies.

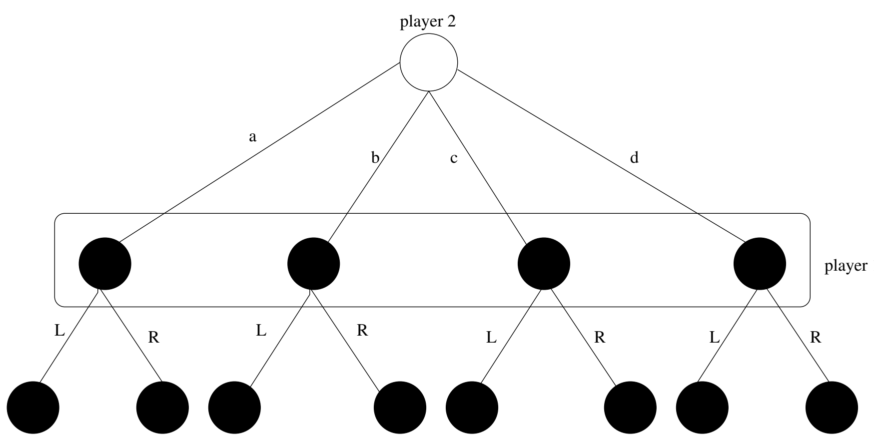

Figure 3: Another extensive form representation for matching pennies with observation

1	2			
	a	b	c	d
L	−1, +1	−1, +1	+1, −1	+1, −1
R	+1, −1	−1, +1	+1, −1	−1, +1

- An Important question: Does the normal form summarize all of the strategically relevant information?
  - This question has been debated quite intensely by game theorists.
  - In static games where all the players (statically) choose their actions at the same time (without observing the choices of other players), the normal form and the extensive form have the same representational power.
  - This is not so in the case of dynamic games.

Interpretations of Strategic Form Games

Osborne and Rubinstein provide the following two interpretations.

Interpretation 1

- A strategic game is a model of an event that occurs only once.
  - Each player knows the details of the game and the fact that all players are rational
  - The players choose their strategies simultaneously and independently
  - Each player is unaware of the choices being made by the other players.

Interpretation 2

- A player can form an expectation of the other players' behavior on the basis of information about the way that the game or a similar game was played in the past.
- A strategic game models a sequence of plays of the game under the condition that there is no strategic link between the plays of the game.
  - That is, a player who plays the game many times should only worry about his own instantaneous payoff and ignore the effects of his current action on the future behavior of the other players.

Note: A class of games called *repeated games* will have to be used if there is a strategic link between plays of a game.

Examples of Normal Form Games

Example 1: Matching Pennies

This is also called the matching coins problem. Two players 1 and 2 put down their respective rupee coins, heads up or tails up. If both the coins match (both heads or both tails), then player 1 pays 1 Rupee to player 2. Otherwise, player 2 pays 1 Rupee to player 1.

$$\begin{aligned} N &= \{1, 2\} \\ S_1 &= S_2 = \{H, T\} \\ S &= S_1 \times S_2 = \{(H,H), (H,T), (T,H), (T,T)\} \end{aligned}$$

The payoff matrix is given by

	2	
1	H	T
H	−1, +1	+1, −1
T	+1, −1	−1, +1

This is a classic example of a two player zero-sum game.

Example 2: BOS Game

This is called the *Battle of Sexes* or the *Batch or Stravinsky* game.

- Proposed by Luce and Raiffa (1957). Two players 1 and 2 wish to go out together for a music concert or a film. Player 1 prefers music concert and player 2 prefers film.

	2	
1	M	F
M	2,1	0,0
F	0,0	1,2

- This game captures a situation where the players want to coordinate but they have conflicting interests.
- The outcomes (M,F) and (F,M) can be ruled out. The question is: which one between (M,M) and (F,F) is the likely outcome?

Example 3: A Coordination Game

This game is similar to BOS but the two players now have preference for the same option, namely music concert. In this case, the natural outcome should be (M,M).

	2	
1	M	F
M	(2,2)	(0,0)
F	(0,0)	(1,1)

Example 4: Hawk-Dove (Chicken)

There are two players who are fighting over a company/prey/property/etc. Each player can behave like a hawk or a dove.

	2	
1	Hawk	Dove
Hawk	3,3	4,1
Dove	1,4	0,0

- The best outcome happens for a player who acts as a hawk while the other acts like a dove.
- When both are hawks, the outcome is least desirable since nobody wins.
- When both are hawks, the outcome is better than if both were doves.
- What is a good prediction for this game?

Example 5: Prisoner's Dilemma

- This is attributed to Raiffa (1951) and Flood and Dresher (1952).
- This is one of the most extensively studied problems in game theory. Significant amount of experimentation has been done.
- Two individuals are arrested for allegedly committing a crime and are lodged in separate prisons. The district attorney (DA) interrogates them separately.
- The DA privately tells each prisoner that if he is the only one to confess, he will get a light sentence of 1 year in jail while the other would be sentenced to 10 years in jail. If each player confesses, then each one would get 5 years in jail. If neither confesses, then each would get 2 years in jail.

	2	
1	NC	C
NC	$-2, -2$	$-10, -1$
C	$-1, -10$	$-5, -5$

- How would the prisoners behave in such a situation? They would obviously like to minimize their stay in the jail.

Observation 1

C is each player's best strategy regardless of what the other player plays:

$$u_1(C, C) > u_1(NC, C)$$
$$u_1(C, NC) > u_1(NC, NC)$$

$$u_2(C, C) > u_2(C, NC)$$
$$u_2(NC, C) > u_2(NC, NC)$$

Thus (C,C) is a natural prediction for this game.

Observation 2

- Though (C,C) is a natural prediction, the outcome (NC, NC) is the best outcome jointly for the players.
- Prisoner's Dilemma is a classic example of a game where rational, intelligent behavior does not lead to a socially optimal result (Pareto efficient outcome).

Observation 3

- Each prisoner has a negative effect on the other. When a prisoner moves away from (NC, NC) to reduce his jail term by 1 year, the jail term of the other player increases by 8 years. This is an example of what is called *externality*.

Observation 4

NC is good for a player only if the other player also plays NC. Thus cooperation leads to a socially optimal outcome.

Example 6: DA's Brother

This is a modification of the PD problem in which prisoner 1 is a brother of the District Attorney. As a result, the DA is lenient towards prisoner 1.

	2	
1	NC	C
NC	$0, -2$	$-10, -1$
C	$-1, -10$	$-5, -5$

Example 7: Cold War

The countries have to decide whether they should emphasize defence spending or on healthcare.

	Pakistan	
India	Healthcare	Defence
Healthcare	10,10	-10, 20
Defence	20,-10	0,0

Observation 1

Each player finds that "defence" is the best response whatever the other player plays.

Observation 2

Healthcare is good only if the other player plays healthcare.

Observation 3

Predicted outcome for the game is (defence, defence) but (healthcare, healthcare) is socially optimal (Pareto efficient).

Observation 4

Like the Prisoner's Dilemma problem, this is an example where rationality leads to an outcome that is not socially optimal.

Observation 5

If the players can cooperate, the outcome will be socially optimal.

Example 8: Tragedy of the Common

This situation was first studied by

- A village has n farmers $N = \{1, 2, \ldots, n\}$.
- Each farmer has the option of keeping a sheep or not. $S_1 = S_2 = \cdots = S_n = \{0, 1\}$.
- Utility from keeping a sheep arises because of milk, wool, etc. Utility = 1.
- The village has grassland (limited) and when a sheep grazes on this, the damage to the environment = 5.
- The damage to the environment is to be shared equally by the farmers.

Let s_i be the strategy of each farmer. Then $s_i = 0, 1$. The payoff to farmer i is given by:

$$
\begin{aligned}
&= u_i(s_1, s_2, \ldots s_i, \ldots, s_n) \\
&= s_i - \left[\frac{5(s_1 + \cdots + s_n)}{n}\right]
\end{aligned}
$$

For the case $n = 2$, the payoff matrix would be:

	2	
1	0	1
0	0,0	−2.5,−1.5
1	−1.5, −2.5	−4, −4

Observation 1

If $n > 5$, keeping a sheep would add more utility to a farmer from milk/wool than subtract utility from him due to environmental damage. If $n < 5$, then the farmer gets less utility from keeping than from not keeping a sheep. If $n = 5$, the farmer has equal benefit and loss.

Observation 2

If $n > 5$, then every farmer would like to keep a sheep. How about $n \leq 5$?

Observation 3

If the Government now imposes a pollution tax of 5 units for every sheep kept, the payoff becomes:

$$u_1(s_1, \ldots, s_n) = s_i - 5s_i - \frac{5(s_1 + \cdots + s_n)}{n}$$

Now every farmer will prefer not to keep a sheep.

Example 9: A Sealed Bid Auction

The problem here is to allocate a unique object to one of n players in exchange for a payment.

$$N = \{1, 2, \ldots, n\}$$

Sealed bids are sought from the n players. Let $v_1, v_2, \ldots, v_n$ be the valuations of the players for the object. Then the strategies of the players are:

$$\begin{aligned} S_1 &\in (0, v_1] \\ S_2 &\in (0, v_2] \\ &\vdots \\ S_n &\in (0, v_n] \end{aligned}$$

Assume that the object is awarded to the agent with the lowest index among those who bid the highest. Let $b_1, b_2, \ldots, b_n$ be the bids from the n players. Then the allocation will be:

$$\begin{aligned} &= 1 \quad && \text{if } b_i > b_j \quad && \forall j = 1, \ldots, i-1 \\ & && \text{and } b_i \geq b_j \quad && \forall j = i+1, \ldots, n \\ &= 0 && \text{otherwise} \end{aligned}$$

Let $p_i(b_1, \ldots, b_n)$ be the payment. Define the payoff to the players as

$$u_i(b_1, b_2, \ldots, b_n) = x_i(b_1, b_2, \ldots, b_n)(v_i - p_i(b_1, \ldots, b_n))$$

The question is: what is the predicted outcome of this game.

First Price Sealed Bid Auction

Here the winner pays as he bids.

Second Price Sealed Bid Auction

Here, the payment by the winner is the highest bid among the players who do not win.

Example 10: A Duopoly Pricing Game (Bertrand Model)

This is due to Bertrand (1883). There are two companies 1 and 2 which wish to maximize their profits. The demand for a price p is given by a continuous and strictly decreasing function $x(p)$. The cost for producing each unit of product $= c > 0$. The companies simultaneously choose their prices p_1 and p_2. The amount of sales for each company is given by

$$\begin{aligned} x_1(p_1, p_2) &= x(p_1) \quad \text{if} \quad p_1 < p_2 \\ &= \frac{1}{2}x(p_1) \quad \text{if} \quad p_1 = p_2 \\ &= 0 \quad \text{if} \quad p_1 > p_2 \end{aligned}$$

$$\begin{aligned} x_2(p_1, p_2) &= x(p_2) \quad \text{if} \quad p_2 < p_1 \\ &= \frac{1}{2}x(p_2) \quad \text{if} \quad p_1 = p_2 \\ &= 0 \quad \text{if} \quad p_2 > p_1 \end{aligned}$$

It is assumed that the firms incur production costs only for an output level equal to their actual sales. Given prices p_1 and p_2, the utilities of the two companies are:

$$u_1(p_1, p_2) = (p_1 - c)x_1(p_1, p_2)$$
$$u_2(p_1, p_2) = (p_2 - c)x_2(p_1, p_2)$$

Note that for this game,

$$N = \{1, 2\}$$
$$S_1 = S_2 = (0, \infty)$$

Result: For the Bertrand duopoly model, the profile (c,c) gives the unique Nash equilibrium. The implication of this result is that in the equilibrium, the companies set their prices equal to the cost.

Example 11: A Duopoly Pricing Game (Cournot Model)

Here, the two companies simultaneously decide how much to produce. This model is due to Cournot (1838).

Example : Agriculturists deciding how much of a perishable crop to pick each morning to send to a market. Here $p(\cdot) = \bar{x}^1(\cdot)$ is the inverse demand function and gives the price that clears the market. Here

$$N = \{1, 2\}$$
$$S_1 = S_2 = \{1, 2, \ldots, Q\}$$

Q is some maximum quantity.

$$u_1(q_1, q_2) = p(q_1, q_2)q_1 - cq_1$$
$$u_2(q_1, q_2) = p(q_1, q_2)q_2 - cq_2.$$